A note on monotone countable paracompactness

Ge Ying, Chris Good

Abstract. We show that a space is MCP (monotone countable paracompact) if and only if it has property $(*)$, introduced by Teng, Xia and Lin. The relationship between MCP and stratifiability is highlighted by a similar characterization of stratifiability. Using this result, we prove that MCP is preserved by both countably biquotient closed and peripherally countably compact closed mappings, from which it follows that both strongly Fréchet spaces and q-space closed images of MCP spaces are MCP. Some results on closed images of wN spaces are also noted.

Keywords: monotone countable paracompactness, MCP, monotone countable metacompactness, MCM, β-space, wN-space, g-functions, stratifiability, countably biquotient mapping, peripherally countably compact mapping, (quasi-)perfect mapping

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A space X is said to be *monotonically countably metacompact* (MCM) (see [1]) if there is an operator U assigning to each decreasing sequence $(D_j)_{j\in\omega}$ of closed sets with empty intersection, a sequence of open sets $U((D_j)) = \big(U(n,(D_j))\big)_{n\in\omega}$ such that

1. $D_n \subseteq U(n,(D_j))$ for each $n \in \omega$,
2. if $D_n \subseteq E_n$, then $U(n,(D_j)) \subseteq U(n,(E_j))$,
3. $\bigcap_{n\in\omega} U(n,(D_j)) = \emptyset$.

X is said to be *monotonically countably paracompact* (MCP) if, in addition,

3′. $\bigcap_{n\in\omega} \overline{U(n,(D_j))} = \emptyset$.

MCP spaces are precisely the monotonically cp spaces of Pan [9]. Stratifiable spaces are MCP and semi-stratifiable spaces are MCM. MCM spaces are equivalent to β-spaces and MCP q-spaces coincide with wN-spaces, mirroring the relationship between stratifiable spaces and Nagata spaces, which are equivalent to stratifiable q-spaces. (Recall that a g-function on a space X with topology $\mathcal{T}$ is a mapping $g : \omega \times X \to \mathcal{T}$ such that $x \in g(n,x)$ for $n \in \omega$. A space X is a q-space [8] if there is a g-function such that whenever $x_n \in g(n,x)$, the sequence $(x_n)_{n\in\omega}$ has a cluster point and is a wN-space [5] if, in addition, whenever $g(n,x) \cap g(n,x_n) \neq \emptyset$, the sequence $(x_n)_{n\in\omega}$ has a cluster point.)

A space is said to have property $(*)$ if there is an operator V assigning to each closed set D a decreasing sequence $V(D) = (V_n(D))_{n\in\omega}$ of open sets such that

1. $D \subseteq V_n(D)$,
2. if $E \subseteq D$, then $V_n(E) \subseteq V_n(D)$ and
3. if (D_n) is a decreasing sequence of closed sets with empty intersection, then $\bigcap_{n\in\omega} \overline{V_n(D_n)} = \emptyset$.

Property $(*)$ was introduced by Teng, Xia and Lin [12], who proved that a space is wN if and only if it is a q-space with property $(*)$. In this short paper, we give a more elegant characterization of MCM and MCP spaces in terms of property $(*)$, highlighting the relationship between MCP and stratifiable spaces, which can be characterized in a similar fashion.

$$\begin{tikzcd}
\text{Nagata} \equiv \text{strat.} + \text{q} \arrow[r] \arrow[d] & \text{stratifiable} \arrow[r] \arrow[d] & \text{semi-stratifiable} \arrow[d] \\
\text{wN} \equiv \text{MCP} + \text{q} \arrow[r] & \text{MCP} \equiv (*) \arrow[r] & \text{MCM} \equiv \beta
\end{tikzcd}$$

It is known that even closed irreducible images of MCP spaces need not be MCP [1], so one can ask which classes of closed mappings do preserve MCP and in which classes of spaces MCP is preserved under closed mappings. Using our characterization, we prove that MCP is preserved by both countably biquotient closed and peripherally countably compact closed mappings. It follows that both strongly Fréchet spaces and q-space closed images of MCP spaces are MCP. We end with some remarks about closed images of wN spaces, showing that an example due to Lutzer [7] answers a question raised in [1].

All spaces are T_1 and regular, ω denotes the first infinite ordinal, $\overline{A}$ denotes the closure of A, $\operatorname{Int} A$ the interior and ∂A the boundary. All mappings are continuous and surjective.

Theorem 1. *A space X is MCM if and only if there is an operator U assigning to each $n \in \omega$ and each closed set D an open set $U(n, D)$ such that*

1. $D \subseteq U(n, D)$,
2. *if $E \subseteq D$ then $U(n, E) \subseteq U(n, D)$ and*
3. *if (D_j) is a decreasing sequence of closed sets with empty intersection, then $\bigcap U(n, D_n) = \emptyset$.*

A space X is MCP if and only if it has property $(*)$ if and only if there is such an operator U satisfying, in addition,

3′. *if (D_j) is a decreasing sequence of closed sets with empty intersection, then $\bigcap \overline{U(n, D_n)} = \emptyset$.*

Proof: We shall deal with the MCP case. Clearly a space has property $(*)$ if and only if it has an operator U satisfying the conditions stated (it does not matter whether the sequences $\{V_n(D)\}$ and $\{U(n,D)\}$ are decreasing or not) and any such space is MCP.

Conversely, suppose that V is an MCP operator on X. We show that there is an operator U on X satisfying conditions (1), (2) and (3′). For each x in X and n in ω, define

$$D_j^n(x) = \begin{cases} \{x\} & \text{if } j \leqslant n, \\ \emptyset & \text{otherwise,} \end{cases}$$

so that, for fixed x and n, $\left(D_j^n(x)\right)_{j\in\omega}$ is a decreasing sequence of closed sets with empty intersection. Now, for each closed set D, define

$$U(n,D) = \bigcup_{x\in D} V(n,(D_j^n(x))).$$

Since $x \in D_n^n \subseteq V(n,(D_j^n(x)))$ for each n, $D \subseteq V(n,D)$ and clearly, if $E \subseteq D$, then $U(n,E) \subseteq U(n,D)$. Now suppose that (D_j) is a decreasing sequence of closed sets with empty intersection. If x is in D_n, then $D_j^n(x) \subseteq D_j$ for all j, so that, by the monotonicity of V, $V(n,(D_j^n(x))) \subseteq V(n,(D_j))$. Hence $U(n,D_n) \subseteq V(n,(D_j))$ and $\bigcap \overline{U(n,D_n)} \subseteq \bigcap \overline{V(n,(D_j))} = \emptyset$, as required. □

Similar characterizations of stratifiability and semi-stratifiability are possible.

Theorem 2. *A space X is semi-stratifiable if and only if there is an operator U assigning to each $n \in \omega$ and each closed set D an open set $U(n,D)$ such that*

1. *$D \subseteq U(n,D)$,*
2. *if $E \subseteq D$ then $U(n,E) \subseteq U(n,D)$ and*
3. *if (D_j) is a decreasing sequence of closed sets, then $\bigcap U(n,D_n) = \bigcap D_n$.*

X is stratifiable if and only if there is such an operator U satisfying, in addition,

3′. *if (D_j) is a decreasing sequence of closed sets, then $\bigcap \overline{U(n,D_n)} = \bigcap D_n$.*

Proof: Again one direction is obvious and we prove only the stratifiable case. But if X is stratifiable with operator H assigning to each $n \in \omega$ and closed set D, an open set $H(n,D)$, then H satisfies conditions (1), (2) and (3′). The first two conditions are obvious. For the third, if (D_j) is a decreasing sequence of closed sets and $y \notin \bigcap D_j$, then for some n, $y \notin D_n$. As $D_j \subseteq D_n$ for all $j \geq n$, the monotonicity of H implies that $H(j,D_j) \subseteq H(j,D_n)$ and hence that $y \notin \bigcap_{j\in\omega} \overline{H(j,D_j)} \subseteq \bigcap_{j\in\omega} \overline{H(j,D_n)} = D_n$ as required. □

The assumption that the sequence (D_n) is a decreasing sequence is essential, however, as the following example shows.

Example 3. Let us say for the moment that a space X is SSS (strongly semi-stratifiable) if it has an operator U satisfying (1), (2) and (3) of Theorem 2 but without the requirement in (3) that the sequence (D_j) is decreasing.

$\mathbb{R}^2$ is not SSS: suppose, to the contrary, that U is an SSS operator for $\mathbb{R}^2$. For each r in $\mathbb{R}$, consider the closed set $D_r = (\mathbb{R} \times \{0\}) \cup (\{r\} \times [0,1])$. For some uncountable subset R_1 of $\mathbb{R}$ and some $n_1 \in \omega$,

$$(r - 1/n_1, r + 1/n_1) \times \{1/2\} \subseteq U(1, D_r)$$

for each r in R_1. Moreover there is some uncountable subset S_1 of R_1 and some closed interval $[p_1, q_1]$ with rational end points such that $[p_1, q_1] \subseteq (r - 1/n_1, r + 1/n_1)$ for all $r \in S_1$. Now we can inductively choose an $n_j \in \omega$, an uncountable $S_j \subseteq S_{j-1}$ and an interval $[p_j, q_j] \subseteq [p_{j-1}, q_{j-1}]$ such that

$$[p_j, q_j] \times \{1/2\} \subseteq (r - 1/n_j, r + 1/n_j) \times \{1/2\} \subseteq U(j, D_r)$$

for each r in S_j. Let $x \in \bigcap [p_j, q_j]$ and for each n choose some $r_n \neq x$ from S_n. Then $(x, 1/2)$ is in $U(n, D_{r_n})$ for all n, but is not in $\bigcap D_{r_n}$.

We mention in passing the following characterizations in terms of g-functions. The proofs are easy and are omitted (see [2], [4]).

Theorem 4. *The following are equivalent for any space X:*

1. *X is MCP;*
2. *there is a g-function g such that for any decreasing sequence of closed sets $\{D_n\}$ with empty intersection and any $y \in X$, there is some $n \in \omega$ and some open U containing y such that $U \cap \bigcup_{x \in D_n} g(n, x) = \emptyset$.*

Theorem 5. *The following are equivalent for any space X:*

1. *X is stratifiable;*
2. *there is a g-function g such that for any decreasing sequence of closed sets $\{D_n\}$ and any $y \notin \bigcap_{n \in \omega} D_n$, there is some $n \in \omega$ and some open U containing y such that $U \cap \bigcup_{x \in D_n} g(n, x) = \emptyset$.*

Theorem 6. *The following are equivalent for any space X:*

1. *X is wN;*
2. *there is a g-function g such that for any decreasing sequence of closed sets $\{D_n\}$ with empty intersection and any $y \in X$, there is some $n \in \omega$ such that $g(n, y) \cap \bigcup_{x \in D_n} g(n, x) = \emptyset$;*
3. *there is a monotone operator U such that for any decreasing sequence of closed sets $\{D_n\}$ with empty intersection and any $y \in X$, there is some $n \in \omega$ such that $U(n, \{y\}) \cap U(n, D_n) = \emptyset$.*

Theorem 7. *The following are equivalent for any space X:*

1. *X is Nagata;*
2. *there is a g-function g such that for any decreasing sequence of closed sets $\{D_n\}$ and any $y \notin \bigcap_{n \in \omega} D_n$, there is some $n \in \omega$ $g(n, y) \cap \bigcup_{x \in D_n} g(n, x) = \emptyset$;*

3. *there is a monotone operator U such that for any decreasing sequence of closed sets $\{D_n\}$ and any $y \notin \bigcap D_n$, there is some $n \in \omega$ such that $U(n, \{y\}) \cap U(n, D_n) = \emptyset$.*

We note that Heath [3] has characterized stratifiable spaces as those spaces with a g-function such that for any open set U and $y \in U$, there is some $n \in \omega$ and some neighbourhood V of y and such that x is in U whenever $W \cap g(n, x) \neq \emptyset$. Heath also has a similar characterization of Nagata spaces.

We now turn our attention to the images of MCP spaces under closed mappings. Recall that a space X is said to be strongly Fréchet [10] if, whenever (A_n) is a decreasing sequence of subsets of X and $x \in \bigcap_n \overline{A_n}$, then there is $x_n \in A_n$ for each $n \in \omega$ such that $x_n \to x$. Moreover if $f : X \to Y$, then f is said to be:

1. (quasi-)perfect if and only if it is closed and has (countably) compact fibres;
2. (countably) biquotient if and only if for each $y \in Y$ and (countable) open cover $\mathcal{U}$ of $f^{-1}(y)$, there is a finite subcollection $\mathcal{U}'$ such that $y \in \operatorname{Int}\big[f(\bigcup \mathcal{U}')\big]$;
3. peripherally (countably) compact if and only if for each $y \in Y$, $\partial f^{-1}(y)$ is (countably) compact.

The following implications for closed maps are obvious.

$$\begin{tikzcd}
 & \text{perfect} \arrow[r] \arrow[d] & \text{quasi-perfect} \arrow[d] \\
\text{open} \arrow[r] & \text{biquotient} \arrow[r] & \text{countably biquotient}
\end{tikzcd}$$

Lemma 8. *If a topological property P is closed hereditary, then quasi-perfect mappings preserve property P if and only if peripherally countably compact closed mappings preserve property P.*

Proof: Since quasi-perfect mappings are peripherally countably compact, we only need to prove the 'if' part.

Note first that if $f : X \to Y$ is a closed mapping and, for each $y \in Y$,

$$A_y = \begin{cases} \partial f^{-1}(y) & \text{if } \ \partial f^{-1}(y) \neq \emptyset \\ \{x_y\} & \text{for some } \ x_y \in f^{-1}(y), \text{ otherwise,} \end{cases}$$

then $Z = \bigcup_{y \in Y} A_y$ is a closed subset of X and $f(Z) = Y$ (see [8]).

Now if $f : X \to Y$ is a peripherally countably compact closed map and P is a closed hereditary property of X, then $Z = \bigcup_{y \in Y} A_y$ is a closed subset of X with property P and $g = f \restriction_Z$ is a closed map from Z onto Y with countably compact fibres, i.e. g is quasi-perfect and Y has property P. $\square$

We use the following characterization, which follows from Theorem 1 trivially by de Morgan's laws and the fact that a finite union of closed sets is closed.

Lemma 9. *A space is MCP, i.e. has property $(*)$, if and only if there is an operator F assigning to $n \in \omega$ and each open set U a closed set $F(n,U)$ such that*

1. $F(n,U) \subseteq F(n+1,U)$ *and* $F(n,U) \subseteq U$,
2. *if* $U \subseteq V$, *then* $F(n,U) \subseteq F(n,V)$ *and*
3. *if* (U_j) *is an increasing open cover of* X, *then* $\bigcup_{n\in\omega} \mathrm{Int}\big[F(n,U_n)\big] = X$.

Theorem 10. *Let X be an MCP space and let $f : X \to Y$. Then Y is MCP if any of the following conditions hold.*

1. f *is countably biquotient closed.*
2. f *is quasi-perfect closed* [1].
3. f *is open and closed* [1].
4. f *is peripherally countably compact closed.*
5. f *is closed and* Y *is strongly Fréchet or a q-space.*

Proof: Parts (2) and (3) follow from (1) and, since MCP is closed hereditary, (4) follows from (1) and Lemma 8. Moreover, if X is countably paracompact, Y is either a q-space or strongly Fréchet and $f : X \to Y$ is closed, then f is peripherally countably compact ([6]). Hence (5) follows from (4).

It remains to prove (1). To this end, suppose that f is countably biquotient and let E be an operator on X satisfying the conditions of Lemma 9. For each $n \in \omega$ and open subset U of Y let $F(n,U) = f\big(E(n,f^{-1}(U))\big)$. We claim that F satisfies the conditions of Lemma 9.

If $y \in F(n,U)$, then $y = f(x)$ for some $x \in E(n,f^{-1}(U))$, which is a subset of $f^{-1}(U)$ and $E(n+1,f^{-1}(U))$, so that $y \in U$ and $y \in F(n+1,U)$. If $U \subseteq V$, then $f^{-1}(U) \subseteq f^{-1}(V)$ and $E(n,f^{-1}(U)) \subseteq E(n,f^{-1}(U))$ for each $n \in \omega$. Hence

$$F(n,U) = f\big(E(n,f^{-1}(U))\big) \subseteq f\big(E(n,f^{-1}(V))\big) = F(n,V).$$

Finally, if $(U_n)_{n\in\omega}$ is an increasing open cover of Y, then $\big(f^{-1}(U_n)\big)_{n\in\omega}$ is an increasing open cover of X and $\bigcup_{n\in\omega} \mathrm{Int}\big[E(n,f^{-1}(U_n))\big] = X$. Hence, for each $y \in Y$, $\big(\mathrm{Int}\big[E(n,f^{-1}(U_n))\big]\big)_{n\in\omega}$ is a countable open cover of $f^{-1}(y)$ and, since f is countably biquotient, $y \in \mathrm{Int}\big[f\big(\bigcup_{n\in\mathcal{F}} E(n,f^{-1}(U_n))\big)\big]$, for some finite $\mathcal{F} \subseteq \omega$. Let $n_0 = \max \mathcal{F}$. As $\big(E(n,f^{-1}(U_n))\big)_{n\in\omega}$ is an increasing sequence of sets,

$$y \in \mathrm{Int}\big[f\big(E(n_0,f^{-1}(U_{n_0}))\big)\big] = \mathrm{Int}\big[F(n_0,U_{n_0})\big].$$

So $\bigcup_{n\in\omega} \mathrm{Int}\big[F(n,U_n)\big] = Y$ as required. $\square$

In [1] the authors ask whether perfect mappings preserve wN-spaces. In fact they do not: Lutzer [7, Example 4.3] (see also [12]) describes a perfect image of a first countable stratifiable space that is not even a q-space. However, since both finite-to-one closed images and open images of q-spaces are q-spaces and since a space is wN if and only if it is an MCP q-space, we obtain the following corollary from Theorem 10, (2) and (3).

Corollary 11. *The open and closed and finite-to-one closed images of wN spaces are wN.*

Tanaka [11, Example 3.2] constructs a completely regular finite-to-one open image of a metrizable space, which nevertheless fails to be countably paracompact (so certainly not MCP or wN), showing that [1, Proposition 18] is wrong.

In [12], it is shown that if Y is the closed image of the wN space X and Y is a q-space, then Y is wN. It is not difficult to see that the assumption that X is a q-spaces can be omitted, so the following theorem is obtained (in fact, it can also be obtained from Theorem 10 (5)).

Theorem 12. *If Y is the closed image of an MCP space and Y is a q-space, then Y is wN.*

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Department of Mathematics, Suzhou University, Suzhou 215006, P.R. China

E-mail: geying@pub.sz.jsinfo.net

School of Mathematics and Statistics, University of Birmingham, Birmingham B15 2TT, United Kingdom

E-mail: c.good@bham.ac.uk

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