The lattice copies of ℓ_1 in Banach lattices

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Abstract. It is known that a Banach lattice with order continuous norm contains a copy of ℓ_1 if and only if it contains a lattice copy of ℓ_1. The purpose of this note is to present a more direct proof of this useful fact, which extends a similar theorem due to R.C. James for Banach spaces with unconditional bases, and complements the c_0- and ℓ_∞-cases considered by Lozanovskii, Mekler and Meyer-Nieberg.

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1. Introduction

This note deals with the known result below a proof of which can be obtained from several dispersed (but fundamental) theorems (see e.g. [12, Propositions 2.5.13 and 2.5.15]; or [16, Theorems 1.5, 3.2, 4.1, 4.6 and 4.7]); we present a more direct approach depending on the notion of a weakly compactly generated Banach space.

Theorem. *Let E be a Banach lattice with an order continuous norm. Then the following two conditions are equivalent.*

(i) *ℓ_1 is embeddable in E.*
(ii) *ℓ_1 is lattice embeddable in E.*

Let us recall that a Banach lattice G is [lattice] embeddable in a Banach lattice E provided that there exists a topological [and lattice] isomorphism from G into E. It is easily seen that $G = \ell_p$ $(1 \leq p < \infty)$ is [lattice] embeddable in E if and only if E has a normalized [and pairwise disjoint] sequence equivalent to the standard basis of ℓ_p (see [3, p. 217]).

The Theorem extends a similar result due to R.C. James (see e.g. [9, Theorem 1.c.9]):

Proposition 1. *If X is a Banach space with an unconditional Schauder basis (x_n), then ℓ_1 is embeddable in X iff there is a normalized block basic sequence (y_n) of (x_n) equivalent to the unit vector basis of ℓ_1 iff every normalized block basic sequence (y_n) of (x_n) (and hence, (x_n) also) weakly tends to* 0.

As is known ([10, p. 2 and Proposition 1.a.7]), every Banach space with an unconditional Schauder basis, endowed with the coordinatewise ordering, can be viewed as a Banach lattice with order continuous norm. Thus, the second condition in Proposition 1 simply means that ℓ_1 is lattice embeddable in X. The Theorem complements also the now classical c_0- and ℓ_∞-cases obtained by Lozanovskii, Mekler, and Meyer-Nieberg ([3, Theorems 14.9 and 14.12]; cf. [6], [15], [17]).

Combining the Theorem with Proposition 2.3.11 of [12] asserting that every lattice copy in a Banach lattice is complementable in it we obtain immediately the following

Corollary 1. *If a Banach lattice E has order continuous norm then E contains a copy of ℓ_1 if and only if E contains a complemented copy of ℓ_1.*

The connections between the lattice embeddability of ℓ_1 into a Banach lattice E and the topological properties of its unit ball B_E provided E has order continuous norm are well described in the monograph [12] (see Theorem 2.4.14, Corollary 2.5.10, Propositions 2.5.13, and 2.5.15). Here we present yet another consequence of the Theorem. Recall ([9, p. 8]) that a Schauder basis (x_n) of a Banach space X is called *shrinking* if it fulfills the third equivalent condition in Proposition 1. For our purposes we generalize this notion by saying that *the unit ball B_E of a Banach lattice E is shrinking if every pairwise disjoint sequence (x_n) in B_E weakly tends to* 0. Since property (i) in the Theorem is invariant under linear homeomorphisms, and since every normalized and pairwise disjoint sequence in a Banach lattice E forms an unconditional Schauder basis, from Proposition 1 we obtain immediately:

Corollary 2. *Let E and F be two Banach lattices with order continuous norms. If E and F are linearly homeomorphic, then the unit balls B_E and B_F are shrinking simultaneously.*

In particular, if a Banach space X has two different partial orderings $\leq_1$ and $\leq_2$, say, under which $X_i := (X, \leq_i)$, $i = 1, 2$ are Banach lattices, then the unit ball B_X is shrinking with respect to both the orderings.

The last corollary extends to the case of Banach lattices with order continuous norms the following known property which follows from Proposition 1: *Let (x_n) and (y_n) be two unconditional Schauder bases of a Banach space X. Then the bases are shrinking simultanously.* The construction of many nonequivalent unconditional Schauder bases in a given Banach space X having at least two nonequivalent unconditional bases is given in ([9, p. 118]; cf. the remark on p. 153 concerning Orlicz spaces).

Remarks. 1. It should be noted that the assumption of the Theorem regarding order continuity of the norm is essential. It is well known that a $C(K)$-lattice, with K infinite compact Hausdorff, has no order continuous norm ([16, Theorem 1.4]), and it is easy to check that every normalized and pairwise disjoint sequence in every infinitely dimensional $C(K)$-lattice spans a lattice copy of c_0. Therefore no $C(K)$-lattice can have a lattice copy of ℓ_1, even if it possesses isometric copies of

ℓ_1 (this is the case for $K = [0,1]$ or $K = \beta\mathbb{N}$). One should also mention the paper by I. Polyrakis [14] in which he proves that $C[0,1]$ contains a *lattice-subspace* (i.e., a closed vector subspace which is a vector lattice with the ordering induced from $C[0,1]$) that is order isomorphic to ℓ_1.

2. The following more subtle example than the above one shows that the notion of the order continuity of the norm in the Theorem cannot be replaced by σ-order continuity (which means that any *sequence* which decreases to 0 necessarily norm converges to 0). Let $E = \ell_\infty/c_0$. It is known that the natural (=quotient) norm of E is σ-order continuous without being order continuous ([13, Example 8]; cf. [2, p. 406]). Moreover, since ℓ_∞ contains a copy of ℓ_1 and c_0 does not, from the result of Diestel (asserting that the noncontainment of a copy of ℓ_1 is a three-space property ([5, Lemma 8 and note]; cf. [4])) it follows that E contains a copy of ℓ_1. On the other hand, E is order isometric to the Banach lattice $C(\beta\mathbb{N} \setminus \mathbb{N})$ and therefore, by Remark 1, ℓ_1 cannot be lattice embeddable in E.

2. Proof of the Theorem

We start with a lemma in the proof of which the WCG-property plays an important role. Let us recall that a Banach space X is said to be weakly compactly generated (WCG) provided that there exists a weakly compact set V such that its linear span $\operatorname{lin} V$ is dense in X. If E is a linear lattice, then A_e [B_e, respectively] denotes the principal ideal [principal band, respectively] in E generated by an element $e \in E$. We have $B_e = A_e^{dd}$, where for a nonempty set $C \subset E$ the symbol C^d denotes the band $\{x \in E : \ |x| \wedge |c| = 0 \text{ for all } c \in C\}$. Note that A_e is always order dense in B_e. If $E = e^{dd}$ then e is called a weak order unit of E.

Lemma 1. *Let E be a Dedekind σ-complete Banach lattice. If E has a weak order unit, then the following two conditions are equivalent.*

(i) *ℓ_∞ is lattice embeddable in E.*
(ii) *E has ℓ_∞ as a quotient.*

Proof: (i) $\Rightarrow$ (ii) Since in every Banach space X a closed copy of ℓ_∞ is complemented in X ([9, p. 105]).

(ii) $\Rightarrow$ (i) By the result of Lozanovskii, Mekler and Meyer-Nieberg (see e.g. [3, Theorem 14.9]), we have to show that the norm of E is not order continuous. To this end, notice that if (ii) holds then E cannot be a WCG-space (since the WCG-property is inherited by quotients ([8, Proposition 2.1]), and ℓ_∞ is not a WCG-space ([12, Lemma 5.4.11)]). On the other hand, every Banach lattice H with order continuous norm and a weak order unit e, say, is generated by the weakly compact set $[-e, e]$ ([12, Theorem 2.4.2(vi)]), since the principal ideal $A_e = \bigcup_{n=1}^\infty n[-e, e]$ is *norm dense* in $e^{dd} = H$ (see [12, Corollary 2.4.4(xiii)]). Consequently, from our assumptions it follows that the norm of E is not order continous indeed. $\square$

We shall also use the following result obtained independently by Lozanovskii (announced in [11] and proved in [1, Theorem, p. 732]) and Kühn ([7]; cf. [12, Proposition 2.3.12]):

Lemma 2. *Let E be a Banach lattice. Then E contains a sublattice order isomorphic to ℓ_1 if and only if E^* contains a sublattice order isomorphic to ℓ_∞.*

Proof of the Theorem: (i) $\Rightarrow$ (ii) Let (x_n) be a Schauder basic sequence in E which is equivalent to the unit vectors of ℓ_1. Put

$$e := \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n|,$$

and $G = e^{dd}$. Then G has both a weak order unit and order continuous norm. It follows that the Banach lattice $F = G^*$ has a weak order unit also (see e.g. [12, Theorem 2.4.9(ii)]). Moreover, the lattice F is Dedekind σ-complete, and since ℓ_1 is evidently embeddable in G, the Banach space F has ℓ_∞ as a quotient. Now Lemmas 1 and 2 imply that ℓ_1 is lattice embeddable in G, and hence in E also.

(ii) $\Rightarrow$ (i) Obvious. $\square$

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