Comment.Math.Univ.Carolin. 42,4 (2001)721–728


# On the Dirichlet problem for functions of the first Baire class

Jiří Spurný

*Abstract.* Let $\mathcal{H}$ be a simplicial function space on a metric compact space $X$. Then the Choquet boundary $\operatorname{Ch} X$ of $\mathcal{H}$ is an $F_\sigma$-set if and only if given any bounded Baire-one function $f$ on $\operatorname{Ch} X$ there is an $\mathcal{H}$-affine bounded Baire-one function $h$ on $X$ such that $h = f$ on $\operatorname{Ch} X$. This theorem yields an answer to a problem of F. Jellett from [8] in the case of a metrizable set $X$.

*Keywords:* weak Dirichlet problem, function space, Choquet simplexes, Baire-one functions

*Classification:* 46A55, 31B05, 26A21

## 1. Introduction

Let $\mathcal{H}$ be a *function space* on a compact metric space $X$. By this we mean a linear subspace of $\mathcal{C}(X)$ (the space of all real-valued continuous functions on $X$ equipped with the sup-norm $||.||$) containing constant functions and separating points of $X$. Let $\mathcal{M}^1(X)$ denote the set of all probability Radon measures on $X$ and $\varepsilon_x$ the Dirac measure at $x \in X$. Let further $\mathcal{M}_x(\mathcal{H})$ be the set of all $\mathcal{H}$*-representing measures* for $x \in X$, i.e.

$$\mathcal{M}_x(\mathcal{H}) = \{\mu \in \mathcal{M}^1(X) : \mu(h) = h(x) \ \text{ for any } \ h \in \mathcal{H}\}.$$

A bounded Borel function $f$ is called $\mathcal{H}$*-affine* if it satisfies $\mu(f) = f(x)$ for any $x \in X$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. The space of all $\mathcal{H}$-affine continuous functions will be denoted by $\mathcal{A}(\mathcal{H})$. *The Choquet boundary* $\operatorname{Ch} \mathrm{X}$ of $\mathcal{H}$ is defined as the set $\{x \in X : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$. The Choquet boundary is a $G_\delta$-set and the Choquet representation theorem guarantees for any $x \in X$ the existence of a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $\mu(X \setminus \operatorname{Ch} \mathrm{X}) = 0$. We say that $(X, \mathcal{H})$ is *a simplicial space* if for any $x \in X$ there is a unique measure representing $x$ carried by the Choquet boundary.

We introduce main examples of function spaces.

---

Research supported in part by grants GA ČR No. 201/00/0767 from the Grant Agency of the Czech Republic, GAUK 165/99 from the Grant Agency of Charles Univesity, and in part by grant number MSM 113200007 from the Czech Ministry of Education.

**Examples.** 1. *Continuous functions.* Let $X$ be a metric compact space. For $\mathcal{H} = \mathcal{C}(X)$ we have $\mathrm{Ch\,X} = X$ and $\mathcal{C}(X)$ is a simplicial space because there are no $\mathcal{H}$-representing measures except Dirac measures.

2. *Affine functions.* Let $X$ be a metrizable convex compact subset of a Hausdorff locally convex space $E$ and $\mathcal{H}$ the linear space $\mathcal{A}(X)$ of all continuous affine functions on $X$. In this case the Choquet boundary $\mathrm{Ch\,X}$ coincides with the set $\mathrm{ext}\,X$ of all extreme points of $X$. Then $(X, \mathcal{A}(X))$ is a simplicial space if and only if $X$ is a Choquet simplex (for a definition of a Choquet simplex see e.g. [1] or [7]).

3. *Harmonic functions.* Let $\Omega$ be a bounded open subset of a Euclidean space $\mathbb{R}^n$, $X$ the closure $\overline{\Omega}$ of $\Omega$ and $\mathcal{H}$ the linear space $H(\Omega)$ of all continuous functions on $\overline{\Omega}$ which are harmonic on $\Omega$. We will study this example more deeply in Section 3.

A well-known theorem (cf. [11]) in the case of affine functions on a Choquet simplex $X$ asserts that $\mathrm{Ch\,X}$ is closed if and only if any continuous function $f$ on $\mathrm{Ch\,X}$ can be extended to an affine continuous function $h$ on $X$. A similar result can be obtained for general function spaces. This paper answers the question (in the case of a metrizable space $X$) asked by F. Jellett in [8]. He posed a problem whether a similar assertion can be proved for $F_\sigma$-sets and functions of the first Baire class. In the sequel we prove a theorem which says that for a simplicial space $(X, \mathcal{H})$, the Choquet boundary is an $F_\sigma$-set if and only if any bounded function of the first Baire class on $\mathrm{Ch\,X}$ can be extended to a bounded $\mathcal{H}$-affine function $h$ of the first Baire class on $X$.

## 2. Results

Let $X$ be a metric space. We write $B^b(X)$ for the space of all bounded real-valued Borel functions on $X$. Let $f$ be a real-valued function on $X$. Then the function $f$ is of *the first Baire class* or *a Baire-one function* (written $f \in B_1(X)$) if $f$ is a pointwise limit of a sequence $\{f_n\}$ of continuous functions on $X$. Let us denote the set of all bounded functions of the first Baire class on $X$ by $B_1^b(X)$. Due to [10, Theorem 2.12], a function $f$ is of the first Baire class on a compact metric space $X$ if and only if for every nonempty closed set $F$ and every couple $a < b$, the sets $\{x \in F : f(x) < a\}$ and $\{x \in F : f(x) > b\}$ are not simultaneously dense in $F$ (*the* [D–P] *condition*). A set $B$ is called *ambivalent* if it is both an $F_\sigma$ and $G_\delta$-set, or equivalently, if the characteristic function $\chi_B$ of the set $B$ is in $B_1(X)$. Due to the [D–P] condition, a subset $B$ of a metric compact space is ambivalent if and only if for every nonempty closed set $F$, the sets $F \cap B$ and $F \setminus B$ are not simultaneously dense in $F$ (*the* [A] *condition*).

A metric space $X$ is said to be *a Baire space* if and only if the intersection of each countable family of dense open sets in $X$ is dense. A set $A \subset X$ is *residual* if its complement $X \setminus A$ is a set of the first category, i.e. $X \setminus A = \bigcup_{n=1}^{\infty} A_n$ where

$A_n$ is a nowhere dense subset of $X$ for every integer $n$. We will employ the fact that a $G_\delta$-subspace $F$ of a complete metric space $X$ is a Baire space. Note also that a residual subset of a Baire space is dense. A suitable reference for details on Baire spaces is [6].

For a set $B$ in a metric space $X$ let us denote by $\operatorname{der}(B)$ the set of all accumulation points of $B$.

**Theorem.** *Let $(X, \mathcal{H})$ be a simplicial space. Then the following assertions are equivalent:*

(i) $\operatorname{Ch} \mathrm{X}$ *is an $F_\sigma$-set,*
(ii) *given $f \in B_1^b(\operatorname{Ch}\mathrm{X})$ there exists an $\mathcal{H}$-affine function $h \in B_1^b(X)$ such that $h = f$ on $\operatorname{Ch}\mathrm{X}$.*

In what follows we assume that $(X, \mathcal{H})$ is a simplicial space. Let us denote by $\mu_x$ the unique probability measure on $X$ representing a point $x$ supported by $\operatorname{Ch}\mathrm{X}$. We will consider the operator $T\colon B^b(X) \to B^b(X)$ defined by $Tf(x) = \int_X f\, d\mu_x$ for $f \in B^b(X)$. According to [11, Proposition 9.10], $T$ maps $\mathcal{C}(X)$ into $B_1^b(X)$. Thus $T$ maps a bounded Borel function $f$ on $X$ onto a bounded Borel function $Tf$. Let us notice that $Tf(x) = f(x)$ for $x \in \operatorname{Ch}\mathrm{X}$.

Let $B$ be a Borel set, $\operatorname{Ch}\mathrm{X} \subset B \subset X$ (in particular $B = \operatorname{Ch}\mathrm{X}$). Given a bounded Borel function $g$ on $B$, define $Tg$ as $Tf$, where a bounded Borel function $f$ on $X$ is defined by $f = g$ on $B$ and $f = 0$ elsewhere. Since any measure $\mu_x$ is carried by the Choquet boundary we see that $Tg(x) = Tf(x) = \mu_x(f) = \mu_x(g)$ for every point $x \in X$.

**Lemma 1.** *Let $f \in B^b(X)$. Then $Tf$ is an $\mathcal{H}$-affine function on $X$.*

Proof: Given $y \in X$ and $\lambda \in \mathcal{M}_y(\mathcal{H})$, define a linear functional $\mu$ on $\mathcal{C}(X)$ by the formula $\mu(g) = \int_X Tg\, d\lambda$, $g \in \mathcal{C}(X)$. Then $\mu$ is obviously a probability measure representing the point $y$. The equality

$$\mu(\operatorname{Ch}\mathrm{X}) = \int_X \mu_x(\operatorname{Ch}\mathrm{X})\, d\lambda = \int_X 1\, d\lambda = 1$$

now implies that $\mu$ is supported by $\operatorname{Ch}\mathrm{X}$. Therefore $\mu = \mu_y$ because $(X, \mathcal{H})$ is a simplicial space. Thus we obtain

$$\lambda(Tf) = \int_X \mu_x(f)\, d\lambda = \mu(f) = \mu_y(f) = Tf(y)$$

and the proof is complete. □

**Lemma 2.** *Suppose that $f \in B^b(\mathrm{Ch\,X})$ and $F \in B^b(X)$ is an $\mathcal{H}$-affine function such that $F = f$ on $\mathrm{Ch\,X}$. Then $F = Tf$.*

PROOF: Pick $y \in X$. Since $F$ is $\mathcal{H}$-affine, we have

$$F(y) = \int_{\mathrm{Ch\,X}} F(x)\, d\mu_y(x) = \int_{\mathrm{Ch\,X}} f(x)\, d\mu_y(x) = Tf(y).$$

□

**Lemma 3.** *Let $\mathrm{Ch\,X}$ be an $F_\sigma$-set and $f \in B_1^b(\mathrm{Ch\,X})$. Then $Tf$ is an $\mathcal{H}$-affine function of the first Baire class.*

PROOF: Due to the assumption we write $\mathrm{Ch\,X} = \bigcup_{n=1}^{\infty} F_n$ where $F_n$ are compact sets such that $F_1 \subset F_2 \subset \cdots \subset \mathrm{Ch\,X}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\mathrm{Ch\,X}$ converging pointwise to $f$. We may assume that $\|f\|$, $\|f_n\|$ are bounded by a positive number $M$. Since $(X, \mathcal{H})$ is a simplicial space, according to [3, Corollary 3.6], there exist $\mathcal{H}$-affine continuous functions $h_n$ on $X$ such that $h_n = f_n$ on $\mathrm{Ch\,X}$ and $\|h_n\| = \|f_n\|$.

The proof will be completed by showing that $h_n(x) \to Tf(x)$ for all $x \in X$. For fixed $x \in X$ and $\varepsilon$ positive choose an integer $n_0$ such that $\int_X |f - f_n|\, d\mu_x < \varepsilon$ and $\mu_x(F_n) > 1 - \varepsilon$ for all $n \geq n_0$. For such $n$ we have

$$\begin{aligned} |Tf(x) - h_n(x)| &= \left| \int_X (f - h_n)\, d\mu_x \right| \\ &\leq \int_X |f - f_n|\, d\mu_x + \int_X |f_n - h_n|\, d\mu_x \\ &\leq \varepsilon + \int_{\mathrm{Ch\,X} \setminus F_{n_0}} 2M\, d\mu_x \leq \varepsilon(1 + 2M), \end{aligned}$$

which proves the lemma. □

We start the main part of the proof of the Theorem with the following lemma.

**Lemma 4.** *Let $F$ be a metric compact space and $G$ be a subset of $F$ such that $\overline{G} = F = \overline{F \setminus G}$. Let $K \subset G$ be a closed subset of $F$. Then $K$ is nowhere dense in $G$.*

PROOF: Since $K$ is a closed set in $F$, it is a closed subset of $G$ as well. Suppose that $K$ is not nowhere dense in $G$. Find a nonempty open set $U \subset F$ such that $U \cap G \neq \emptyset$ and $U \cap G \subset K$. Since $F \setminus G$ is dense in $F$, we may find a point $x \in U \cap (F \setminus G)$. Due to density of $G$ in $F$, there is a sequence $\{x_n\}$ of points of $G$ such that $x = \lim_{n \to \infty} x_n$. Since $x \in U$ and $U$ is open in $F$, we may assume that $x_n \in U \cap G$ for each integer $n$. Since $U \cap G \subset K$ and $K$ is a closed set, $x \in K \subset G$. This contradiction concludes the proof. □

**Lemma 5.** *If* $\mathrm{Ch\,X}$ *is not an* $F_\sigma$*-set, then there exists a function* $f \in B_1^b(X)$ *such that* $Tf \notin B_1^b(X)$.

Proof: Suppose that the Choquet boundary $\mathrm{Ch\,X}$ of $\mathcal{H}$ is not an $F_\sigma$-set. Thus it is not an ambivalent set and according to condition [A] we can find a nonempty closed set $F$ satisfying $F = \overline{F \cap \mathrm{Ch\,X}} = \overline{F \setminus \mathrm{Ch\,X}}$. Let $B$ denote the set $\{x \in F \setminus \mathrm{Ch\,X} : \mu_x(F) \geq \frac{1}{2}\}$. Suppose that $B$ is not dense in $F$. Then there exists an open set $U \subset X$ satisfying $U \cap F \neq \emptyset$ and $U \cap F \cap B = \emptyset$. The function $f = \chi_F$ is of the first Baire class. Since

$$Tf(x)\begin{cases} = 1 & \text{for } \ x \in F \cap \mathrm{Ch\,X} \cap U, \\ \leq \frac{1}{2} & \text{for } \ x \in (F \setminus \mathrm{Ch\,X}) \cap U, \end{cases}$$

we see that $Tf$ is not in $B_1^b(X)$ due to condition [D–P] applied to the set $\overline{U \cap F}$. Thus we may suppose that $B$ is dense in $F$.

Choose a countable set $S_1 \subset B$ dense in $B$, $S_1 = \{x_n\}_{n=1}^\infty$. Denote $\mu_n = \mu_{x_n}$. Fix an integer $n$. Since

$$\mu_n(F) \geq \frac{1}{2} \quad \text{ and } \quad \mu_n(F \setminus \mathrm{Ch\,X}) = 0,$$

inner reqularity of Radon measures allows us to find a compact subset $K_n$ of $X$ such that $K_n \subset F \cap \mathrm{Ch\,X}$ and $\mu_n(K_n) \geq \frac{1}{4}$.

Set $Y = F \cap \mathrm{Ch\,X}$ and $K = \bigcup_{n=1}^\infty K_n$. Due to Lemma 4 the set $K$ is a countable union of closed nowhere dense subsets of $Y$. Hence $K$ is of the first Baire category in $Y$. Since $Y$ is a $G_\delta$-subset of a compact metric space, it is a Baire space. Since the set $Y \setminus K$ is residual in $Y$, it is dense in $Y$. Due to density of $Y$ in $F$ we obtain that $Y \setminus K$ is dense in $F$. Find a countable set $S_2 \subset Y \setminus K$ such that $S_2$ is dense in $F$.

Thus we have two countable sets $S_1,\ S_2$ such that

$$\begin{aligned} &S_1 \subset F \setminus \mathrm{Ch\,X}, \\ &S_2 \subset F \cap (\mathrm{Ch\,X} \setminus K), \end{aligned}$$

and both of them are dense in $F$. Let us denote $F_0 = \{x_1\}$. We will construct by induction nonempty sets $\{F_n\}_{n=1}^\infty$ and nonempty open sets $\{V_n\}_{n=1}^\infty$, $\{U_n\}_{n=1}^\infty$ such that for every integer $n$

(i) $\bigcup_{k=0}^n F_k$ is closed,
(ii) $\bigcup_{k=0}^n F_k \subset \bigcap_{k=1}^n U_k$,
(iii) $K_n \subset V_n$,
(iv) $U_n \cap V_n = \emptyset$,
(v) $\mathrm{der}(F_n) \cap S_1 = F_{n-1}$ and $\mathrm{der}(F_n) \cap S_2 = F_{n-1}$,
(vi) $F_n \subset S_1 \cup S_2$.

First, let us find disjoint open sets $U_1$, $V_1$ such that $x_1 \in U_1$ and $K_1 \subset V_1$. Since $S_1$ and $S_2$ are dense in $F$, there exists a set $F_1 \subset S_1 \cup S_2$ with $F_1 \subset U_1$, $\operatorname{der}(F_1 \cap S_1) = \{x_1\}$ and $\operatorname{der}(F_1 \cap S_2) = \{x_1\}$. Then all the required conditions are clearly satisfied.

Suppose that $F_j$, $V_j$, $U_j$ with desired properties have been constructed for $j \leq n$. Since $S_1 \cup S_2$ is disjoint from $K$, condition (vi) implies that $K_{n+1}$ is disjoint from $\bigcup_{k=0}^{n} F_k$. Find two disjoint open sets $U_{n+1}$, $V_{n+1}$ satisfying $\bigcup_{k=0}^{n} F_k \subset U_{n+1}$ and $K_{n+1} \subset V_{n+1}$. Let us construct $F_{n+1} \subset S_1 \cup S_2$ such that $F_{n+1} \subset \bigcap_{k=1}^{n+1} U_k$ and $\operatorname{der}(F_{n+1} \cap S_1) = F_n$, $\operatorname{der}(F_{n+1} \cap S_2) = F_n$. Then all the required conditions are satisfied.

Put $H = \bigcup_{n=0}^{\infty} F_n$. Conditions (ii) and (iv) imply that $H \cap \bigcup_{n=1}^{\infty} V_n = \emptyset$. Thus the set $\overline{H}$ is a closed set disjoint with $K$. Moreover, by (v) both sets $H \cap S_1$ and $H \cap S_2$ are dense in $H$. Thus $\overline{H \cap S_1} = \overline{H} = \overline{H \cap S_2}$. Set $f = \chi_{\overline{H}}$. Then $f$ is a function of the first Baire class on $X$. If $x$ is in $H \cap S_1$ then

$$\mu_n(\overline{H}) \leq \mu_n(X \setminus K) \leq \mu_n(X \setminus K_n) \leq \frac{3}{4},$$

which implies

$$Tf(x) \begin{cases} = 1, & x \in H \cap S_2, \\ \leq \frac{3}{4}, & x \in H \cap S_1. \end{cases}$$

By applying condition [D–P] to the set $\overline{H}$, we get that $Tf$ is not a function of the first Baire class and the proof is complete. $\square$

Proof of the Theorem: The implication (i)$\Rightarrow$(ii) is a consequence of Lemma 1 and Lemma 3. For the converse, suppose that $\operatorname{Ch} X$ is not an $F_\sigma$-set. Due to Lemma 5 there exists a function $f \in B_1^b(X)$ such that $Tf$ is not in $B_1^b(X)$. Then $g = f|_{\operatorname{Ch} X}$ is clearly a Baire-one function on $\operatorname{Ch} X$. If $F$ is an $\mathcal{H}$-affine Borel function equal to $g$ on $\operatorname{Ch} X$ then Lemma 2 yields $F = Tg = Tf$. But $Tf$ is not a function of the first Baire class and this proves the Theorem. $\square$

## 3. An application in potential theory

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let the function space $\mathcal{H}$ consist of all functions continuous on $\overline{\Omega}$ harmonic on $\Omega$. For a real-valued function $f$ defined on the boundary $\partial\Omega$ we denote by $Hf$ the PWB-solution of the Dirichlet problem on $\Omega$ with the boundary condition $f$ provided it exists. Given $x \in \Omega$, we have $Hf(x) = \lambda_x(f)$ where $\lambda_x$ is a harmonic measure representing the point $x$. In this case the Choquet boundary of $\mathcal{H}$ coincides with the set $\partial_{\mathrm{reg}}\Omega$ of all regular points of $\Omega$. According to a deep result of J. Bliedtner and W. Hansen [4] the function space $(\overline{\Omega}, \mathcal{H})$ is simplicial. Moreover, $\mathcal{H} = \mathcal{A}(\mathcal{H})$ and for any $x \in \Omega$ the measure $\mu_x$ equals $\lambda_x$.

If we reformulate the general results into the language of potential theory we get the following assertions.

**Proposition 1.** *The set of regular points $\partial_{\mathrm{reg}}\Omega$ is closed if and only if for any continuous function $f$ defined on $\partial_{\mathrm{reg}}\Omega$ there exists a function $h$ continuous on $\overline{\Omega}$ and harmonic on $\Omega$ such that $h = f$ on $\partial_{\mathrm{reg}}\Omega$.*

PROOF: Follows by [1, Theorem II.4.3]. □

**Proposition 2.** *The set of all regular points $\partial_{\mathrm{reg}}\Omega$ is an $F_\sigma$-set if and only if for any bounded function $f$ of the first Baire class defined on $\partial_{\mathrm{reg}}\Omega$ there exists a bounded $H(\Omega)$-affine function $h$ of the first Baire class on $\overline{\Omega}$ such that $h = f$ on $\partial_{\mathrm{reg}}\Omega$.*

PROOF: The proof is a direct consequence of the Theorem. □

## 4. Final remarks and open problems

1. It seems to be an open problem whether or not the Theorem is valid if we omit the condition of metrizability of the space $X$. If $X$ is a compact Hausdorff space only then the Choquet boundary $\mathrm{Ch}\,\mathrm{X}$ need not be a Borel set and the situation is much more complicated.

2. The first implication of the Theorem has been known since sixties. The proof can be found e.g. in [5] and [9].

3. Consider again the function space of Example 3 (harmonic functions). Following a definition of H. Bauer [2], the set $\Omega$ is termed *semiregular* if the PWB-solution $Hf$ can be continuously extended to the closure $\overline{\Omega}$ of $\Omega$ for any continuous function $f$ on $\partial\Omega$. Proposition 1 tells us that $\Omega$ is semiregular if and only if the set $\partial_{\mathrm{reg}}\Omega$ is closed.

4. Let $X$ be a compact convex subset of a locally convex space $E$. If $X$ is a Choquet simplex and the set of all extreme point $\operatorname{ext} X$ is closed we call $X$ *a Bauer simplex*. Alfsen [1] is a suitable reference for further details on Bauer simplexes.

**Acknowledgment.** The author wishes to express his thanks to Prof. J. Lukeš for drawing his attention to [8], many helpful comments and encouragements. He would also like to extend his thanks to the referee without whose critical comments this improved version would not have been possible.

## References

[1] Alfsen E.M., *Compact convex sets and boundary integrals*, Springer-Verlag, New York-Heidelberg, 1971.
[2] Bauer H., *Axiomatische behandlung des Dirichletschen problems fur elliptische und parabolische differentialgleichungen*, Math. Ann. **146** (1962), 1–59.
[3] Boboc N., Cornea A., *Convex cones of lower semicontinuous functions on compact spaces*, Rev. Roum. Math. Pures. App. **12** (1967), 471–525.
[4] Bliedtner J., Hansen W., *Simplicial cones in potential theory*, Invent. Math. (2) **29** (1975), 83–110.

[5] Capon M., *Sur les fonctions qui vérifient le calcul barycentrique*, Proc. London Math. Soc. (3) **32** (1976), 163–180.
[6] Engelking R., *General Topology*, Heldermann, Berlin, 1989.
[7] Choquet G., *Lectures on analysis vol. II: Representation theory*, W.A. Benjamin, Inc., New York-Amsterdam, 1969.
[8] Jellett F., *On affine extensions of continuous functions defined on the extreme boundary of a Choquet simplex*, Quart. J. Math. Oxford (2) **36** (1985), 71–73.
[9] Lacey H.E.. Morris P.D., *On spaces of type $A(K)$ and their duals*, Proc. Amer. Math. Soc. **23** (1969), 151–157.
[10] Lukeš J., Malý J., Zajíček L., *Fine topology methods in real analysis and potential theory*, Lecture Notes in Math. 1189, Springer-Verlag, 1986.
[11] Phelps R.R., *Lectures on Choquet's theorem*, D. Van Nostrand Co., 1966.

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic

*E-mail*: spurny@karlin.mff.cuni.cz

(*Received* May 17, 2001, *revised* June 27, 2001)