Permitted trigonometric thin sets and infinite combinatorics

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Abstract. We investigate properties of permitted trigonometric thin sets and construct uncountable permitted sets under some set-theoretical assumptions.

Keywords: permitted trigonometric thin sets, set of perfect measure zero, set of uniform measure zero, s-set

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1. Introduction

This work is motivated by properties of the family of $\mathcal{N}$-permitted sets. A set E is an N-set if there exists a trigonometric series

$$\sum_{n=1}^{\infty} \rho_n \sin n\pi x$$

absolutely converging on E with $\sum_{n=1}^{\infty} \rho_n = \infty$, $\rho_n \geq 0$. It is a well known fact that the family $\mathcal{N}$ of N-sets is not an ideal and this is the reason to look for conditions when the union of two N-sets is an N-set and, in particular, the reason for the study of sets which can be adjoined to any N-set so that the resulting set is again an N-set. Such sets are called $\mathcal{N}$-permitted sets. This notion was introduced by Arbault.

The classical Arbault-Erdős Theorem (see [1]) says that every countable set is an $\mathcal{N}$-permitted set. It is also well known an (unsuccessful) Arbault's attempt to construct a perfect $\mathcal{N}$-permitted set:

Let G be the set of convergence of a series $\sum_{n=1}^{\infty} \rho_n |\sin n\pi x|$ with $\sum_{n=1}^{\infty} \rho_n = \infty$. Let us find a condition posed on a perfect nowhere dense set P so that the union $G \cup P$ is again an N-set.

Let $s_n = \sum_{k=1}^{n} \rho_k$. Let $\{\eta_n\}_{n=1}^{\infty}$, $\{\varepsilon_n\}_{n=1}^{\infty}$ be decreasing sequences of positive reals converging to 0. Let q_n be the least natural number such that P can be covered by q_n intervals of length η_n. Let a_i^n, $i = 1, \ldots, q_n$, be the left end-points of these intervals. By Dirichlet-Minkowski Theorem there are natural numbers

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$1 \le \lambda_n \le (1/\varepsilon_n)^{q_n}$ such that $|\sin \lambda_n n\pi\alpha_i^n| < \pi\varepsilon_n$. Moreover, if we assume that $|\sin \lambda_n n\pi\eta_n| < \pi\varepsilon_n$ then since for every $y \in P$, $|y - \alpha_i^n| \le \eta_n$ for some i, we have $|\sin \lambda_n n\pi y| < 2\pi\varepsilon_n$ for every $y \in P$.

Let us determine the values of η_n, ε_n, q_n, ρ_n' so that $\sum_{n=1}^{\infty} \rho_n' = \infty$, and $\sum_{n=1}^{\infty} \rho_n' |\sin \lambda_n n\pi x| < \infty$ for $x \in G \cup P$. The convergence on the set G will be ensured by the condition $\rho_n' \lambda_n \le \rho_n$. To fulfill this let us set $\rho_n' = \rho_n \varepsilon_n^{q_n}$. The convergence on the set P will be ensured by the conditions $\sum_{n=1}^{\infty} \rho_n' \varepsilon_n < \infty$ and $|\sin \lambda_n n\pi\eta_n| \le \pi\varepsilon_n$ which follow from next two conditions

$$\sum_{n=1}^{\infty} \rho_n \varepsilon_n^{q_n+1} < \infty \quad \text{and} \quad n\eta_n \le \varepsilon_n^{q_n+1}, \tag{1.1}$$

since then $|\sin \lambda_n n\pi\eta_n| \le \lambda_n \pi \varepsilon_n^{q_n+1} \le \pi\varepsilon_n$. The divergence $\sum_{n=1}^{\infty} \rho_n' = \infty$ is now expressed by

$$\sum_{n=1}^{\infty} \rho_n \varepsilon_n^{q_n} = \infty. \tag{1.2}$$

Let us set $\varepsilon_n^{q_n} = 1/s_n$, $n\eta_n = 1/s_n^2$. This fulfills the second part of (1.1), and conditions (1.1) and (1.2) are expressed by

$$\sum_{n=1}^{\infty} \frac{\rho_n}{s_n^{1+1/q_n}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\rho_n}{s_n} = \infty. \tag{1.3}$$

Now we use the fact that if $f(n)$ is a monotone function and $\sum_{n=1}^{\infty} \rho_n = \infty$, then $\sum_{n=1}^{\infty} 1/f(n)$ converges if and only if $\sum_{n=1}^{\infty} \rho_n/f(s_n)$ converges. Therefore the second series diverges. The first one converges if we set, for example, $q_n \le (\log s_n)/\log(\log s_n)^2$.

Arbault [1] used these arguments to claim that the set P is an $\mathcal{N}$-permitted set. However, these arguments are not enough to derive this fact because the choice of the set P (namely η_n and q_n) depends on ρ_n. Although we are able to choose η_n independently on ρ_n by taking $n\eta_n = 1/n^2$, since we can always assume that $s_n \le n$, this is not possible for q_n. Arbault in fact took $q_n = (\log n)/\log(\log n)^2$ which does not depend on ρ_n but in this case Bary [4, Chapter XIII, §8] showed that the first series in (1.3) need not converge.

So far it is not known whether there exists a perfect $\mathcal{N}$-permitted set and even whether there exists an $\mathcal{N}$-permitted set of the size of the continuum.

We shall need the following small uncountable cardinals:

$\mathfrak{m}$ is the least cardinal κ for which Martin's Axiom MA_κ fails;

$\mathfrak{p}$ is the least size of a family $\mathcal{F} \subseteq [\omega]^\omega$ whose every finite subfamily has an infinite intersection but $\mathcal{F}$ has no infinite pseudo-intersection (i.e., there is no infinite set $A \in [\omega]^\omega$ such that $A - X$ is finite for every $X \in \mathcal{F}$);

$\mathfrak{t}$ is the least size of a tower, i.e., a family $\mathcal{F} \subseteq [\omega]^\omega$ well-ordered by $\supseteq^*$ without an infinite pseudo-intersection;

$\mathfrak{h}$ is the minimal cardinal κ such that the algebra $\mathcal{P}(\omega)/\mathrm{fin}$ is not κ-distributive;

$\mathfrak{s}$ is the least size of a splitting family $\mathcal{F} \subseteq [\omega]^\omega$, i.e., for every $A \in [\omega]^\omega$ there exists $B \in \mathcal{F}$ such that $|A - B| = |A \cap B| = \omega$;

$\mathfrak{b}$ is the cardinality of an unbounded subset of ${}^\omega\omega$ ordered by $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many n;

$\mathfrak{d}$ is the least cardinality of a cofinal subset of ${}^\omega\omega$ ordered by $\leq^*$.

It is well known that $\mathfrak{m} \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{h} \leq \min\{\mathfrak{s}, \mathfrak{b}\}$.

If I is an ideal of sets of reals we define:

add(I) is the minimal size of a set $A \subseteq I$ such that $\bigcup A \notin I$;

non(I) is the minimal size of a set $A \subseteq \mathbb{R}$ such that $A \notin I$;

cov(I) is the minimal size of a set $A \subseteq I$ such that $\bigcup A = \mathbb{R}$;

cof(I) is the minimal size of a cofinal subset A of I ordered by $\subseteq$.

The letters $\mathcal{L}$ and $\mathcal{M}$ denote the σ-ideal of Lebesgue measure zero sets and the σ-ideal of meager sets on the real line, respectively. We identify the real line $\mathbb{R}$ with the interval $[0, 1]$.

2. Permitted sets

In this work we deal with permitted sets for these families of trigonometric thin sets: $\mathcal{A}$, $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$ (the family of A-sets, the family of pseudo-Dirichlet sets, the family of N-sets, and the family of N_0-sets, respectively). They are all included in $\mathcal{L}$ and $\mathcal{M}$, i.e., in the σ-ideal of Lebesgue measure zero sets and the σ-ideal of meager sets on the real line.

Let us recall these notions. Let E be a set of reals.

(1) E is a pseudo-Dirichlet set (pD-set) if there exists an increasing sequence of integers $\{n_k\}_{k=0}^\infty$ such that the sequence $\{\sin n_k\pi x\}_{k=0}^\infty$ converges quasi-normally to 0, i.e., there exists a decreasing sequence of positive reals $\{\varepsilon_n\}_{n=0}^\infty$ converging to 0 such that $(\forall\, x \in E)(\forall^\infty k)\, |\sin n_k\pi x| < \varepsilon_k$.

(2) E is an N_0-set if there exists an increasing sequence of integers $\{n_k\}_{k=0}^\infty$ such that $\sum_{k=0}^\infty |\sin n_k\pi x| < \infty$ for $x \in E$.

(3) E is an A-set if there exists an increasing sequence of integers $\{n_k\}_{k=0}^\infty$ such that the sequence $\{\sin n_k\pi x\}_{k=0}^\infty$ converges pointwise to 0 on E.

(4) E is an N-set if there exists a sequence of nonnegative reals ρ_n such that $\sum_{n=0}^\infty \rho_n = \infty$ and $\sum_{n=0}^\infty \rho_n |\sin n\pi x| < \infty$ for $x \in E$.

For all basic facts and definitions we refer to the expository paper [9].

In general the words *family of thin sets* mean any family of sets of reals $\mathcal{F}$ such that (i) $\mathcal{F}$ contains all singletons, (ii) with every set, $\mathcal{F}$ contains all its subsets, and (iii) $\mathcal{F}$ does not contain an interval of reals.

Let $\mathcal{F}$ be a family of thin sets. A set A of reals is $\mathcal{F}$-permitted if $A \cup B \in \mathcal{F}$ for every $B \in \mathcal{F}$. For any family $\mathcal{F}$ of thin sets or reals let us denote

$$\mathrm{Perm}(\mathcal{F}) = \{A \subseteq \mathbb{R} : (\forall\, B \in \mathcal{F})\, A \cup B \in \mathcal{F}\}.$$

Each of the families $\mathcal{F} = p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$ has this property:

If $A \in \mathcal{F}$, then also the group generated by A belongs to $\mathcal{F}$.

It follows that $\mathrm{Perm}(\mathcal{F})$ has the same property. Moreover, a vector space over $\mathbb{Q}$ generated by an N-set is an N-set (see [1], page 267). Therefore, a vector space over $\mathbb{Q}$ generated by an $\mathcal{N}$-permitted set is an $\mathcal{N}$-permitted set.

A well known Marcinkiewicz's Theorem says that there are two perfect Dirichlet sets A, B such that $A + B$ is the real line $\mathbb{R}$ and hence $A \cup B \notin \mathcal{A}$ and $A \cup B \notin \mathcal{N}$. Therefore none of the considered families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, is an ideal and they are all families of thin sets in the above sense.

The study of permitted sets was started by Arbault in [1] and independently by Erdős who proved that every countable set is $\mathcal{N}$-permitted. Bukovská in [5] proved that every set of cardinality $< \mathfrak{p}$ is $p\mathcal{D}$-permitted. Bartoszyński and Scheepers [3] improved this by showing that every set of cardinality $< \mathfrak{h}$ is $p\mathcal{D}$-permitted and $\mathcal{N}_0$-permitted. Bukovský, Kholshchevnikova, and Repický in [9] proved that even every set of size $< \min\{\mathfrak{s}, \mathfrak{b}\}$ is $p\mathcal{D}$-permitted and $\mathcal{N}_0$-permitted. Bukovský and Bukovská in [7], and Kholshchevnikova in [15] and [16] proved that every set of size $< \mathfrak{p}$ is $\mathcal{N}$-permitted and every set of size $< \mathfrak{m}$ is $\mathcal{A}$-permitted. Also in the case of $\mathcal{N}$ and $\mathcal{A}$, Bartoszyński and Scheepers obtained improvements and showed that every set of size $< \mathfrak{t}$ is $\mathcal{N}$-permitted and every set of size $< \mathfrak{s}$ is $\mathcal{A}$-permitted.

An open cover $\mathcal{U}$ of a set A is an ω-cover if for every finite set $B \subseteq A$ there is $U \in \mathcal{U}$ such that $B \subseteq \mathcal{U}$. A set A is a γ-set if for every ω-cover $\mathcal{U}$ of A there is a sequence $\{U_n\}_{n=1}^{\infty}$ of sets from $\mathcal{U}$ such that $A \subseteq \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} U_n$. It is well known that the minimal cardinality of a set which is not a γ-set is $\mathfrak{p}$.

Galvin and Miller [11] assuming $\mathfrak{p} = \mathfrak{c}$ proved the existence of a γ-set of size continuum. Bukovský, Kholshchevnikova, and Repický [9] proved that every γ-set is permitted for the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, and $\mathcal{A}$, proving so the consistency of the existence of permitted sets of the size of the continuum.

Unfortunately, it is consistent with ZFC that every γ-set is countable, and so the question of the existence of large permitted sets in ZFC remained open.

3. Properties of permitted sets

The gap in Arbault's proof probably cannot be corrected. The natural question is how large $\mathcal{N}$-permitted sets can be. We describe a σ-ideal of sets which are $\mathcal{F}$-permitted for all families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, and $\mathcal{A}$. This σ-ideal helps to improve known lower estimates for the minimal size of a non-permitted set.

Definition 3.1. Let A be a set of reals.

(i) A has perfect measure zero if for every sequence of positive reals $\{\varepsilon_n\}_{n=1}^{\infty}$ there are an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ and a sequence of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that

  (1) $|\mathcal{I}_n| \le n$,

  (2) $|I| < \varepsilon_n$ for every $I \in \mathcal{I}_n$, and

  (3) $A \subseteq \bigcup_m \bigcap_{k>m} \bigcup \mathcal{I}_{n_k}$.

(ii) A has uniformly measure zero if for every sequence of positive reals $\{\varepsilon_n\}_{n=1}^{\infty}$ there is a sequence of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that the above conditions (1)–(3) are satisfied for $n_k = k$.

(iii) A has strong measure zero if for every sequence of positive reals $\{\varepsilon_n\}_{n=1}^{\infty}$ there exists a sequence of intervals $\{I_n\}_{n=1}^{\infty}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $|I_n| < \varepsilon_n$ for each $n \ge 1$.

Let $\mathcal{L}_{\text{u.m.z.}}$, $\mathcal{L}_{\text{p.m.z.}}$, and $\mathcal{L}_{\text{s.m.z.}}$ denote the families of sets having uniform measure zero, perfect measure zero, and strong measure zero, respectively. Let us recall the main results of [19] and [20]:

Theorem 3.2.

(i) *Every γ-set has perfect measure zero and* $\mathcal{L}_{\text{u.m.z.}} \subseteq \mathcal{L}_{\text{p.m.z.}} \subseteq \mathcal{L}_{\text{s.m.z.}}$.

(ii) $\operatorname{add}(\mathcal{L}_{\text{p.m.z.}}) \ge \min\{\mathfrak{h}, \operatorname{add}(\mathcal{L})\}$ *and* $\operatorname{add}(\mathcal{L}_{\text{u.m.z.}}) \ge \operatorname{add}(\mathcal{L})$.

(iii) *The group generated by a set of uniform measure zero* (*perfect measure zero*) *is a set of uniform measure zero* (*perfect measure zero*).

(iv) *Every set of reals of perfect measure zero is $\mathcal{F}$-permitted for $\mathcal{F} = p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$.* □

Moreover, each of the following inequalities is consistent with ZFC: $\mathcal{L}_{\text{u.m.z.}} \subsetneqq \mathcal{L}_{\text{p.m.z.}}$, $\mathcal{L}_{\text{p.m.z.}} \subsetneqq \mathcal{L}_{\text{s.m.z.}}$, $\mathcal{L}_{\text{s.m.z.}} \not\subseteq \operatorname{Perm}(\mathcal{F})$, $\mathcal{F} = p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$ (see the discussion in [19]).

Notice that condition (iv) of Theorem 3.2 generalizes Theorems 13.3 and 13.4 in [9] which say that γ-sets are permitted for $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$.

All families $\operatorname{Perm}(\mathcal{F})$ are closed under group generation. From this point of view Theorem 3.2(iii) is not surprising.

The following cardinals are connected with perfect measure zero and uniform measure zero:

$$\mathfrak{f} = \min\{|A| : A \subseteq {}^{\omega}\omega \wedge A \text{ is bounded} \wedge$$
$$(\forall \varphi \in \prod_{n=0}^{\infty} [\omega]^{\le n})(\forall X \in [\omega]^{\omega})(\exists f \in A)(\exists^{\infty} n \in X)\, f(n) \notin \varphi(n)\},$$
$$\mathfrak{k} = \min\{|A| : A \subseteq {}^{\omega}\omega \wedge A \text{ is bounded} \wedge$$
$$(\forall \varphi \in \prod_{n=0}^{\infty} [\omega]^{\le n})(\exists f \in A)(\exists^{\infty} n)\, f(n) \notin \varphi(n)\}.$$

It is immediate from the definitions that $\mathfrak{k} \leq \mathfrak{f}$, $\mathrm{non}(\mathcal{L}_{\mathrm{p.m.z.}}) = \mathfrak{f}$ and $\mathrm{non}(\mathcal{L}_{\mathrm{u.m.z.}}) = \mathfrak{k}$. Consequently, every set of cardinality $< \mathfrak{f}$ is $\mathcal{F}$-permitted for $\mathcal{F} = p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$. Since $\mathfrak{p} = \mathrm{non}(\gamma\text{-set})$, by condition (i) of Theorem 3.2, we obtain $\mathfrak{p} \leq \mathfrak{f}$.

In connections with trigonometric thin sets, $\mathfrak{k}$ and $\mathfrak{f}$ were both studied by Kada and Kamo [13], the cardinal $\mathfrak{f}$ was introduced independently by C. Laflamme [17] (see [8]).

For the sake of completeness let us mention also the following Miller's characterization (see [18]) of the least cardinality of a set which has not strong measure zero:

$$\begin{aligned}\mathrm{non}(\mathcal{L}_{\mathrm{s.m.z.}}) = \min\{|A| : A \subseteq {}^{\omega}\omega \wedge A \text{ is bounded} \wedge \\ (\forall\, g \in {}^{\omega}\omega)(\exists f \in A)(\forall^{\infty} n)\, f(n) \neq g(n)\}.\end{aligned}$$

Let us note that the cardinals $\mathfrak{k}$ and $\mathfrak{f}$ are not explicitly mentioned in [19] and although they were originally a motivation for introduction of the notions "perfect measure zero" and "uniform measure zero," the results of the paper were obtained independently of the results of [17] and [13].

In connection with Bartoszyński and Scheeper's result saying that every set of size $< \mathfrak{s}$ is $\mathcal{A}$-permitted, we introduce the following definition.

Definition 3.3. A set A of reals is an s-set if for every sequence of open sets $\{U_n : n \in \omega\}$ there is an increasing sequence of integers $\{n_k : k \in \omega\}$ such that $A \subseteq \bigcup_m \bigcap_{k \geq m} U_{n_k} \cup \bigcup_m \bigcap_{k \geq m} \mathbb{R} \setminus U_{n_k}$.

We can easily observe that $\mathfrak{s}$ is the minimal size of a set which is not an s-set.

Theorem 3.4. *Every s-set is $\mathcal{A}$-permitted.* □

Theorem 3.2 and the inequality $\mathfrak{t} \leq \mathrm{non}(\mathrm{Perm}\,\mathcal{N})$ proved by Bartoszyński and Scheepers, have the following generalization, proved in [20]. For a cardinal number κ let $\mathcal{L}^{\kappa}_{\mathrm{p.m.z.}}$ be the system of sets which are the union of less then κ sets of perfect measure zero.

Theorem 3.5. $\mathcal{L}^{\mathfrak{t}}_{\mathrm{p.m.z.}} \subseteq \mathrm{Perm}(\mathcal{F})$ *for* $\mathcal{F} = p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$. □

Although it is well known that

$$p\mathcal{D} \subseteq \mathcal{N}_0 \subseteq \mathcal{N}, \qquad \mathcal{N}_0 \subseteq \mathcal{A},$$

no inclusions (and even no relationship) between $\mathrm{Perm}(\mathcal{F})$'s are known. Everything we know about cardinal characterizations of $\mathcal{F}$-permitted sets is expressed by the following diagram (here $\rightarrow$ means $\leq$ and $\mathcal{L}_0$ is the σ-ideal generated by closed sets of Lebesgue measure zero):

$$
\begin{tikzcd}
\mathrm{non}(\mathcal{L}_0) \arrow[rrrr] & & & & \mathrm{non}(\mathcal{L}\cap\mathcal{M}) & \\
\mathrm{non}(\mathcal{N}) \arrow[u] & \mathrm{non}(\mathcal{N}_0) \arrow[l] & & \mathrm{non}(p\mathcal{D}) \arrow[ll] \arrow[r] & \mathrm{non}(\mathcal{A}) \arrow[u] & \mathrm{non}(\mathcal{N}_0) \arrow[l] \\
\mathrm{non}(\mathrm{Perm}\,\mathcal{N}) \arrow[u] & \mathrm{non}(\mathrm{Perm}\,\mathcal{N}_0) \arrow[u] & & \mathrm{non}(\mathrm{Perm}\,p\mathcal{D}) \arrow[u] & \mathrm{non}(\mathrm{Perm}\,\mathcal{A}) \arrow[u] & \\
\mathfrak{f} \arrow[u] \arrow[rr] & & \mathfrak{f}\cdot\min\{\mathfrak{s},\mathfrak{b}\} \arrow[ul] \arrow[ur] \arrow[rr] & & \mathfrak{f}\cdot\mathfrak{s} \arrow[u] & \\
\mathfrak{t} \arrow[u] \arrow[rr] & & \min\{\mathfrak{s},\mathfrak{b}\} \arrow[u] \arrow[rr] & & \mathfrak{s} \arrow[u] &
\end{tikzcd}
$$

Notice that for every set $A \in \mathcal{F}$, the family $\mathcal{F}_A = \{A \cup B : B \in \mathrm{Perm}(\mathcal{F})\}$ is a directed subset of $\mathcal{F}$. It follows that if the union of every countable directed subset of $\mathcal{F}$ belongs to $\mathcal{F}$ then $\mathrm{Perm}(\mathcal{F})$ is a σ-ideal. This reminds the following result of Bukovská in [5]: The union of every directed family of Dirichlet sets of size $< \mathfrak{p}$ is a pseudo-Dirichlet set. On the other hand S. Kahane in [14] proved that $p\mathcal{D}$, $\mathcal{N}_0$, and $\mathcal{N}$ are not closed on countable increasing unions. More exactly, for $\alpha < \omega_1$ let $\mathcal{F}^{\alpha\uparrow}$ denote the αth iteration of the operation "the set of all increasing unions of ω-sequences of sets." Then for every $\alpha < \beta$ there exists an $F_{\sigma\delta}$ set in $\mathcal{D}^{\beta\uparrow}$ which is not in $\mathcal{N}^{\alpha\uparrow}$.

This shows that the reduction of the problem of σ-additivity of permittedness to directed sets does not lead to a trivial solution.

Bukovský in [8] considers arbitrary functions instead of sin function in the definition of trigonometric thin set (for the sake of completeness let us say that this research was started by Bukovská in [6]). Instead of real line he considers torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The above results allow some generalizations. Let $f : \mathbb{T} \to [0, \infty)$ be a continuous function, $f(0) = 0$, and $f(x-y) \le f(x)+f(y)$ for $x, y \in \mathbb{T}$. Then if in the definition of trigonometric thin sets the functions $\sin n\pi x$ are replaced by $f(nx)$, then Theorem 3.2(iv) remains true for modified $p\mathcal{D}$, $\mathcal{N}_0$, and $\mathcal{A}$. In the case of $\mathcal{N}$ an additional hypothesis on the continuity of f is needed: $|f(x) - f(y)| < \psi(\delta)$ whenever $|x-y| < \delta$ where $\psi : (0, 1) \to (0, \infty)$ is any nondecreasing function such that $\lim_{x\to 0^+} \psi(x) = 0$ and $\sum_{n=1}^{\infty} n^{-1}\psi(n^{-a}) < \infty$ for each $a > 0$.

4. Criteria of permittedness

Let us mention that each of the mentioned results about permitted sets uses one of the following criteria for a set $A \subseteq \mathbb{R}$ to be $\mathcal{F}$-permitted:

The case $\mathcal{F} = p\mathcal{D}$.

(1d) Every increasing sequence of integers $\{m_l\}_{l=0}^{\infty}$ has a subsequence $\{m_{l_n}\}_{n=0}^{\infty}$ such that both sequences $\{\sin m_{l_n}\pi x\}_{n=0}^{\infty}$ and $\{\cos m_{l_n}\pi x\}_{n=0}^{\infty}$ converge quasinormally on A (not necessarily to 0).

(2d) For every increasing sequence of integers $\{m_l\}_{l=0}^{\infty}$ there are increasing sequences of integers $l'_n < l_n \le l'_{n+1}$, $n \in \omega$, such that $\{\sin(m_{l_n} - m_{l'_n})\pi x\}_{n=0}^{\infty}$ converges quasinormally to 0 on A.

The case $\mathcal{F} = \mathcal{A}$.

(1a) Every increasing sequence of integers $\{m_l\}_{l=0}^{\infty}$ has a subsequence $\{m_{l_n}\}_{n=0}^{\infty}$ such that both sequences $\{\sin m_{l_n}\pi x\}_{n=0}^{\infty}$ and $\{\cos m_{l_n}\pi x\}_{n=0}^{\infty}$ converge pointwise on A (not necessarily to 0).

(2a) For every increasing sequence of integers $\{m_l\}_{l=0}^{\infty}$ there are increasing sequences of integers $l'_n < l_n \le l'_{n+1}$, $n \in \omega$, such that $\{\sin(m_{l_n} - m_{l'_n})\pi x\}_{n=0}^{\infty}$ converges pointwise to 0 on A.

The case $\mathcal{F} = \mathcal{N}_0$.

(n_0) For every increasing sequence of integers $\{m_l\}_{l=0}^{\infty}$ there are increasing sequences of integers $l'_n < l_n \le l'_{n+1}$, $n \in \omega$, such that for all $x \in A$, $\sum_{n=0}^{\infty} |\sin(m_{l_n} - m_{l'_n})\pi x| < \infty$.

The case $\mathcal{F} = \mathcal{N}$.

(1n) For every sequence of nonnegative reals a_n such that $\sum_{n=0}^{\infty} a_n = \infty$ and for every increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $n_{k+1} > g(n_k)$ where $g(n) = \min\{m : \sum_{k=n}^{m} a_k/s_k \ge 1\}$ there exist integers $0 < \lambda_n \le s_n$ for each k and for each $n_k \le n \le g(n_k)$ such that for every $x \in A$, $\sum_{k=0}^{\infty} \sum_{n=n_k}^{g(n_k)} (a_n/s_n)|\sin \lambda_n n\pi x| < \infty$.

(2n) For every sequence of nonnegative reals a_n such that $\sum_{n=0}^{\infty} a_n = \infty$ and for every increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $n_{k+1} > g(n_k)$ where $g(n) = \min\{m : \sum_{k=n}^{m} a_k/s_k \ge 1\}$ there exist integers $0 < \lambda_n \le s_n$ for each k and for each $n_k \le n \le g(n_k)$, there exists an infinite set $b \subseteq \omega$ such that for every $x \in A$, $\sum_{k\in b} \sum_{n=n_k}^{g(n_k)} (a_n/s_n)|\sin \lambda_n n\pi x| < \infty$.

(3n) For every sequence of nonnegative reals a_n such that $\sum_{n=0}^{\infty} a_n = \infty$ there exists an infinite set $b \subseteq \omega$ and there exist integers $0 < \lambda_n \le s_n$, $n \in b$ such that $\sum_{n\in b} a_n/s_n = \infty$ and for every $x \in A$, $\sum_{n\in b}(a_n/s_n)|\sin \lambda_n n\pi x| < \infty$.

(4n) For every series $\sum_{n=0}^{\infty} a_n|\sin n\pi x|$ with $\sum_{n=0}^{\infty} a_n = \infty$ there exist reals $b_n \ge 0$ and natural numbers $\lambda_n \to \infty$ (for n with $b_n > 0$) such that $b_n|\sin \lambda_n \pi x| \le a_n|\sin n\pi x|$ for all n, $\sum_{n=0}^{\infty} b_n|\sin \lambda_n \pi x| < \infty$ for $x \in A$, and $\sum_{n=0}^{\infty} b_n = \infty$.

Arbault calls the sets satisfying condition (4n) $\mathcal{N}$-permitted sets in the restricted sense and proves that the family of such sets is a σ-ideal.

The following implications hold true between these conditions:

$$\begin{array}{ccccccccc}
(1a) & \longrightarrow & (2a) & & (1n) & \longrightarrow & (2n) & \longrightarrow & (3n) & \longrightarrow & (4n) \\
\uparrow & & \uparrow & & & & & & & & \\
(1d) & \longrightarrow & (2d) & \longrightarrow & (n_0) & & & & & &
\end{array}$$

Theorem 3.2(iv), Theorem 3.4, Theorem 3.5, and all mentioned cardinality results on permitted sets in fact say the following:

Theorem 4.1.

(i) *Every set of size* $< \min\{\mathfrak{s}, \mathfrak{b}\}$ *satisfies condition* (1d) ([9, Corollary 12.3]). *More generally, every s-set which is a wQN-set satisfies condition* (2d) ([9, Theorem 12.2] and [19, Theorem 3.1]).

(ii) *Every set of size* $< \mathfrak{s}$ *satisfies condition* (1a) ([9, Theorem 12.2(2)]). *More generally, every s-set satisfies condition* (1a) ([19, Theorem 3.1]).

(iii) *Every set of perfect measure zero satisfies condition* (2d) ([19, Theorem 2.2]).

(iv) *Every set of size* $< \mathfrak{t}$ *satisfies condition* (3n) ([3, Theorem 1(1)]).

(v) *Every set of perfect measure zero satisfies condition* (1n) ([19, Theorem 2.1]).

(vi) *Every union of* $< \mathfrak{t}$ *sets of perfect measure zero satisfies condition* (2n) ([20, Proposition 2.4]).

(vii) *Every union of* $< \mathfrak{t}$ *sets of perfect measure zero satisfies condition* (2d) ([20, Proposition 3.2]). □

Notice that if A is such that for every increasing sequence $\{m_l\}_{l=0}^{\infty}$ there is a subsequence $\{m_{l_n}\}_{n=0}^{\infty}$ such that $\sin m_{l_n}\pi x$ converges (either quasinormally, pointwise, or the series of these terms converges absolutely) on a set A, then $A \subseteq \mathbb{Z}$ and so this condition is not a reasonable criterion.

5. Uncountable sets of perfect measure zero

By Theorem 3.2(i), every set of perfect measure zero has strong measure zero and hence it is not possible to construct an uncountable set of perfect measure zero in ZFC. But under Martin's Axiom, especially under the assumption $\mathfrak{p} = \mathfrak{c}$, there is a γ-set of size continuum and every γ-set is a set of perfect measure zero. We shall show that a slightly weaker assumption is enough to derive the existence of an uncountable set of perfect measure zero.

Our example is easier to formulate in the Cantor space ${}^{\omega}2$ or in the homeomorphic space $\mathcal{P}(\omega)$ with the topology inherited from ${}^{\omega}2$ via characteristic functions. Therefore we identify subsets of ω with their characteristic functions and hence we adopt this notation: For $x \subseteq \omega$ and $n \in \omega$, $[x{\restriction}n] = \{y \subseteq \omega : x \cap n = y \cap n\}$ is a basic clopen set in $\mathcal{P}(\omega)$.

So let us translate the definition of a set of perfect measure zero in this space. We apply the natural measure preserving reduction $g : \mathcal{P}(\omega) \to [0,1]$ defined by

$$g(x) = \textstyle\sum_{k\in x} 2^{-k-1}. \tag{5.1}$$

We say that a set $A \subseteq \mathcal{P}(\omega)$ is a set of perfect measure zero if for every increasing function $f : \omega \to \omega$ there exist sets $S_n \in [\mathcal{P}(f(n))]^n$, $n \in \omega$, and an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $A \subseteq \bigcup_m \bigcap_{k\geq m}[S_{n_k}, f(n_k)]$ where $[S,m]$ denotes the set $\bigcup_{s\in S}[s{\restriction}m]$.

It is quite easy to see that $A \subseteq \mathcal{P}(\omega)$ is a set of perfect measure zero if and only if $g(A) \subseteq [0,1]$ is a set of perfect measure zero.

Theorem 5.1. *If $\mathfrak{t} = \mathfrak{b}$, then there exists a set of perfect measure zero of size $\mathfrak{b}$.*

Proof: For an infinite set $x \subseteq \omega$ let $e_x : \omega \to \omega$ be the increasing enumeration function of the set x, i.e., $\operatorname{rng} e_x = x$. The proof of the theorem is given by the following lemma which is based on the same ideas as the proof of Theorem 5.1 in [12]. □

Lemma 5.2. *The family of infinite sets $A = \{x_\alpha : \alpha < \mathfrak{b}\}$ is a set of perfect measure zero whenever this family satisfies the next two conditions:*

(i) $\alpha < \beta < \mathfrak{b}$ *implies* $x_\beta \subseteq^* x_\alpha$, *and*
(ii) $\{e_{x_\alpha} : \alpha < \mathfrak{b}\}$ *is an unbounded family of functions.*

Proof: Let $f : \omega \to \omega$ be an arbitrary increasing function. We show that there are an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ and $S_{n_k} \in [\mathcal{P}(f(n_k))]^{n_k}$ such that $A \subseteq \bigcup_m \bigcap_{k\geq m}[S_{n_k}, f(n_k)]$.

Let us define by induction $m_0 = 0$, $m_{n+1} = f(2^{m_n+1})$. Due to condition (ii) there exists $\alpha < \mathfrak{b}$ such that $[m_n, m_{n+1}) \cap x_\alpha = \emptyset$ for infinitely many $n \in \omega$. Let $\{i_j\}_{j=0}^{\infty}$ be the increasing enumeration of all n with $[m_n, m_{n+1}) \cap x_\alpha = \emptyset$. Due to condition (i), for every $\beta \geq \alpha$ for all but finitely many $j \in \omega$, $[m_{i_j}, m_{i_j+1}) \cap x_\beta = \emptyset$ and therefore

$$\{x_\beta : \beta \geq \alpha\} \subseteq \bigcup_m \bigcap_{j>m} [\mathcal{P}(m_{i_j}), f(2^{m_{i_j}+1})].$$

Since $\alpha < \mathfrak{t}$, the set $\{x_\beta : \beta < \alpha\}$ is a set of perfect measure zero. Therefore there exist $S'_j \in [\mathcal{P}(f(2^{m_{i_j}+1}))]^j$ for $j \in \omega$ and an infinite set $a \subseteq \omega$ such that

$$\{x_\beta : \beta < \alpha\} \subseteq \bigcup_m \bigcap_{j\in a\setminus m} [S'_j, f(2^{m_{i_j}+1})].$$

Let us set $S_{2^{m_{i_j}+1}} = S'_j \cup \mathcal{P}(m_{i_j})$. Then $|S_{2^{m_{i_j}+1}}| \leq j + 2^{m_{i_j}} \leq 2^{m_{i_j}+1}$. Let $\{n_k\}_{k=0}^{\infty}$ be the increasing enumeration of the set $\{2^{m_{i_j}+1} : j \in a\}$. Then n_k and S_{n_k} are as promised and so A is a set of perfect measure zero. □

Notice that the set $A\cup[\omega]^{<\omega}$ for the set A from the above proof is a wQN-set which is $\mathfrak{b}$-concentrated on the countable set $[\omega]^{<\omega}$ (see [10], Theorem 7.1(ii)).

It is said that a set of reals $A\subseteq\mathbb{R}$ is strongly meager if for every set G of Lebesgue measure zero $A+G\neq\mathbb{R}$. Again the function $g:\mathcal{P}(\omega)\to[0,1]$ defined by (5.1) enables to equivalently define this notion in the space $\mathcal{P}(\omega)$ with the group operation $a\oplus b=(a-b)\cup(b-a)$. T. Bartoszyński and I. Recław [2] under the assumption $\mathfrak{p}=\mathfrak{c}$ have constructed a γ-set in ${}^{\omega}2$ of size $\mathfrak{c}$ which is not strongly meager. A similar construction works also for sets of perfect measure zero under a somewhat weaker assumption.

Theorem 5.3. *If $\mathfrak{t}=\mathfrak{c}$ then there exists a set of size $\mathfrak{c}$ of perfect measure zero which is not strongly meager.*

Proof: Let p_n, $n<\omega$ be an increasing sequence of integers such that $p_{n+1}-p_n\geq n$ for all n. First let us show that there exists a set $\{x_\alpha:\alpha<\mathfrak{c}\}$ of infinite subsets of ω such that

(i) $\alpha<\beta<\mathfrak{c}$ implies $x_\beta\subseteq^* x_\alpha$;
(ii) $\{e_{x_\alpha}:\alpha<\mathfrak{c}\}$ is an unbounded family of functions;
(iii) for every $\alpha<\mathfrak{c}$ the set $x'_\alpha=\{n:[p_n,p_{n+1})\subseteq x_\alpha\}$ is infinite;
(iv) $(\forall\, z\in\mathcal{P}(\omega))(\exists\alpha<\mathfrak{c})(\exists^\infty n)\,[p_n,p_{n+1})\cap x_\alpha=[p_n,p_{n+1})\cap z$.

Let $\{f_\alpha:\alpha<\mathfrak{c}\}$ be a dominating family in ${}^{\omega}\omega$ consisting of strictly increasing functions and let $\{z_\alpha:\alpha<\mathfrak{c}\}$ be an enumeration of $\mathcal{P}(\omega)$. We construct the sets x_α by induction on $\alpha<\mathfrak{c}$ so that conditions (i)–(iii) and the following condition (iv$'$) are satisfied:

(iv$'$) $(\forall\,\alpha<\mathfrak{c})(\exists^\infty n)\,[p_n,p_{n+1})\cap x_\alpha=[p_n,p_{n+1})\cap z_\alpha$.

Let us assume that $\{x_\beta:\beta<\alpha\}$ have been constructed. By conditions (i) and (iii) the set $\{x'_\beta:\beta<\alpha\}$ is a decreasing chain with respect to $\subseteq^*$ and since $\alpha<\mathfrak{t}$ there exists $y'_\alpha\in[\omega]^\omega$ such that $y'_\alpha\subseteq x'_\beta$ for all $\beta<\alpha$. Let $y_\alpha=\bigcup_{n\in y'_\alpha}[p_n,p_{n+1})$. Let us choose y'_α in such a way that $e_{y_\alpha}\not\leq f_\alpha$, i.e., $e_{y_\alpha}(n)\geq f_\alpha(n)$ for infinitely many n. Now let $a_0\cup a_1=y'_\alpha$ be a partition of y' into two infinite sets. We set $x_\alpha=(\bigcup_{n\in a_0}[p_n,p_{n+1}))\cup(\bigcup_{n\in a_1}[p_n,p_{n+1})\cap z_\alpha)$. Then $x_\alpha\subseteq y_\alpha$ and conditions (i)–(iv) are fulfilled.

Since the set $A=\{x_\alpha:\alpha<\mathfrak{c}\}$ satisfies assumptions of Lemma 5.2, the set A is a set of perfect measure zero.

Let $G=\bigcap_m\bigcup_{n>m}U_n$ where $U_n=\{x\in\mathcal{P}(\omega):x\cap[p_n,p_{n+1})=\emptyset\}$. Then $\mu(G)=0$ because $\mu(U_n)\leq 2^{-n}$. We show that $A\oplus G=\mathcal{P}(\omega)$ and hence the set A is not strongly meager. Let $\alpha<\mathfrak{c}$ be arbitrary. By (iv) for infinitely many n, $(x_\alpha\oplus z_\alpha)\cap[p_n,p_{n+1})=\emptyset$. Therefore $x_\alpha\oplus z_\alpha\in G$ and so $z_\alpha\in A\oplus G$. □

6. s-sets

Theorem 6.1. *The family of s-sets is an $\mathfrak{h}$-complete ideal.*

Proof: Let $\langle A_\xi : \xi < \kappa \rangle$, be a sequence of s-sets where $\kappa < \mathfrak{h}$. Let $\{U_n\}_{n=0}^\infty$ be a sequence of open sets. The family $\mathcal{X}_\xi = \{a \in [\omega]^\omega : A_\xi \subseteq \bigcup_m \bigcap_{k\in a\setminus m} U_k \cup \bigcup_m \bigcap_{k\in a\setminus m} \mathbb{R} \setminus U_k\}$ for $\xi < \kappa$ is an open subset of $[\omega]^\omega$ and therefore there is a set $a \in \bigcap_{\xi<\kappa} \mathcal{X}_\xi$ which witnesses that the union $\bigcup_{\xi<\kappa} A_\xi$ is an s-set. $\square$

Theorem 6.2.

(i) *If a set $A \subseteq \mathbb{R}$ is an s-set then for every sequence of continuous functions $f_n : A \to [0,1]$ there exists an increasing sequence of integers $\{n_k\}_{k=0}^\infty$ such that the sequence of functions $\{f_{n_k}\}_{k=0}^\infty$ converges pointwise.*

(ii) *If $|A| < \mathfrak{b}$, then A is an s-set if and only if for every sequence of continuous functions $f_n : A \to [0,1]$ there exists an increasing sequence of integers $\{n_k\}_{k=0}^\infty$ such that the sequence of functions $\{f_{n_k}\}_{k=0}^\infty$ converges pointwise.*

Proof: (i) Let A be an s-set and let $f_n : A \to [0,1]$ be continuous for $n \in \omega$. Let $\{q_n : n \in \omega\}$ be an enumeration of rationals. The sets $U_{n,k} = \{x \in A : f_k(x) < q_n\}$ are relatively open and since A is an s-set we can find a decreasing sequence of sets $a_n \subseteq \omega$ such that $A \subseteq \bigcup_m \bigcap_{k\in a_n\setminus m} U_{n,k} \cup \bigcup_m \bigcap_{k\in a_n\setminus m} A \setminus U_{n,k}$. For any infinite pseudo-intersection a of a_n, $n \in \omega$, the sequence of functions $\{f_n\}_{n\in a}$ converges pointwise.

(ii) Conversely, let a sequence of open sets U_n, $n \in \omega$, be given. Since $|A| < \mathfrak{b}$, there exists a sequence of positive reals $\{\varepsilon_n\}_{n=0}^\infty$ such that for every $x \in A$ for all but finitely many $n \in \omega$, either $\operatorname{dist}(x, \mathbb{R} \setminus U_n) > \varepsilon_n$ or $x \in \mathbb{R} \setminus U_n$. For every $n \in \omega$ there is a continuous function $g_n : \mathbb{R} \to [0,1]$ such that $g_n(x) = 1$ whenever $\operatorname{dist}(x, \mathbb{R} \setminus U_n) \leq \varepsilon_n/2$, and $g_n(x) > 0$ if and only if $\operatorname{dist}(x, \mathbb{R} \setminus U_n) < \varepsilon_n$. Now let us assume that $\{g_{n_k}\}_{k=0}^\infty$ converges pointwise on a set $A \subseteq \mathbb{R}$. Then if $\lim_{k\to\infty} g_{n_k}(x) = 0$, then for all but finitely many $k \in \omega$, $\operatorname{dist}(x, \mathbb{R} \setminus U_{n_k}) > \varepsilon_{n_k}/2$ and hence $x \in U_{n_k}$ for all but finitely many k. If $\lim_{k\to\infty} g_{n_k}(x) > 0$, then for all but finitely many $k \in \omega$, $\operatorname{dist}(x, \mathbb{R} \setminus U_{n_k}) < \varepsilon_{n_k}$. Hence by the choice of the sequence $\{\varepsilon_n\}_{n=0}^\infty$, $x \in \mathbb{R} \setminus U_{n_k}$ for all but finitely many k. Therefore $A \subseteq \bigcup_m \bigcap_{k>m} U_{n_k} \cup \bigcup_m \bigcap_{k>m} \mathbb{R} \setminus U_{n_k}$. $\square$

The following notion estimates $\mathcal{A}$-permitted sets better than s-sets do.

Definition 6.3. A set $A \subseteq \mathbb{R}$ is an s$'$-set if for every sequence of disjoint couples of open sets $\langle U_n, V_n : n \in \omega \rangle$ for which there exists a countable set S such that $\mathbb{R} \setminus S \subseteq U_n \cup V_n$ for all n, there exists an increasing sequence of integers $\{n_k\}_{k=0}^\infty$ such that $A \setminus S \subseteq \bigcup_m \bigcap_{k>m} U_{n_k} \cup \bigcup_m \bigcap_{k>m} V_{n_k}$.

Theorem 6.4.

(i) *Every s-set is an s$'$-set.*

(ii) *Every s$'$-set is $\mathcal{A}$-permitted.*

Proof: (i) Assume that U_n, V_n, S are as in the definition of s$'$-set and let A be an s-set. Then there exists a sequence $\{n_k\}_{k=0}^\infty$ such that $A \subseteq \bigcup_m \bigcap_{k>m} U_{n_k} \cup$

$\bigcup_m \bigcap_{k>m} \mathbb{R} \setminus U_{n_k}$. But clearly, $\bigcup_m \bigcap_{k>m} \mathbb{R} \setminus U_{n_k} \subseteq \bigcup_m \bigcap_{k>m} (V_{n_k} \cup S) = S \cup \bigcup_m \bigcap_{k>m} V_{n_k}$.

(ii) We proceed exactly like in [19] in the case of s-sets.

Let E be an A-set and let $\{\sin m_k \pi x\}_{k=0}^{\infty}$ converge on E. Let $\{q_n : n \in \omega\}$ be an enumeration of $\mathbb{Q}$. The set $S = \{x \in \mathbb{R} : (\exists n) \sin n\pi x \in \mathbb{Q} \vee \cos n\pi x \in \mathbb{Q}\}$ is countable and the couples of disjoint open sets

$$\begin{aligned} U_k &= \{x : \cos k\pi x > 0\}, & V_k &= \{x : \cos k\pi x < 0\} \\ U_{n,k} &= \{x : \sin k\pi x > q_n\}, & V_{n,k} &= \{x : \sin k\pi x < q_n\} \end{aligned}$$

be such that $\mathbb{R} \setminus S \subseteq U_i \cup V_i$ for all $i \in \omega \cup (\omega \times \omega)$. Since A is an s$'$-set we can construct a sequence of infinite sets $a_n \subseteq \omega$ as follows:

(1) $a_{n+1} \subseteq a_n \subseteq \{m_k : k \in \omega\}$,
(2) $A \setminus S \subseteq \bigcup_m \bigcap_{k \in a_0 \setminus m} U_k \cup \bigcup_m \bigcap_{k \in a_0 \setminus m} V_k$, and
(3) $A \setminus S \subseteq \bigcup_m \bigcap_{k \in a_{n+1} \setminus m} U_{n,k} \cup \bigcup_m \bigcap_{k \in a_{n+1} \setminus m} V_{n,k}$.

Let $a \in [\omega]^{\omega}$ be an infinite pseudo-intersection of the system $\{a_n : n \in \omega\}$ and let $\{n_k\}_{k=0}^{\infty}$ be the strictly increasing enumeration of a. We can find a so that the sequence $\{n_{k+1} - n_k\}_{k=0}^{\infty}$ is strictly increasing. This sequence witnesses that the set $E \cup (A \setminus S)$ is an A-set. But as S is countable and hence $\mathcal{A}$-permitted it follows that also the set $E \cup A$ is an A-set. Therefore A is $\mathcal{A}$-permitted. $\square$

7. Uncountable small sets

A set $p \subseteq {}^{<\omega}2$ is a perfect tree if (1) $\emptyset \in p$, (2) $(\forall\, s \in p)(\forall\, n)\, s{\restriction}n \in p$, and (3) $(\forall\, s \in p)(\exists t \in p)(s \subseteq t \wedge t^\frown 0 \in p \wedge t^\frown 1 \in p)$.

Let P be the set of all perfect trees in ${}^{<\omega}2$ ordered by $p \leq q$ if and only if $p \subseteq q$. Perfect trees $p, q \in P$ are incompatible if there is no $r \in P$ such that $r \leq p$ and $r \leq q$. For $n \in \omega$ and $p \in P$ we define $p \leq_n q$ if $p \leq q$ and $p \cap {}^n2 = q \cap {}^n2$. A set $D \subseteq P$ is said to be ω-dense if for every tree $p \in P$ and every $n \in \omega$ there is $q \in D$ such that $q \leq_n p$.

Let us start with the following observations:

Fact 7.1.

(i) *There is no perfect set of perfect measure zero.*
(ii) *There is no perfect s$'$-set.*

Proof: (i) Every set of perfect measure zero has strong measure zero and it is a well known fact that there is no perfect set of strong measure zero.

(ii) Let us work in the space ${}^{\omega}2$. Let $p \subseteq {}^{<\omega}2$ be a perfect tree and let $\pi : {}^{<\omega}2 \to p$ be the natural embedding which is defined by induction on $|s|$ for $s \in {}^{<\omega}2$ as follows: $\pi(\emptyset) = \emptyset$, $\pi(s^\frown i) = t_s^\frown i$, where t_s is the first splitting node above $\pi(s)$ in p. Now let $U_n = \bigcup\{[\pi(s)] : s(n) = 1 \wedge s \in {}^{n+1}2\}$. Then there is no sequence $\{n_k\}_{k=0}^{\infty}$ such that $[p] \subseteq \bigcup_m \bigcap_{k>m} U_n \cup \bigcup_m \bigcap_{k>m} ({}^{\omega}2 \setminus U_n)$, and hence $[p]$ is not an s$'$-set. □

Let us recall that an Aronszajn tree T is an ω_1-tree with no ω_1-branches whose all levels T_α, $\alpha < \omega_1$, are countable. We denote $T_{<\alpha} = \bigcup_{\beta<\alpha} T_\beta$.

Lemma 7.2. *Let $\langle D_\alpha : \alpha < \omega_1\rangle$ be a sequence of open ω-dense subsets of P. There is an Aronszajn tree $T \subseteq P$ such that $T_\alpha \subseteq D_\alpha$ for all $\alpha < \omega_1$. Moreover, for any two incompatible perfect trees $p, q \in T$, $[p] \cap [q] = \emptyset$.*

Proof: We use a modification of a fairly known construction of an Aronszajn tree. We construct levels T_α of the tree T by induction on $\alpha < \omega_1$ so that

(i) T_α is countable, and
(ii) $(\forall\, q \in T_{<\alpha})(\forall\, n)(\exists p \in T_\alpha)\, q \leq_n p$.

Let $p_0 \in D_0$ be arbitrary and we let $T_0 = \{p_0\}$. Let us assume that $T_{<\alpha}$ has been constructed and we construct T_α.

Case 1. α is a successor ordinal, $\alpha = \beta + 1$. For every $p \in T_\beta$ let us fix a sequence of perfect trees $q_n^p \leq_n p$, $n \in \omega$, such that the perfect sets $[q_n^p]$, $n \in \omega$, are pairwise disjoint. Since D_α is ω-closed we can find them in D_α. We then set $T_\alpha = \{q_n^p : p \in T_\beta \wedge n \in \omega\}$.

Case 2. α is a limit ordinal. For every $p \in T_{<\alpha}$ and $n \in \omega$ let us fix an increasing sequence of ordinals $\{\alpha_m\}_{m\in\omega}$, such that $\lim_{m\to\infty} \alpha_m = \alpha$ and using the induction hypothesis find a sequence of perfect trees $q_m \in T_{\alpha_m}$ and an increasing sequence of integers $\{k_m\}_{m=0}^{\infty}$ such that $k_m \geq n$, $q_{m+1} \leq_{k_m} q_m$ and for all $s \in q_{m+1} \cap {}^{k_m}2$ there are two incompatible extensions s_0, s_1 of s in $q_{m+1} \cap {}^{k_{m+1}}2$. Then $r_n^p = \bigcap_{m\in\omega} q_m$ is a perfect tree and $r_n^p \leq_n p$. Since the set $\{r_n^p : p \in T_{<\alpha} \wedge n \in \omega\}$ is countable we can refine this system of perfect trees by a system $\{q_n^p : p \in T_{<\alpha} \wedge n \in \omega\}$ so that $q_n^p \leq_n r_n^p$, $q_n^p \in D_\alpha$ and the perfect sets $[q_n^p]$, $(p, n) \in T_{<\alpha} \times \omega$, are pairwise disjoint. We let $T_\alpha = \{q_n^p : p \in T_{<\alpha} \wedge n \in \omega\}$.

Since there is no decreasing chain of closed sets of length ω_1, T has no ω_1-branch.

□

The Aronszajn tree T constructed in the previous proof has every level countable infinite and there exists a natural enumeration $\{p_\alpha : \alpha < \omega_1\}$ of elements of T so that $T_\alpha = \{p_{\omega\alpha+n} : n \in \omega\}$. The above lemma can be easily generalized so that we can require $p_\alpha \in D_\alpha$ for $\alpha < \omega_1$.

We say that an ideal $\mathcal{I}$ on ${}^{\omega}2$ is tall if it contains all singletons and $(\forall\, p \in P)$ $(\exists q \leq p)\, [q] \in \mathcal{I}$. If $\mathcal{I}$ is a tall ideal on ${}^{\omega}2$ then for every $p \in P$ there exists $q \leq_n p$ such that $[q] \in \mathcal{I}$, i.e., the family of perfect trees $D_{\mathcal{I}} = \{p \in P : [p] \in \mathcal{I}\}$ is ω-dense in P (for $s \in p \cap {}^{n}2$ let $q_s \leq p_s$ be such that $[q_s] \in \mathcal{I}$ and let $q = \bigcup_{s\in p\cap {}^{n}2} q_s$).

We do not know to whom to attribute the following theorem although we think that it should be well known.

Theorem 7.3. *The intersection of a sequence of ω_1 many tall σ-ideals on ${}^\omega 2$ contains an uncountable set.*

Proof: Let $\langle \mathcal{I}_\alpha : \alpha < \omega_1 \rangle$ be any sequence of tall σ-ideals on ${}^\omega 2$. Let $T \subseteq P$ be an Aronszajn tree such that $T_\alpha \subseteq D_{\mathcal{I}_\alpha}$ for all $\alpha < \omega_1$ and $[p] \cap [q] = \emptyset$ for incompatible $p, q \in T$. Let A be any selector of the family $\{[p] : p \in T\}$.

Let $\alpha < \omega_1$ be arbitrary. Clearly, the set $B_\alpha = \bigcup_{p \in T_\alpha} [p]$ belongs to $\mathcal{I}_\alpha$ since it is a σ-ideal, and $A \setminus B_\alpha$ is countable and hence also in $\mathcal{I}_\alpha$. Therefore A belongs to $\mathcal{I}_\alpha$ for all $\alpha < \omega_1$ and A is clearly uncountable. □

If $\mathfrak{t} = \mathfrak{b}$ then there exists a set of perfect measure zero of size $\mathfrak{b}$ (Theorem 5.1). Under the assumption $\mathfrak{b} = \omega_1$ we can obtain an uncountable set of perfect measure zero by another construction (Theorem 7.5). We need the following two characterizations.

Lemma 7.4. *Let $g : \omega \to \omega$ be monotone unbounded and let $g(n) \leq n$ for all n. Let $A \subseteq {}^\omega 2$.*

(i) *A is a set of uniform measure zero if and only if the following condition holds:*

(7.1) *For every increasing $f : \omega \to \omega$ there are $S_n \in [{}^{f(n)}2]^{\leq g(n)}$ such that $A \subseteq \bigcup_m \bigcap_{n>m} \bigcup_{s \in S_n} [s]$.*

(ii) *A is a set of perfect measure zero if and only if the following condition holds:*

(7.2) *For every increasing $f : \omega \to \omega$ and every set $a \in [\omega]^\omega$ there are $b \in [a]^\omega$ and $S_n \in [{}^{f(n)}2]^{\leq g(n)}$ such that $A \subseteq \bigcup_m \bigcap_{n \in b \setminus m} \bigcup_{s \in S_n} [s]$.*

Proof: We prove the case (ii) only. The proof of the case (i) can be easily reduced from the proof of (ii).

Obviously, (7.2) is stronger than the definition of "A is a set of perfect measure zero." So let us assume that $A \subseteq {}^\omega 2$ is a set of perfect measure zero and we prove (7.2). Let $f : \omega \to \omega$ be increasing and let $\{m_k : k \in \omega\}$ be the increasing enumeration of a set $a \in [\omega]^\omega$. Let i_k be the largest integer i with the minimal value $g(i) \geq k$, i.e., $g^{-1}(\{k\}) = (i_{k-1}, i_k]$ for $k > 0$ (and some of these intervals may be empty). Let us define the following two monotone functions:

$$f'(n) = f(m_k) \qquad \text{for } m_{k-1} < n \leq m_k,\ k \in \omega,$$
$$f''(n) = f'(i_n) \qquad \text{for all } n \in \omega.$$

As A is a set of perfect measure zero there are $S''_n \in [{}^{f''(n)}2]^n$ and $b'' \in [\omega]^\omega$ such that $A \subseteq \bigcup_m \bigcap_{n \in b'' \setminus m} \bigcup_{s \in S''_n} [s]$. Let $S'n = \{s \restriction f'(n) : s \in S''_{g(n)}\}$ and

$b' = g^{-1}(b'')$. If $g(n) = k$ then $f'(n) \leq f'(i_k) = f''(k)$ and hence $S'_n \subseteq {}^{f'(n)}2$, $|S'_n| \leq g(n)$ and $A \subseteq \bigcup_m \bigcap_{n \in b' \setminus m} \bigcup_{s \in S'_n} [s]$. As the function f' is constant on intervals $(m_{k-1}, m_k]$ we can define a sequence $\langle S_n : n \in \omega \rangle$ and a set $b \subset a = \{m_k : k \in \omega\}$ such that $\langle S_n : n \in b \rangle$ and $\langle S'_n : n \in b' \rangle$ consist of the same terms. It follows that condition (7.2) holds. $\square$

Theorem 7.5.

(i) *The ideal $\mathcal{L}_{\text{u.m.z.}}$ is the intersection of $\mathfrak{d}$ many tall σ-ideals.*

(ii) *The ideal $\mathcal{L}_{\text{p.m.z.}}$ is the intersection of $\mathfrak{b}$ many tall σ-ideals.*

Hence, if $\mathfrak{d} = \omega_1$ (resp. $\mathfrak{b} = \omega_1$) then there exists an uncountable set of uniform measure zero (resp. of perfect measure zero).

Proof: Let

$$\mathcal{F} = \{g \in {}^{\omega}\omega : \lim_{n \to \infty} g(n)/n = 0\},$$

and for any increasing $f : \omega \to \omega$ let

$$\begin{aligned}\mathcal{I}_f =& \{A \subseteq {}^{\omega}2 : (\exists g \in \mathcal{F})(\exists S_n \in [{}^{f(n)}2]^{g(n)})\, A \subseteq \textstyle\bigcup_m \bigcap_{n > m} \bigcup_{s \in S_n} [s]\},\\ \mathcal{J}_f =& \{A \subseteq {}^{\omega}2 : (\exists g \in \mathcal{F})(\forall\, a \in [\omega]^{\omega})(\exists b \in [a]^{\omega})(\exists S_n \in [{}^{f(n)}2]^{g(n)})\\ & \qquad A \subseteq \textstyle\bigcup_m \bigcap_{n \in b \setminus m} \bigcup_{s \in S_n} [s]\}.\end{aligned}$$

Notice that whenever $g_k : \omega \to [\omega]^{<\omega}$, $k \in \omega$, are such that $\lim_{n\to\infty} |g_k(n)|/n = 0$ for all $k \in \omega$ then there is $g : \omega \to [\omega]^{<\omega}$ such that $\lim_{n\to\infty} |g(n)|/n = 0$ and $(\forall\, k)(\forall^{\infty} n)\, g_k(n) \subseteq g(n)$. Also if $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of infinite subsets of ω then there is an infinite set a such that $a \subseteq^* a_n$ for all n. Therefore $\mathcal{I}_f$ and $\mathcal{J}_f$ are σ-ideals. Easily it can be seen that the ideals $\mathcal{I}_f$ are tall and as $\mathcal{I}_f \subseteq \mathcal{J}_f$ the ideals $\mathcal{J}_f$ are such too. Using Lemma 7.4 we can easily see that if $D \subseteq {}^{\omega}\omega$ is a dominating family and $B \subseteq {}^{\omega}\omega$ is an unbounded family both consisting of increasing functions then $\mathcal{L}_{\text{u.m.z.}} = \bigcap_{f \in D} \mathcal{I}_f$ and $\mathcal{L}_{\text{p.m.z.}} = \bigcap_{f \in B} \mathcal{J}_f$. $\square$

Notice that the above proof gives estimations for additivities of the ideals $\mathcal{L}_{\text{u.m.z.}}$ and $\mathcal{L}_{\text{p.m.z.}}$: $\operatorname{add}(\mathcal{L}_{\text{u.m.z.}}) \geq \min_f \operatorname{add}(\mathcal{I}_f)$ and $\operatorname{add}(\mathcal{L}_{\text{p.m.z.}}) \geq \min_f \operatorname{add}(\mathcal{J}_f)$.

Our next aim is to prove the existence of an uncountable s-set. At first attempt to prove this using the same arguments as in Theorem 7.5 it is natural to ask about the following σ-ideals defined from sequences $\mathcal{X} = \langle U_n : n \in \omega \rangle$ of open sets:

$$\mathcal{I}_{\mathcal{X}} = \{A \subseteq {}^{\omega}2 : (\forall\, a \in [\omega]^{\omega})(\exists b \in [a]^{\omega})\, A \subseteq \textstyle\bigcup_m \bigcap_{n \in b \setminus m} U_n \cup \bigcup_m \bigcap_{n \in b \setminus m} {}^{\omega}2 \setminus U_n\}.$$

These ideals can contain perfect sets but it is not clear whether all of them and it is not clear at all whether they are tall (see [21] for a partial information).

As we are not able to prove that the ideals $\mathcal{I}_{\mathcal{X}}$ are tall, for every sequence of open sets $\mathcal{X} = \langle U_n : n \in \omega \rangle$ and every $a \in [\omega]^{\omega}$ we define

$\mathcal{I}_{\mathcal{X},a} = \{A \subseteq {}^\omega 2 : (\exists b \in [a]^\omega)\, A \subseteq (\bigcup_m \bigcap_{n\in b\setminus m} U_n) \cup (\bigcup_m \bigcap_{n\in b\setminus m} {}^\omega 2 \setminus U_n)\}$,
$D_{\mathcal{X},a} = \{p \in P : [p] \in \mathcal{I}_{\mathcal{X},a}\}$.

Lemma 7.6. *$D_{\mathcal{X},a}$ is an ω-dense subset of P.*

Proof: Let $p \in P$ and $k \in \omega$. We find $q \leq_k p$, $q \in D_{\mathcal{X},a}$.

For $s \in p$ let $a_s = \{n : [p_s] \cap U_n$ is dense in $[p_s]\}$ and let $S = \{a_s : s \in p\}$. There is $a' \in [a]^\omega$ such that a' refines all sets in S, i.e., $a' \subseteq^* u$ or $a' \subseteq^* \omega \setminus u$ for every $u \in S$. We prove that for every $s \in p$ and every set $u \in [a']^\omega$ there exist $v \in [u]^\omega$ and a perfect tree $q^{(s)} \subseteq p_s$ such that $[q^{(s)}] \subseteq \bigcup_m \bigcap_{n\in v\setminus m} U_n \cup \bigcup_m \bigcap_{n\in v\setminus m} {}^\omega 2 \setminus U_n$. Then the lemma follows because if $p \cap {}^k 2 = \{s_i : i \leq n_0\}$ then we inductively find decreasing sequence of infinite sets $v_i \subseteq a'$ and $q_i \leq p_{s_i}$ such that $[q_i] \subseteq \bigcup_m \bigcap_{n\in v_i\setminus m} U_n \cup \bigcup_m \bigcap_{n\in v_i\setminus m} {}^\omega 2 \setminus U_n$. Now it is enough to take $q = \bigcup_{i\leq n_0} q_i$ and $b = v_{n_0}$.

Let $s \in p$ and $u \in [a']^\omega$ be arbitrary. There are two possibilities:

(1) There is $t \in p_s$ and m such that $u \setminus m \subseteq a_t$.
(2) For every $t \in p_s$, $(\forall^\infty n \in u)\, [p_t] \cap U_n$ is not dense in $[p_t]$.

If (1) holds true then we set $v = u$ and by induction for $n \in u \setminus m$ we define nth branching levels of a tree $q^{(s)} \subseteq p_t$ consisting of those $r \in p_t$ for which $[p_r] \subseteq U_n$.

If (2) holds true then by induction on $n \in \omega$ we define an increasing sequence of $k_n \in u$ and nth branching levels of a tree $q^{(s)} \subseteq p_s$ consisting of those $r \in p_s$ for which $[p_r] \cap U_{k_n} = \emptyset$. Let $v = \{k_n : n \in \omega\}$. □

In general we do not know whether the families $\mathcal{I}_{\mathcal{X},a}$ are σ-ideals. However, all these families possess some weak form of σ-additivity which is sufficient for our needs:

(7.3) If $\langle A_n : n \in \omega\rangle$ is a sequence of elements of $\mathcal{I}_{\mathcal{X},a}$ such that $A_n \in \mathcal{I}_{\mathcal{X},a}$ is witnessed by some $b_n \in [a]^\omega$ such that the system $\{b_n : n \in \omega\}$ is centered, then $\bigcup_{n\in\omega} A_n \in \mathcal{I}_{\mathcal{X},a}$ is witnessed by any infinite pseudo-intersection of $\{b_n : n \in \omega\}$.

Theorem 7.7. *If $\mathfrak{c} = \omega_1$ then there exists an uncountable s-set.*

Proof: Let $\mathcal{X}_\alpha$, $\alpha < \omega_1$ be an enumeration of all sequences of open sets. Like in the lemma before Theorem 7.3, by induction on $\alpha < \omega_1$ we construct levels $T_\alpha = \{p_{\omega\alpha+n} : n \in \omega\}$ of an Aronszajn tree T consisting of perfect trees and decreasing sequences of sets $a_\alpha^{n+1} \subseteq a_\alpha^n \subseteq \omega$, $n \in \omega$, so that a_α^{n+1} witnesses that $p_{\omega\alpha+n} \in D_{\mathcal{X}_{\alpha},a_\alpha^n}$, and let a_α be an infinite pseudo-intersection of $\{a_\alpha^n : n \in \omega\}$. Let A be any selector of the family $\{[p] : p \in T\}$. We prove that A is an s-set.

Let $\mathcal{X}$ be any sequence of open sets. There is $\alpha < \omega_1$ such that $\mathcal{X} = \mathcal{X}_\alpha$. By condition (7.3) the set $B_\alpha = \bigcup_{p\in T_\alpha} [p]$ belongs to $\mathcal{I}_{\mathcal{X},a_\alpha}$ and let $a \in [a_\alpha]^\omega$ witness that. Since $A \setminus B_\alpha$ is countable, $A \setminus B_\alpha \in \mathcal{I}_{\mathcal{X},a}$. Consequently, $A \in \mathcal{I}_{\mathcal{X},a}$. It follows that A is an s-set. □

8. Some open questions

Let $\mathcal{F}$ be any of the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$. The following questions are open:

(i) Is there a perfect $\mathcal{F}$-permitted set?
(ii) Is there an $\mathcal{F}$-permitted set of the size of continuum?
(iii) Is there an uncountable set of perfect measure zero?
(iv) Is there an s-set of the size of continuum?
(v) Are there any inclusions between $\mathrm{Perm}(\mathcal{F})$'s?
(vi) All known lower bounds for $\mathrm{non}(\mathcal{F})$ and $\mathrm{non}(\mathrm{Perm}\,\mathcal{F})$ are the same. Is $\mathrm{non}(\mathcal{F}) = \mathrm{non}(\mathrm{Perm}\,\mathcal{F})$?
(vii) Is $\mathrm{Perm}(\mathcal{F})$ a σ-ideal? What is the additivity of $\mathrm{Perm}(\mathcal{F})$?
(viii) In the opposite direction to condition (iv) of Theorem 3.2, in Laver's model $\mathcal{L}_{\text{s.m.z.}} = [\mathbb{R}]^{\leq\omega}$. Is $\mathrm{Perm}(\mathcal{F}) = [\mathbb{R}]^{\leq\omega}$ consistent with set theory?
(ix) Is there an $\mathcal{F}$-permitted set which has not strong measure zero?
(x) Is there an $\mathcal{N}$-permitted set not satisfying criterion (4n)?
(xi) Let $p(x_1, \dots, x_k)$ be any polynomial over $\mathbb{Z}$ without absolute terms. Let $\{l_n^{(i)}\}_{n=0}^{\infty}$ denote an increasing sequence of natural numbers for $1 \leq i \leq k$. If in criteria (2d), (2a), (n_0) we replace term $\sin(m_{l_n} - m_{l'_n})\pi x$ by $\sin p(m_{l_n^{(1)}}, \dots, m_{l_n^{(k)}})\pi x$ we obtain generalized criteria for permittedness (provided that the set of all values $p(m_{l_n^{(1)}}, \dots, m_{l_n^{(k)}})$ is infinite). Some of these criteria may be trivial, like the criterion in the remark after Theorem 4.1. The question: Is there a permitted set (for family $p\mathcal{D}$, $\mathcal{N}_0$, or $\mathcal{A}$) which does not satisfy the generalized criterion?
(xii) Is $\mathfrak{P}_2$ a tall ideal?

Questions (ii) and (iv) are open even if we replace the phrase "set of size of continuum" by the phrase "uncountable set."

Fact 8.1. *If* $\mathrm{Perm}(\mathcal{F})$ *is not a σ-ideal, then there exists an uncountable (in fact of size $\geq \mathfrak{f}$ and in the case $\mathcal{F} = \mathcal{A}$ also of size $\geq \mathfrak{s}$) $\mathcal{F}$-permitted set.*

PROOF: $\mathcal{L}_{\text{p.m.z.}} \subseteq \mathrm{Perm}(\mathcal{F})$ and $\mathcal{L}_{\text{u.m.z.}}$ is a σ-ideal containing all sets of reals of size $< \mathfrak{f}$. □

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