A characterization of $C_2(q)$ where $q > 5$

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Abstract. The order of every finite group G can be expressed as a product of coprime positive integers $m_1, \ldots, m_t$ such that $\pi(m_i)$ is a connected component of the prime graph of G. The integers $m_1, \ldots, m_t$ are called the order components of G. Some non-abelian simple groups are known to be uniquely determined by their order components. As the main result of this paper, we show that the projective symplectic groups $C_2(q)$ where $q > 5$ are also uniquely determined by their order components. As corollaries of this result, the validities of a conjecture by J.G. Thompson and a conjecture by W. Shi and J. Be for $C_2(q)$ with $q > 5$ are obtained.

Keywords: prime graph, order component, finite group, simple group

Classification: 20D05, 20D60

1. Introduction

If n is an integer, $\pi(n)$ is the set of prime divisors of n and if G is a finite group $\pi(G)$ is defined to be $\pi(|G|)$. The prime graph $\Gamma(G)$ of a group G is a graph whose vertex set is $\pi(G)$, and two distinct primes p and q are linked by an edge if and only if G contains an element of order pq. Let π_i, $i = 1, 2, \ldots, t(G)$ be the connected components of $\Gamma(G)$. For $|G|$ even, π_1 will be the connected component containing 2. Then $|G|$ can be expressed as a product of some positive integers m_i, $i = 1, 2, \ldots, t(G)$ with $\pi(m_i) =$ the vertex set of π_i. The integers m_i's are called the order components of G. The set of order components of G will be denoted by $\mathrm{OC}(G)$. If the order of G is even, then m_1 is the even order component and $m_2, \ldots, m_{t(G)}$ will be the odd order components of G. The order components of non-abelian simple groups having at least three prime graph components are obtained by G.Y. Chen [8, Tables 1, 2, 3]. The order components of non-abelian simple groups with two order components are illustrated in Table 1 according to [13], [20]. The following groups are uniquely determined by their order components: Suzuki-Ree groups [6], Sporadic simple groups [3], $PSL_2(q)$ [8], $E_8(q)$ [7], $G_2(q)$ where $q \equiv 0 \pmod 3$ [2], $F_4(q)$ where q is even [12], $PSL_3(q)$ where q is an odd prime power [11] and A_p where p and $p-2$ are primes [10]. In this paper, we prove that the projective symplectic groups $C_2(q)$ where $q > 5$ are also uniquely determined by their order components. In other words we have:

The Main Theorem. *Let G be a finite group, $M = C_2(q)$ where $q > 5$. If $\mathrm{OC}(G) = \mathrm{OC}(M)$ then $G \cong M$.*

2. Preliminary results

Definition 2.1 ([9])**.** A finite group G is called a 2-Frobenius group if it has a normal series $G > K > H > 1$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.2 ([20, Theorem A])**.** *If G is a finite group with its prime graph having more than one component, then G is one of the following groups:*

(a) *a Frobenius or 2-Frobenius group;*
(b) *a simple group;*
(c) *an extension of a π_1-group by a simple group;*
(d) *an extension of a simple group by a π_1-solvable group;*
(e) *an extension of a π_1-group by a simple group by a π_1-group.*

Lemma 2.3 ([20, Lemma 3])**.** *If G is a finite group with more than one prime graph component and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, then H is a nilpotent group.*

The next lemma follows from Theorem 2 in [1]:

Lemma 2.4. *Let G be a Frobenius group of even order and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(\Gamma(G)) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$ and G has one of the following structures:*

(a) $2 \in \pi(K)$ *and all Sylow subgroups of H are cyclic;*
(b) $2 \in \pi(H)$, *K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of G are cyclic groups and the 2-Sylow subgroups of G are cyclic or generalized quaternion groups;*
(c) $2 \in \pi(H)$, *K is an abelian group and there exists $H_0 \leq H$ such that $|H : H_0| \leq 2$, $H_0 = Z \times SL(2,5)$, $(|Z|, 2.3.5) = 1$ and the Sylow subgroups of Z are cyclic.*

The next lemma follows from Theorem 2 in [1] and Lemma 2.3:

Lemma 2.5. *Let G be a 2-Frobenius group of even order. Then $t(\Gamma(G)) \geq 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that*

(a) $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi(K/H) = \pi_2$;
(b) *G/K and K/H are cyclic, $|G/K|$ divides $|\operatorname{Aut}(K/H)|$, $(|G/K|, |K/H|) = 1$ and $|G/K| < |K/H|$;*
(c) *H is nilpotent and G is a solvable group.*

Lemma 2.6 ([5, Lemma 8])**.** *Let G be a finite group with $t(\Gamma(G)) \geq 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not a π_i-number, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.*

The next lemma follows from Lemma 1.4 in [4].

Lemma 2.7. *Suppose G and M are two finite groups satisfying $t(\Gamma(M)) \geq 2$, $N(G) = N(M)$, where $N(G) = \{n \mid G$ has a conjugacy class of size $n\}$, and $Z(G) = 1$. Then $|G| = |M|$.*

Lemma 2.8 ([4, Lemma 1.5])**.** *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(\Gamma(G_1)) = t(\Gamma(G_2))$ and $\mathrm{OC}(G_1) = \mathrm{OC}(G_2)$.*

Lemma 2.9. *Let G be a finite group and let M be a non-abelian simple group with $t(M) = 2$ satisfying $\mathrm{OC}(G) = \mathrm{OC}(M)$.*

(1) *Let $|M| = m_1m_2$, $\mathrm{OC}(M) = \{m_1, m_2\}$, and $\pi(m_i) = \pi_i$ for $i = 1$ or 2. Then $|G| = m_1m_2$ and one of the following holds:*

- (a) *G is a Frobenius or 2-Frobenius group;*
- (b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group. Moreover $\mathrm{OC}(K/H) = \{m_1', m_2', \ldots, m_s', m_2\}$, $|K/H| = m_1'm_2'\ldots m_s'm_2$ and $m_1'm_2'\ldots m_s' \mid m_1$ where $\pi(m_j') = \pi_j'$, $1 \leq j \leq s$.*

(2) $|G/K| \mid |\mathrm{Out}(K/H)|$.

Proof: (1) follows from the above lemmas. Since $t(G) \geq 2$, we have $t(G/H) \geq 2$. Otherwise $t(G/H) = 1$, so that $t(G) = 1$. Since $2 \mid |H|$ and H is a π_i-group, we arrive to a contradiction. Moreover, we have $Z(G/H) = 1$. For any $xH \in G/H$ and $xH \notin K/H$, xH induces an automorphism of K/H and this automorphism is trivial if and only if $xH \in Z(G/H)$. Therefore, $G/K \leq \mathrm{Out}(K/H)$ and since $Z(G/H) = 1$, (2) follows. □

Lemma 2.10. *Let $M = C_2(q)$. Suppose $D(q) = \frac{q^2+1}{k}$, where $k = (2, q-1)$.*

- (a) *If $p \in \pi(M)$, then $|S_p| \leq q^4$ where $S_p \in \mathrm{Syl}_p(M)$;*
- (b) *If $p \in \pi_1(M)$, $p^\alpha \mid |M|$ and $p^\alpha - 1 \equiv 0 \pmod{D(q)}$, then $p^\alpha = q^4$ or $(q, p^\alpha) = (3, 2^4)$.*
- (c) *If $p \in \pi_1(M)$, $p^\alpha \mid |M|$ and $p^\alpha + 1 \equiv 0 \pmod{D(q)}$ then $p^\alpha = q^2$ or $(q, p^\alpha) = (2, 3^2), (3, 2^2), (3, 2^6), (3, 3^2)$ or $(5, 2^6)$.*

Proof: **(a)** Observe that $|M| = q^4(q+1)^2(q-1)^2\frac{(q^2+1)}{k}$ and $(q-1, q+1) = 1$ or 2. Thus if q is even, the factors are coprime and if q is odd and $p^\alpha \mid |M|$, thus $p^\alpha \mid q^4$ or $p^\alpha \mid 4(q+1)^2$ or $p^\alpha \mid 4(q-1)^2$ or $p^\alpha \mid (q^2+1)$. Therefore (a) follows.

(b) Let $p^\alpha \mid |M|$ and $p \in \pi_1(M)$ with $p^\alpha - 1 \equiv 0 \pmod{D(q)}$. Consider the following two cases:

Case 1. q is even:

(1.1) If $p^\alpha \mid q^4$ then $p^\alpha - 1 \geq q^2 + 1$ and hence $q^2 \mid p^\alpha$. Since $p^\alpha - 1 = t(q^2+1)$, we have $q^2 \mid t + 1$ or $q^2 - 1 \leq t$ which means that $p^\alpha = q^4$.

(1.2) If $p^\alpha \mid (q+1)^2$ then since $\frac{(q+1)^2}{2} < q^2 + 1$, p^α must be equal to $(q+1)^2$. Thus $p^\alpha - 1 = q^2 + 1 + 2q - 1$, hence $q^2 + 1 = 2q - 1$ which has no solution.

(1.3) If $p^\alpha \mid (q-1)^2$ then $p^\alpha < (q-1)^2 < q^2+1$, but $p^\alpha - 1 \geq q^2+1$, which is a contradiction.

Case 2. q is odd:

(2.1) If $p^\alpha \mid q^4$ then $p^\alpha > \frac{q^2+1}{2} > \frac{q^2}{2}$ and hence $q^2 \mid p^\alpha$. Since $p^\alpha - 1 = t\frac{(q^2+1)}{2}$, we have $q^2 \mid t+2$ or $q^2-2 \leq t$, therefore $q^2-2 \leq t \leq 2(q^2-1)$ or $t = (q^2-2)+s$, where $0 \leq s \leq q^2$. Similarly to Case 1 we conclude that $p^\alpha = q^4$.

(2.2) If $p^\alpha \mid 4(q-1)^2$ then since $\frac{4(q-1)^2}{8} - 1 < \frac{q^2+1}{2}$, p^α must be equal to $\frac{4(q-1)^2}{s}$ where $1 \leq s \leq 7$, but s cannot be equal to 3, 5, 6, 7. Easy calculations show that if $s = 1$ then $(q, p^\alpha) = (3, 2^4)$ and in the other cases $p^\alpha - 1 \not\equiv 0 \pmod{\frac{q^2+1}{2}}$.

(2.3) If $p^\alpha \mid 4(q+1)^2$ and $p^\alpha - 1 \equiv 0 \pmod{\frac{q^2+1}{2}}$, then since $\frac{4(q+1)^2}{14} - 1 < \frac{q^2+1}{2}$, p^α must be equal to $\frac{4(q+1)^2}{s}$ where $1 \leq s \leq 13$, but s can only be equal to 1, 2, 4, 8, 9. Again easy calculations show that if $s = 4$ then $(q, p^\alpha) = (3, 2^4)$ and in the other cases $p^\alpha - 1 \not\equiv 0 \pmod{\frac{q^2+1}{2}}$.

(c) Similar arguments show that (c) holds. $\square$

Lemma 2.11. *Let G be a finite group and $M = C_2(q)$ where $q > 5$ and $\mathrm{OC}(G) = \mathrm{OC}(M)$. Then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof: G is not a Frobenius group otherwise by Lemma 2.4, $\mathrm{OC}(G) = \{|H|, |K|\}$ where H and K are Frobenius kernel and Frobenius complement of G, respectively. If $2 \mid |H|$ then $|K| = \frac{q^2+1}{k}$, and $|H| = q^4(q+1)^2(q-1)^2$. Since $4(q-1)^2 > 1$, there exists a prime p such that $p^\alpha \mid 4(q-1)^2$. If P is a p-Sylow subgroup of H, then since H is nilpotent, $P \lhd G$ and hence by Lemma 2.6, $\frac{q^2+1}{k} \mid |P| - 1$. By Lemma 2.10(b) this implies that $p^\alpha = q^4$. But $q^4 \nmid 4(q-1)^2$ which is a contradiction. If $2 \mid |K|$ then $|H| = \frac{q^2+1}{k}$ and $|K| = q^4(q+1)^2(q-1)^2$. Now if P is a p-Sylow subgroup of H, then $|P| < |K|$, but $|K| \mid (|P|-1)$, which is a contradiction. Therefore, G is not a Frobenius group.
Let G be a 2-Frobenius group and let q be odd. By Lemma 2.5 there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = \frac{q^2+1}{k} < 4(q+1)^2$ and $|G/K| < |K/H|$. Thus there exists a prime p such that $p \mid 4(q+1)^2$ and $p \mid |H|$. If P is a p-Sylow subgroup of H, since H is nilpotent, P must be a normal subgroup of K with $P \subseteq H$ and $|K| = \frac{q^2+1}{k}|H|$. Therefore, $\frac{q^2+1}{k} \mid (|P|-1)$ by Lemma 2.6 and hence $p^\alpha - 1 \equiv 0 \pmod{D(q)}$, so $|P| = q^4$ which is impossible since $q^4 \nmid 4(q+1)^2$. If q is even, then we consider $(q+1)^2$ instead of $4(q+1)^2$ and proceed similarly. $\square$

Lemma 2.12. *Let G be a finite group and $M = C_2(q)$, where $q > 5$. If $\mathrm{OC}(G) = \mathrm{OC}(M)$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover, the odd order component of M is equal to an odd order component of K/H. In particular, $t(\Gamma(K/H)) \geq 2$.*

PROOF: The first part of the lemma follows from the above lemmas since the prime graph of M has two prime graph components. For primes p and q, if K/H has an element of order pq, then G has one. Hence, by the definition of prime graph component, the odd order component of G must be an odd order component of K/H. □

3. Proof of the main theorem

By Lemma 2.12, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, $t(\Gamma(K/H)) \geq 2$ and the odd order component of M is an odd order component of K/H. We summarize the relevant information in Tables 1–3 below:

Table 1
The order components of simple groups[1] with $t(G) = 2$

Group	Orcmp 1	Orcmp 2
A_p, $p \neq 5, 6$ p and $p-2$ not both prime	$3 \cdot 4 \cdots (p-3)(p-2)(p-1)$	p
A_{p+1}, $p \neq 4, 5$ $p-1$ and $p+1$ not both prime	$3 \cdot 4 \cdots (p-2)(p-1)(p+1)$	p
A_{p+2}, $p \neq 3, 4$ p and $p+2$ not both prime	$3 \cdot 4 \cdots (p-1)(p+1)(p+2)$	p
$A_{p-1}(q)$, $(p,q) \neq (3,2), (3,4)$	$q^{\frac{p(p-1)}{2}} \Pi_{i=1}^{p-1}(q^i - 1)$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$, $q-1 \mid p+1$	$q^{\frac{p(p+1)}{2}}(q^{p+1}-1)\Pi_{i=2}^{p-1}(q^i-1)$	$\frac{q^p-1}{q-1}$
${}^2A_{p-1}(q)$	$q^{\frac{p(p-1)}{2}} \Pi_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{q^p+1}{(q+1)(p,q+1)}$

[1] p is an odd prime number.

Table 1 (continued)

Group	Orcmp 1	Orcmp 2
${}^2A_p(q)$, $q+1\mid p+1$	$q^{\frac{p(p+1)}{2}}(q^{p+1}-1)\Pi_{i=2}^{p-1}(q^i-(-1)^i)$	$\frac{q^p+1}{q+1}$
$(p,q)\neq(3,3),(5,2)$		
${}^2A_3(2)$	$2^6\cdot 3^4$	5
$B_n(q)$, $n=2^m\geq 4, q$ odd	$q^{n^2}(q^n-1)\Pi_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{2}$
$B_p(3)$	$3^{p^2}(3^p+1)\Pi_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p-1}{2}$
$C_n(q)$, $n=2^m\geq 2$	$q^{n^2}(q^n-1)\Pi_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$, $q=2,3$	$q^{p^2}(q^p+1)\Pi_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$, $p\geq 5$, $q=2,3,5$	$q^{p(p-1)}\Pi_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$, $q=2,3$	$\frac{1}{(2,q-1)}q^{p(p+1)}(q^p+1)$	$\frac{q^p-1}{(2,q-1)}$
	$\times(q^{p+1}-1)\Pi_{i=1}^{p-1}(q^{2i}-1)$	
${}^2D_n(q)$, $n=2^m\geq 4$	$q^{n(n-1)}\Pi_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$, $n=2^m+1\geq 5$	$2^{n(n-1)}(2^n+1)$	$2^{n-1}+1$
	$\times(2^{n-1}-1)\Pi_{i=1}^{n-2}(2^{2i}-1)$	
${}^2D_p(3)$, $p\neq 2^m+1$, $p\geq 5$	$3^{p(p-1)}\Pi_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$, $n=2^m+1\neq p, m\geq 2$	$\frac{1}{2}3^{n(n-1)}(3^n+1)$	$\frac{3^{n-1}+1}{2}$
	$\times(3^{n-1}-1)\Pi_{i=1}^{n-2}(3^{2i}-1)$	
$G_2(q)$, $q\equiv\epsilon(\mathrm{mod}\ 3), \epsilon=\pm 1$, $q>2$	$q^6(q^3-\epsilon)(q^2-1)(q+\epsilon)$	$q^2-\epsilon q+1$
${}^3D_4(q)$	$q^{12}(q^6-1)(q^2-1)(q^4+q^2+1)$	q^4-q^2+1
$F_4(q)$, q odd	$q^{24}(q^8-1)(q^6-1)^2(q^4-1)$	q^4-q^2+1
${}^2F_4(2)'$	$2^{11}\cdot 3^3\cdot 5^2$	13
$E_6(q)$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5-1)$	$\frac{q^6+q^3+1}{(3,q-1)}$
	$\times(q^3-1)(q^2-1)$	
${}^2E_6(q)$, $q>2$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5+1)$	$\frac{q^6-q^3+1}{(3,q+1)}$
	$\times(q^3+1)(q^2-1)$	
M_{12}	$2^6\cdot 3^3\cdot 5$	11
J_2	$2^7\cdot 3^3\cdot 5^2$	7
Ru	$2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13$	29
He	$2^{10}\cdot 3^3\cdot 5^2\cdot 7^3$	17
Mcl	$2^7\cdot 3^6\cdot 5^3\cdot 7$	11
Co_1	$2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13$	23
Co_3	$2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11$	23
Fi_{22}	$2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11$	13
$F_5=HN$	$2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11$	19

Table 2
The order components of simple groups[1] with $t(G) \geq 3$

Group	Orcmp 1	Orcmp 2	Orcmp 3	Orcmp 4
A_p, p and $p-2$ are primes	$3 \cdot 4 \cdots (p-3)(p-1)$	$p-2$	p	
$A_1(q)$, $4 \mid q+1$	$q+1$	q	$(q-1)/2$	
$A_1(q)$, $4 \mid q-1$	$q-1$	q	$(q+1)/2$	
$A_1(q)$, $2 \mid q$	q	$q+1$	$q-1$	
$A_2(2)$	8	3	7	
$A_2(4)$	2^6	5	7	9
${}^2A_5(2)$	$2^{15} \cdot 3^6 \cdot 5$	7	11	
${}^2B_2(q)$ $q = 2^{2n+1} > 2$	q^2	$q - \sqrt{2q} + 1$	$q + \sqrt{2q} + 1$	$q-1$
${}^2D_p(3)$ $p = 2^n + 1$, $n \geq 2$	$2 \cdot 3^{p(p-1)}(3^{p-1} - 1)$ $\times \Pi_{i=1}^{p-2}(3^{2i} - 1)$	$(3^{p-1} + 1)/2$	$(3^p + 1)/4$	
${}^2D_{p+1}(2)$ $p = 2^n - 1$, $n \geq 2$	$2^{p(p+1)}(2^p - 1)$ $\times \Pi_{i=1}^{p-1}(2^{2i} - 1)$	$2^p + 1$	$2^{p+1} + 1$	
$E_7(2)$	$2^{63} \cdot 3^{11} \cdot 5^2 \cdot 7^3$ $\cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	73	127	
$F_4(q)$ $2 \mid q$, $q > 2$	$q^{24}(q^6 - 1)^2(q^4 - 1)^2$	$q^4 + 1$	$q^4 - q^2 + 1$	
${}^2F_4(q)$ $q = 2^{2n+1} > 2$	$q^{12}(q^4 - 1)(q^3 + 1)$ $\times (q^2 + 1)(q - 1)$	$q^2 - \sqrt{2q^3}$ $+q - \sqrt{2q} + 1$	$q^2 + \sqrt{2q^3}$ $+q + \sqrt{2q} + 1$	
$G_2(q)$, $3 \mid q$	$q^6(q^2 - 1)^2$	$q^2 + q + 1$	$q^2 - q + 1$	
${}^2G_2(q)$, $q = 3^{2n+1}$	$q^3(q^2 - 1)$	$q - \sqrt{3q} + 1$	$q + \sqrt{3q} + 1$	
$E_7(3)$	$2^{23} \cdot 3^{63} \cdot 5^2 \cdot 7^3$ $\cdot 11^2 \cdot 13^3 \cdot 19 \cdot 37 \cdot 41$ $\cdot 61 \cdot 73 \cdot 547$	757	1093	
${}^2E_6(2)$	$2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11$	13	17	19
M_{11}	$2^4 \cdot 3^2$	5	11	
M_{22}	$2^7 \cdot 3^2$	5	7	11
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	11	23	
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	11	23	
J_1	$2^3 \cdot 3 \cdot 5$	7	11	19
J_3	$2^7 \cdot 3^5 \cdot 5$	17	19	

[1] p is an odd prime number.

Table 2 (continued)

Group	Orcmp 1	Orcmp 2	Orcmp 3	Orcmp 4	Orcmp 5	Orcmp 6
J_4	$2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3$	23	29	31	37	43
HS	$2^9\cdot 3^2\cdot 5^3$	7	11			
Sz	$2^{13}\cdot 3^7\cdot 5^2\cdot 7$	11	13			
ON	$2^9\cdot 3^4\cdot 5\cdot 7^3$	11	19	31		
Ly	$2^8\cdot 3^7\cdot 5^6\cdot 7\cdot 11$	31	37	67		
Co_2	$2^{18}\cdot 3^6\cdot 5^3\cdot 7$	11	23			
F_{23}	$2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13$	17	23			
F'_{24}	$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13$	17	23	29		
$F_1 = M$	$2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3$ $\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 47$	41	59	71		
$F_2 = B$	$2^{41}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13$ $\cdot 17\cdot 19\cdot 23$	31	47			
$F_3 = Th$	$2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13$	19	31			

Table 3
The order components of $\mathbf{E}_8(\mathbf{q})$

Group	$E_8(q)$, $q \equiv 0, 1, 4 \pmod 5$
Orcmp 1	$q^{120}(q^{18}-1)(q^{14}-1)(q^{12}-1)^2(q^{10}-1)^2(q^8-1)^2(q^4+q^2+1)$
Orcmp 2	$q^8+q^7-q^5-q^4-q^3+q+1$
Orcmp 3	$q^8-q^7+q^5-q^4+q^3-q+1$
Orcmp 4	$q^8-q^6+q^4-q^2+1$
Orcmp 5	q^8-q^4+1

Group	$E_8(q)$, $q \equiv 2, 3 \pmod 5$
Orcmp 1	$q^{120}(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^4+1)$ $\times(q^4+q^2+1)$
Orcmp 2	$q^8+q^7-q^5-q^4-q^3+q+1$
Orcmp 3	$q^8-q^7+q^5-q^4+q^3-q+1$
Orcmp 4	q^8-q^4+1

We now proceed with the proof in the following steps:

Step 1. Let $K/H \cong A_n$ where $n = p,\ p+1,\ p+2$ and $p \geq 5$ is a prime number. If $k = 1$ and $q^2+1 = p$ then $|C_2(q)| = p(p-1)^2(p-2)^2$, and hence $(p-3, |C_2(q)|) \mid 2$ which is a contradiction. If $q^2+1 = p-2$ then $|C_2(q)| = (p-2)(p-3)^2(p-4)^2$ and hence $p \nmid |C_2(q)|$ which is a contradiction. If $k = 2$ and $\frac{q^2+1}{2} = p$ then $(p-2, |C_2(q)|) \mid 9$ which implies that $p = 5$ or 11 which is impossible. If $\frac{q^2+1}{2} = p-2$ then $p \nmid |C_2(q)|$ which is a contradiction.

Step 2. If $K/H \cong A_r(q')$ then we distinguish the following 6 cases:

2.1. $K/H \cong A_{p'-1}(q')$ where $(p', q') \neq (3,2), (3,4)$. Then $q'^{p'} - 1 \equiv 0 \pmod{D(q)}$ which implies that $q'^{p'} = q^4$. Since p' is an odd prime, if $p' > 3$, then K/H has a Sylow subgroup of size greater than q^4, which is a contradiction by Lemma 2.10(a). If $p' = 3$, then we have $q'^3 = q^4$ and $(q'-1)(3, q'-1) = (q^2-1)(2, q-1)$. But easy calculations show that these two equations have no common solution.

2.2. $K/H \cong A_{p'}(q')$ where $(q'-1) \mid (p'+1)$, then similarly to 2.1, K/H has a Sylow subgroup of size greater than q^4, and it is a contradiction by Lemma 2.10(a).

2.3. $K/H \cong A_1(q')$, where $4 \mid (q'+1)$. If $D(q) = \frac{q'-1}{2}$ then $q' = q^4$. But $\frac{q^2+1}{(2,q-1)} = \frac{q'-1}{2}$ and so $q^2 - 1 = 1$ or 2 which is impossible. If $D(q) = q'$ and $k = 1$ then $q' = q^2+1$ but $4 \nmid q^2+2$. If $k = 2$ then

$$|K/H| = |A_1(q')| = \frac{q^2+1}{2} \cdot \frac{q^2+3}{2} \cdot \frac{q^2-1}{4},$$

but this is a contradiction since $\frac{q^2+3}{4} \nmid |G|$.

2.4. $K/H \cong A_1(q')$ where $4 \mid (q'-1)$. If $D(q) = \frac{q'+1}{2}$ then $q' = q^2$. But q' is odd so q is odd and hence $k = 2$. Therefore, $|A_1(q^2)| = q^2(q^2-1)(q^2+1)/2$ and so $|G/K| \cdot |H| = q^2(q^2-1)$. But $|G/K| \mid |\operatorname{Out}(A_1(q^2))|$ by Lemma 2.9(3), and if $q = p'^n$ then $|\operatorname{Out}(A_1(q^2))| = 4n$ ([19]), which implies that $|H| \neq 1$. Thus we can consider a p-Sylow subgroup P of H. Since H is nilpotent, $P \lhd G$ and hence $D(q) \mid (|P|-1)$, but $|P| \mid q^2$ or $|P| \mid q^2-1$. If $|P| \mid q^2$ then $|P| = q^2$ or $|P| \leq \frac{q^2}{3}$. But $\frac{q^2+1}{2} \nmid q^2-1$ and $\frac{q^2+1}{2} \geq \frac{q^2}{3} - 1 \geq |P| - 1$ which are contradictions. Similarly $|P| \mid q^2-1$ is not possible. If $D(q) = q'$ then similarly to 2.3, we get a contradiction.

2.5. $K/H \cong A_1(q')$ where $4 \mid q'$. If $D(q)$ equals $q'-1$, then $q' = q^4$ and $|A_1(q')| = q^4(q^4-1)(q^4+1)$, which is impossible. If $D(q) = q'+1$, by Lemma 2.10(c), $q' = q^2$ and since q' is even, q is even. Since $K/H \cong A_1(q^2)$, we get a contradiction similar to 2.4.

2.6. $K/H \cong A_2(2)$ or $A_2(4)$ then $D(q)$ must be equal to 3, 5, 7, 9, none of which is possible.

Step 3. If $K/H \cong {}^2A_r(q')$ then we consider 2 cases:

3.1. $K/H \cong {}^2A_{p'-1}(q')$ or ${}^2A_{p'}(q')$ where $(q'+1) \mid (p'+1)$ and $(p',q') \neq (3,3),(5,2)$. Then $q'^{p'}+1 \equiv 0 \pmod{D(q)}$. By Lemma 2.10(c), $q'^{p'} = q^2$. Since

$$\frac{q'^{p'}+1}{(q'+1)(q'+1,p')} = \frac{q^2+1}{(2,q-1)},$$

so $(2,q-1) = (q'+1)(q'+1,p')$, which is impossible.

3.2. $K/H \cong {}^2A_3(2)$ or ${}^2A_5(2)$. Then $D(q)$ must be equal to 5, 7, 11, none of which is possible.

Step 4. If $K/H \cong B_r(q')$ then we consider 2 cases:

4.1. $K/H \cong B_r(q')$ where $r = 2^t \geq 4$ and q' is odd. Then $q'^r+1 \equiv 0 \pmod{D(q)}$. By Lemma 2.10(c), $q'^r = q^2$. But since $r \geq 4$, we have $q'^{r^2} > q^4$, which is a contradiction by Lemma 2.10(a).

4.2. $K/H \cong B_p(3)$. Then $3^p = q^4$, which is impossible since 3^p is not a square number.

Step 5. If $K/H \cong C_r(q')$ then we consider 2 cases:

5.1. $K/H \cong C_r(q')$ where $r = 2^t \geq 2$. Then $q'^r = q^2$. Since $q'^{r^2} \geq q^4$, we conclude that $r = 2$ and hence $q = q'$, so $K/H = C_2(q)$. Then $|G| = |C_2(q)| = |K/H| = |K|/|H|$ which implies that $|H| = 1$ and $|K| = |G| = |C_2(q)|$. Therefore, $K = C_2(q)$ and hence $G = C_2(q)$.

5.2. $K/H \cong C_{p'}(q')$ where $q' = 2$, 3. Then $q'^{p'} = q^4$, which is a contradiction since $q'^{p'}$ is not a square number.

Step 6. If $K/H \cong D_r(q')$ where $(r,q') = (p',q')$ (with $p' \geq 5$, $q' = 2,3,5$) or, $(r,q') = (p'+1,q')$ (with $q' = 2,3$). Thus $q'^{p'} = q^4$ and since p' is an odd prime, K/H has a Sylow subgroup of size greater than q^4, which is a contradiction by Lemma 2.10(a).

Step 7. Let $K/H \cong {}^2B_2(q')$ where $q' = 2^{2t+1} > 2$.
If $D(q) = q'-1$ then $q' = q^4$ which is a contradiction since $q'^2 > q^4$.
If $D(q) = q' \pm \sqrt{2q'}+1$. Then $q'^2+1 \equiv 0 \pmod{D(q)}$. Therefore, $q^2 = q'^2$ and hence $q = q'$. But $q^2+1 = q \pm \sqrt{2q}+1$, which is impossible.

Step 8. If $K/H \cong {}^2D_r(q')$ then we consider 6 cases:

8.1. $K/H \cong {}^2D_r(q')$ where $r = 2^t > 2$. Then $q'^r = q^2$. Since $r-1 \geq 3$ we have $q^6 \mid |G|$ which is a contradiction by Lemma 2.10(a).

8.2. $K/H \cong {}^2D_r(2)$ where $r = 2^t + 1 \geq 5$. Then $2^{r-1} = q^2$. Since $r \geq 5$ we have $q^{10} \mid |G|$, which is a contradiction by Lemma 2.10(a).

8.3. $K/H \cong {}^2D_p(3)$ where $5 \leq p \neq 2^r + 1$. Then $3^p = q^2$, but 3^p is not a square number.

8.4. $K/H \cong {}^2D_r(3)$ where $r = 2^t + 1 \neq p$, $t \geq 2$. Then $3^{r-1} = q^2$. But $3^{r(r-1)} > q^4$, which is a contradiction by Lemma 2.10(a).

8.5. $K/H \cong {}^2D_p(3)$ where $p = 2^t + 1$, $t \geq 2$. Then we proceed similarly to 8.3 and 8.4.

8.6. $K/H \cong {}^2D_{p+1}(2)$ where $p = 2^r - 1$, $r \geq 2$ then $2^p = q^2$ or $2^{p+1} = q^2$, but similarly to last cases they are impossible.

Step 9. If $K/H \cong G_2(q')$ then we consider 3 cases:

9.1. $K/H \cong G_2(q')$ where $2 < q' \equiv 1 \pmod 3$. Then $D(q) = {q'}^2 - q' + 1$ and hence ${q'}^3 + 1 \equiv 0 \pmod{D(q)}$, so ${q'}^3 = q^2$, and thus $(2, q-1) = q' + 1$ which is a contradiction.

9.2. $K/H \cong G_2(q')$ where $2 < q' \equiv -1 \pmod 3$. Then ${q'}^3 = q^4$, and hence $q^8 \mid |G|$ which is a contradiction.

9.3. $K/H \cong G_2(q')$ where $3 \mid q'$. Then $D(q) = {q'}^2 \pm q' + 1$. This is similar to Cases 9.1 and 9.2.

Step 10. If $K/H \cong E_7(2)$ or $E_7(3)$ or ${}^2E_6(2)$ or ${}^2F_4(2)'$ then $D(q)$ must be equal to 13, 17, 19, 73, 127, 757, 1093, none of which has a solution in $\mathbb{Z}$.

Step 11. If $K/H \cong {}^3D_4(q')$ then $D(q) = {q'}^4 - {q'}^2 + 1$, and hence ${q'}^6 + 1 \equiv 0 \pmod{D(q)}$ which implies that ${q'}^3 = q$, and this implies that ${q'}^2 + 1 = 1$ or 2 which is impossible.

Step 12. If $K/H \cong F_4(q')$ then we consider 2 cases:

12.1. If $D(q) = {q'}^4 - {q'}^2 + 1$ then we proceed similarly to Step 11.

12.2. If $D(q) = {q'}^4 + 1$, then ${q'}^4 = q^2$ and $q^{12} \mid |G|$ which is again impossible.

Step 13. If $K/H \cong {}^2F_4(q')$ where $q' = 2^{2r+1} > 2$ then ${q'}^6 = q^2$ and hence $q = {q'}^3$ and q is even. But ${q'}^6 + 1$ cannot be equal to ${q'}^2 \pm \sqrt{2{q'}^3} + q' \pm \sqrt{2q'} + 1$.

Step 14. If $K/H \cong {}^2G_2(q')$ where $q' = 3^{2r+1}$ then $D(q) = q' \pm \sqrt{3q'} + 1$. If $D(q) = q' - \sqrt{3q'} + 1$ then ${q'}^3 = q^2$ and q is odd. But $q' - \sqrt{3q'} + 1$ cannot be equal to $\frac{{q'}^3+1}{2}$. If $D(q) = q' + \sqrt{3q'} + 1$ then ${q'}^3 = q^4$ but ${q'}^3$ is not a square number and we have a contradiction.

Step 15. If $K/H \cong E_6(q')$ then ${q'}^9 = q^4$ and hence $q^{16} \mid |G|$, which is impossible.

Step 16. If $K/H \cong {}^2E_6(q')$ then ${q'}^9 = q^2$. But $D(q)$ cannot be equal to $({q'}^9 + 1)/(2, q' - 1)$, and we have a contradiction.

Step 17. If K/H is a sporadic simple group then $D(q)$ must be equal to 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71. There is a solution greater than 5 in the form of power of a prime number if $D(q)$ =41 and $q = 9$. By the table of sporadic simple groups, 41 is an odd order component of F_1. But $29 \mid |F_1|$ and $29 \nmid |C_2(9)|$ which is a contradiction.

The proof of the main theorem is now completed. □

Remark 3.1. It is a well known conjecture of J.G. Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We can give a positive answer to this conjecture for the groups under discussion by our characterization of these groups.

Corollary 3.2. *Let G be a finite group with $Z(G) = 1$, $M = C_2(q)$ where $q > 5$ and $N(G) = N(M)$, then $G \cong M$.*

Proof: By Lemmas 2.7 and 2.8, if G and M are two finite groups satisfying the conditions of Corollary 3.2, then $\mathrm{OC}(G) = \mathrm{OC}(M)$. So the main theorem implies this corollary. □

Remark 3.3. Wujie Shi and Bi Jianxing in [17] put forward the following conjecture:

Conjecture. *Let G be a group, M a finite simple group, then $G \cong M$ if and only if*

(i) $|G| = |M|$, *and,*
(ii) $\pi_e(G) = \pi_e(M)$, *where $\pi_e(G)$ denotes the set of orders of elements in G.*

This conjecture is valid for sporadic simple groups ([14]), *groups of alternating type* ([18]), *and some simple groups of Lie type* ([15], [16], [17]). *As a consequence of the main theorem, we prove the validity of this conjecture for the groups under discussion.*

Corollary 3.4. *Let G be a finite group and $M = C_2(q)$ where $q > 5$. If $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.*

Proof: By assumption we must have $\mathrm{OC}(G) = \mathrm{OC}(M)$. Thus the corollary follows by the main theorem. □

Acknowledgment. The first author would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM) Tehran, Iran, for the financial support.

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(*Received* August 1, 2001, *revised* November 8, 2001)