Weighted Miranda–Talenti inequality and applications to equations with discontinuous coefficients

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Abstract. Let Ω be an open bounded set in $\mathbb{R}^n$ $(n \geq 2)$, with C^2 boundary, and $N^{p,\lambda}(\Omega)$ $(1 < p < +\infty,\ 0 \leq \lambda < n)$ be a weighted Morrey space.

In this note we prove a weighted version of the Miranda-Talenti inequality and we exploit it to show that, under a suitable condition of Cordes type, the Dirichlet problem:

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \end{cases}$$

has a unique strong solution in the functional space

$$\left\{ u \in W^{2,p} \cap W_o^{1,p}(\Omega) : \frac{\partial^2 u}{\partial x_i \partial x_j} \in N^{p,\lambda}(\Omega), \quad i,j = 1, 2, \dots, n \right\}.$$

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1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^n$ $(n \geq 2)$, with C^2 boundary, and $N^{p,\lambda}(\Omega)$ $(1 < p < +\infty,\ 0 \leq \lambda < n)$ be the weighted Morrey space formed by the functions $u : \Omega \to \mathbb{R}$ for which

$$\|u\|_{N^{p,\lambda}(\Omega)} = \sup_{x_o \in \Omega} \left\{ \int_\Omega |x - x_o|^{-\lambda} |u(x)|^p \, dx \right\}^{1/p} < +\infty.$$

Also, let $W^{k,p,(\lambda)}(\Omega)$ be the linear space of functions $u \in W^{k,p}(\Omega)$ such that $D^\alpha u \in N^{p,\lambda}(\Omega)$ for $|\alpha| = k$.

In this note we will prove, at first, a weighted version of the Miranda-Talenti inequality (see [35]), namely we will demonstrate the following

Theorem. *Let $1 < p < +\infty$ and $0 \leq \lambda < n$. Then there exists a constant $C_{MT} = C_{MT}(n, p, \lambda, \partial\Omega) > 0$ such that, for any $u \in W^{2,p} \cap W_o^{1,p}(\Omega)$ for which $\Delta u \in N^{p,\lambda}(\Omega)$, we have*

$$\|u\|_{W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)} \leq C_{MT} \, \|\Delta u\|_{N^{p,\lambda}(\Omega)} \, .$$

Next, we exploit the previous result to show that, under a suitable condition of Cordes type, the Dirichlet problem:

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique strong solution in the functional space $W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)$.

2. Notations, assumptions, auxiliary results

In $\mathbb{R}^n$ $(n \geq 2)$, with a generic point $x = (x_1, x_2, \dots, x_n)$, we shall denote by Ω an open nonempty bounded set with C^2-boundary $\partial\Omega$ ([1]).

For $\rho > 0$ we define

$$B(x_0, \rho) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$$
$$\Omega(x_0, \rho) = \Omega \cap B(x_0, \rho).$$

If $u \in L^1(A)$, A being an open nonempty bounded set of $\mathbb{R}^n$, then we will set

$$u_A = \frac{1}{|A|}\int_A u(x)\,dx \ (^2),$$

if moreover $u \in L^1(\mathbb{R}^n)$ we recall the definition of the Hardy-Littlewood maximal function

$$Mu(x) = \sup_{\rho>0}\ u_{B(x,\rho)}.$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multiindex we set

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\dots\partial x_n^{\alpha_n}} \quad (^3).$$

Moreover let $p \in]1, +\infty[$ and $\lambda \in [0, n[$ ([1]).

Definition 2.1. Let $k \in \mathbb{N}$. By $W^{k,p}(\Omega)$ (respectively $W_o^{k,p}(\Omega)$) we denote the closure of $C^\infty(\Omega)$ (respectively $C_o^\infty(\Omega)$) with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|(\sum_{|\alpha|=k} |D^\alpha u|^2)^{1/2}\|_{L^p(\Omega)}.$$

[1] This hypothesis will always be implicitly used.
[2] $|A|$ is the n-dimensional Lebesgue measure of A.
[3] For the sake of simplicity we will denote the gradient $(D^\alpha u)_{|\alpha|=1}$ by Du and the Hessian matrix $(D^\alpha u)_{|\alpha|=2}$ by $H(u)$.

Definition 2.2 (Morrey's space). By $L^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that

$$\|u\|_{L^{p,\lambda}(\Omega)} = \sup_{x_0 \in \Omega, \rho > 0} \left\{ \rho^{-\lambda} \int_{\Omega(x_o,\rho)} |u(x)|^p \, dx \right\}^{1/p} < +\infty. \tag{1}$$

$L^{p,\lambda}(\Omega)$ equipped with the norm (1) is a Banach space.

Definition 2.3 (Weighted Morrey's space [27]). By $N^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$ such that

$$\|u\|_{N^{p,\lambda}(\Omega)} = \left\{ \sup_{x_0 \in \Omega} \int_{\Omega} |x - x_0|^{-\lambda} |u|^p \, dx \right\}^{1/p} < +\infty. \tag{2}$$

Remark 2.1. Fixed $x_o \in \mathbb{R}^n$, set

$$\nu_{x_o}(x) = |x - x_o|^{-\lambda}.$$

The weight $\nu_{x_o}(x)$ satisfies the following properties:

(i) $\frac{1}{\nu_{x_o}(x)} \in L^\infty_{\text{loc}}(\mathbb{R}^n)$,

(ii) $\nu_{x_o}(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$,

(iii) $\nu_{x_o}(x)$ is an A_p (or Muckenhoupt) weight i.e. $\nu_{x_o}(x)$ satisfies the condition

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{x_o}(x) \, dx \right) \left(\frac{1}{|Q|} \int_Q \nu_{x_o}(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} < +\infty$$

where the supremum is taken over all cubes Q (see [37, Corollary 4.4, p. 236 and Proposition 3.2, p. 229]).

Properties (i), (ii) imply respectively that $N^{p,\lambda}(\Omega)$ equipped with the norm (2) is a Banach space and that $C_o^\infty(\Omega)$ is dense in $N^{p,\lambda}(\Omega)$.

Proposition 2.1 ([22]). *It holds*

$$N^{p,\lambda}(\Omega) \subset L^{p,\lambda}(\Omega).$$

Proposition 2.2 ([22]). *If*

$$\frac{\lambda_2 - n}{p} \leq \frac{\lambda_1 - n}{q},$$

with $1 \leq p \leq q < \infty$, *then*

$$N^{q,\lambda_1}(\Omega) \subset N^{p,\lambda_2}(\Omega).$$

Definition 2.4. By $W^{k,p,(\lambda)}(\Omega)$ we denote the linear space of functions $u \in W^{k,p}(\Omega)$ such that $D^\alpha u \in N^{p,\lambda}(\Omega)$ for $|\alpha| = k$.

$W^{k,p,(\lambda)}(\Omega)$ equipped with the norm

$$\|u\|_{W^{k,p,(\lambda)}(\Omega)} = \|u\|_{L^p(\Omega)} + \|(\sum_{|\alpha|=k} |D^\alpha u|^2)^{1/2}\|_{N^{p,\lambda}(\Omega)} \tag{3}$$

is a Banach space.

Proposition 2.3 (Weighted Poincaré's inequality)**.** *Let $u \in W^{1,p,(\lambda)}(\Omega)$. Then there exists a constant $C = C(n, p, \lambda, |\Omega|) > 0$ such that*

$$\|u - u_\Omega\|_{N^{p,\lambda}(\Omega)} \le C\|D\,u\|_{N^{p,\lambda}(\Omega)}.$$

Proof: From Lemma 3.4 in [26] (see also [15, p. 162]) we deduce

$$|u(x) - u_\Omega| \le C(n) \int_\Omega |x - y|^{1-n} |D\,u(y)|\,dy =: C(n)\,I(x) \quad \text{a.a. } x \in \Omega. \tag{4}$$

After extending Du to the whole $\mathbb{R}^n$ by assuming $Du = 0$ in $\mathbb{R}^n \setminus \Omega$ we get

$$\begin{aligned} I(x) &\le \int_{|x-y|\le d_\Omega} |x - y|^{1-n} |D\,u(y)|\,dy \\ &\le \sum_{j=0}^{+\infty} \int_{d_\Omega 2^{-j-1} \le |x-y| < d_\Omega 2^{-j}} |x - y|^{1-n} |D\,u(y)|\,dy \\ &\le \sum_{j=0}^{+\infty} (d_\Omega 2^{-j-1})^{1-n} \int_{|x-y|<d_\Omega 2^{-j}} |D\,u(y)|\,dy \\ &\le C(n, d_\Omega) M|Du(x)| \sum_{j=0}^{+\infty} 2^{-j}. \end{aligned} \tag{5}$$

The thesis now follows from the weighted norm estimate for the maximal function (see [24] or Theorem 1 from [9]) and Remark 2.1(iii); indeed we have

$$\|u - u_\Omega\|_{N^{p,\lambda}(\Omega)} \le C\,\|I\|_{N^{p,\lambda}(\mathbb{R}^n)} \le C\,\|Du\|_{N^{p,\lambda}(\mathbb{R}^n)} = C\,\|Du\|_{N^{p,\lambda}(\Omega)}\,.$$

□

Proposition 2.4. *Let $u \in W^{2,p,(\lambda)}(\Omega)$. Then $D^\alpha u \in N^{p,\lambda}(\Omega)$ for $|\alpha| \le 1$.*

PROOF: Poincaré's inequality gives

$$\|Du - (Du)_\Omega\|_{N^{p,\lambda}(\Omega)} \le C(n,p,\lambda,|\Omega|)\|H(u)\|_{N^{p,\lambda}(\Omega)}. \tag{6}$$

On the other hand by Hölder's inequality and (6) we infer

$$\begin{aligned}\|Du\|_{N^{p,\lambda}(\Omega)} &\le \left[\|Du - (Du)_\Omega\|_{N^{p,\lambda}(\Omega)} + |(Du)_\Omega| \sup_{x_o\in\Omega}\left(\int_\Omega |x-x_o|^{-\lambda}\,dx\right)^{1/p}\right] \\ &\le C(n,p,\lambda,|\Omega|)\left(\|H(u)\|_{N^{p,\lambda}(\Omega)} + \|Du\|_{L^p(\Omega)}\right) < +\infty \qquad (7)\end{aligned}$$

whence, using again Poincaré's inequality,

$$\begin{aligned}\|u\|_{N^{p,\lambda}(\Omega)} &\le \left[\|u - u_\Omega\|_{N^{p,\lambda}(\Omega)} + |u_\Omega| \sup_{x_o\in\Omega}\left(\int_\Omega |x-x_o|^{-\lambda}\,dx\right)^{1/p}\right] \\ &\le C(n,p,\lambda,|\Omega|)\left(\|Du\|_{N^{p,\lambda}(\Omega)} + \|u\|_{L^p(\Omega)}\right) \\ &\le C(n,p,\lambda,|\Omega|)\left(\|H(u)\|_{N^{p,\lambda}(\Omega)} + \|Du\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right).\end{aligned} \tag{8}$$

□

A consequence of the above proposition is the following interpolation inequality.

Proposition 2.5. *Let $u \in W^{2,p,(\lambda)}(\Omega)$. Then for any $\varepsilon > 0$ one has*

$$\|Du\|_{N^{p,\lambda}(\Omega)} \le C(\varepsilon)\,\|u\|_{N^{p,\lambda}(\Omega)} + \varepsilon\,\|H(u)\|_{N^{p,\lambda}(\Omega)} \tag{9}$$

where $C(\varepsilon) > 0$ is independent of u.

PROOF: It is enough to establish (9) for $u \in C^2(\Omega)$.

For $y \in \Omega$ fixed, let us introduce radial and angular coordinates $\rho = |x-y|$, $\omega = \frac{x-y}{\rho}$.

Then we have for $x \in \Omega$,

$$Du(y) = Du(x) - \int_0^\rho D_r^2 u(y + r\omega)\,dr$$

whence

$$|Du(y)|^p \le 2^{p-1}\left[|Du|^p\,dx + \left|\int_0^\rho D_r^2 u(y+r\omega)\,dr\right|^p\right].$$

Fixing $\delta_o > 0$ and integrating with respect to x over $\Omega(y,\delta_o)$ we obtain

$$
\begin{aligned}
(10)\qquad & |Du(y)|^p \le 2^{p-1}C(n)\delta_o^{-n}\Big[\int_\Omega |Du|^p\,dx \\
&\quad + \int_{\Omega(y,\delta_o)}\Big|\int_0^\rho D_r^2 u(y+r\omega)\,dr\Big|^p\,dx\Big] \\
&= 2^{p-1}C(n)\delta_o^{-n}\Big[\int_\Omega |Du|^p\,dx \\
&\quad + \int_0^{\delta_o}\int_{|\omega|=1}\Big|\int_0^\rho D_r^2u(y+r\omega)\,dr\Big|^p \rho^{n-1}\,d\omega\,d\rho\Big] \\
&= 2^{p-1}C(n)\delta_o^{-n}\Big[\int_\Omega |Du|^p\,dx \\
&\quad + \int_0^{\delta_o}\int_{|\omega|=1}\rho^\lambda\rho^{n-1-\lambda}\Big|\int_0^\rho D_r^2u(y+r\omega)\,dr\Big|^p\,d\omega\,d\rho\Big] \\
&\le 2^{p-1}C(n)\delta_o^{-n}\Big[\int_\Omega |Du|^p\,dx \\
&\quad + \delta_o^\lambda\int_0^{\delta_o}\int_{|\omega|=1}\rho^{p-1}\rho^{n-1-\lambda}\int_0^\rho |D_r^2u(y+r\omega)|^p\,dr\,d\omega\,d\rho\Big] \\
&\le 2^{p-1}C(n)\delta_o^{-n}\Big[\int_\Omega |Du|^p\,dx \\
&\quad + \delta_o^{\lambda+p}\int_0^{\delta_o}\int_{|\omega|=1}\rho^{n-1-\lambda}|D_r^2u(y+\rho\omega)|^p\,d\omega\,d\rho\Big] \\
&= 2^{p-1}C(n)\delta_o^{-n}\Big[\int_\Omega |Du|^p\,dx + \delta_o^{\lambda+p}\int_\Omega |x-y|^{-\lambda}|H(u)|^p\,dx\Big] \\
&\le 2^{p-1}C(n)\delta_o^{-n}\Big[\|Du\|^p_{L^p(\Omega)} + \delta_o^{\lambda+p}\,\|H(u)\|^p_{N^{p,\lambda}(\Omega)}\Big].
\end{aligned}
$$

Multiplying both sides of (10) by $|y-x_o|^{-\lambda}$ and integrating with respect to y over $\Omega(x_o,\delta_o)$, for fixed $x_o\in\Omega$, we get

$$
\int_{\Omega(x_o,\delta_o)} |Du(y)|^p|y-x_o|^{-\lambda}\,dy \le C(n,p,\lambda)\delta_o^{-\lambda}\Big[\|Du\|^p_{L^p(\Omega)} + \delta_o^{\lambda+p}\,\|H(u)\|^p_{N^{p,\lambda}(\Omega)}\Big]
$$

whence, using Theorem 7.28 from [15],

$$
\begin{aligned}
&\sup_{\delta\le\delta_o}\int_{\Omega(x_o,\delta)}|Du(y)|^p|y-x_o|^{-\lambda}\,dy\\
(11)\quad &\le C(n,p,\lambda,|\Omega|)\left[\delta_o^{-p-2\lambda}\|u\|^p_{L^p(\Omega)}+\delta_o^p\|H(u)\|^p_{L^p(\Omega)}+\delta_o^p\,\|H(u)\|^p_{N^{p,\lambda}(\Omega)}\right]\\
&\le C(n,p,\lambda,|\Omega|)\left[\delta_o^{-p-2\lambda}\|u\|^p_{N^{p,\lambda}(\Omega)}+\delta_o^p\,\|H(u)\|^p_{N^{p,\lambda}(\Omega)}\right].
\end{aligned}
$$

The thesis now follows from the equivalence of norms as in [20, p. 25]. □

3. Weighted Miranda-Talenti inequality

Before proving a weighted version of the Miranda-Talenti inequality we will premise some useful propositions.

Proposition 3.1. *Let $u\in W_o^{2,p}(\Omega)$ such that $\Delta u\in N^{p,\lambda}(\Omega)$. Then $H(u)\in N^{p,\lambda}(\Omega)$ and there exists a constant $C=C(n,p,\lambda)>0$ such that*

$$
\|H(u)\|_{N^{p,\lambda}(\Omega)}\le C\,\|\Delta u\|_{N^{p,\lambda}(\Omega)}\,. \tag{12}
$$

Proof: We will proceed as in the proof of Proposition 3, p. 57 from [32].

Denoted by $R_j(v)$, $j=1,\dots,n$, the j-th Riesz transform of a function $v\in C_o^2(\mathbb{R}^n)$ (see [32, pp. 57 and 68]). By a density argument and Theorem 3, p. 39 from [32] we get the identity

$$
H(u)=-R_i(R_j(\Delta u)),\quad \forall\,u\in W_o^{2,p}(\Omega). \tag{13}
$$

If we now extend Δu to the whole $\mathbb{R}^n$ by setting $\Delta u=0$ in $\mathbb{R}^n\setminus\Omega$, the thesis is then an immediate consequence of (13), the properties of the kernel of the Riesz transform (see also [34, pp. 220 and 243]) and the weighted L^p inequality from [9, p. 244] (see also [25] and [31]).

Namely we have

$$
\|H(u)\|_{N^{p,\lambda}(\Omega)}\le C\,\|\Delta u\|_{N^{p,\lambda}(\mathbb{R}^n)}=C\,\|\Delta u\|_{N^{p,\lambda}(\Omega)}\,.
$$

□

The above proposition allows us to prove the following interior estimate.

Theorem 3.1. *Let $u\in W^{2,p}(\Omega)$ such that $\Delta u\in N^{p,\lambda}(\Omega)$. Then, for any domains $\Omega'\subset\subset\Omega''\subset\subset\Omega$, $H(u)\in N^{p,\lambda}(\Omega')$ and there exists a constant $C=C(n,p,\lambda,\operatorname{dist}(\Omega',\partial\Omega''))>0$ such that*

$$
\|H(u)\|_{N^{p,\lambda}(\Omega')}\le C\left(\|u\|_{N^{p,\lambda}(\Omega'')}+\|\Delta u\|_{N^{p,\lambda}(\Omega)}\right). \tag{14}
$$

Proof: Suppose $0 < \lambda < n$ (if $\lambda = 0$ see e.g. Theorem 9.11 from [15]).

Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, $0 < R \leq \mathrm{dist}(\Omega', \partial\Omega'')$; set $B_R \equiv B(y_o, R)$, $y_o \in \overline{\Omega'}$ and, for $\sigma \in]0,1[$, let us introduce a cutoff function $\eta \in C_o^2(B_R)$ satisfying

$$\begin{aligned}
&0 \leq \eta \leq 1, \ \ \forall x \in B_R \\
&\eta = 1 \ \text{ in } B_{\sigma R} \\
&\eta = 0 \ \text{ for } |x - y_o| \geq \sigma' R, \ \ \sigma' = \frac{1+\sigma}{2} \\
&|D\eta| \leq \frac{4}{(1-\sigma)R}, \ \ |H(\eta)| \leq \frac{16}{(1-\sigma)^2 R^2} \,.
\end{aligned}$$

Then, if $v = \eta u$ we also have $v \in W_o^{2,p}(B_R)$. We want to prove that $\Delta v \in N^{p,\lambda}(B_R)$.

As a matter of fact, being $u \in W^{2,p}(\Omega)$, one obtains $u, Du \in N^{p,\mu}(\Omega)$ for some $\mu > 0$ (4). Thus, since $\Delta u \in N^{p,\lambda}(\Omega)$ it follows $\Delta v \in N^{p,\mu}(B_R)$ for some $\mu \in]0, \lambda]$.

Let us suppose $\mu \in]0, \lambda[$.

In this case the previous observations together with Proposition 3.1 imply $H(v) \in N^{p,\mu}(B_R)$ and thus $H(u) \in N^{p,\mu}(B_{\sigma R})$, $\mu \in]0, \lambda[$.

Starting now from the fact that $u \in W^{2,p,(\mu)}(B_{\sigma R})$ and repeating the above argument we get $u, Du \in N^{p,\mu_1}(B_{\sigma R})$, for some $\mu_1 \in]\mu, \lambda]$ (4), and $\Delta v \in N^{p,\mu_1}(B_R)$.

If still $\mu_1 \neq \lambda$ we iterate a finite number of times the previous procedure up obtaining $\Delta v \in N^{p,\lambda}(B_R)$.

Thus another application of Proposition 3.1 gives

$$H(v) \in N^{p,\lambda}(B_R) \Rightarrow H(u) \in N^{p,\lambda}(B_{\sigma R})$$

and

$$\begin{aligned}
&\|H(u)\|_{N^{p,\lambda}(B_{\sigma R})} = \|H(v)\|_{N^{p,\lambda}(B_R)} \leq C\, \|\Delta v\|_{N^{p,\lambda}(B_R)} \\
&\leq C \left[\frac{1}{(1-\sigma)^2 R^2} \|u\|_{N^{p,\lambda}(B_R)} + \frac{1}{(1-\sigma)R} \|Du\|_{N^{p,\lambda}(B_{\sigma' R})} + \|\Delta u\|_{N^{p,\lambda}(B_R)} \right].
\end{aligned} \tag{15}$$

Proceeding now as in the proof of Theorem 9.11 from [15] and taking into account Proposition 2.5, we then obtain, for $\sigma = 1/2$,

$$\|H(u)\|_{N^{p,\lambda}(B_{R/2})} \leq \frac{C}{R^2} \left[\|u\|_{N^{p,\lambda}(B_R)} + R^2\, \|\Delta u\|_{N^{p,\lambda}(B_R)} \right].$$

---

(4) Using Sobolev and Hölder inequalities and Proposition 2.2.

The required estimate follows once more from the above one by covering Ω' with a finite number of balls of radius $R/2$. $\square$

In order to extend Theorem 3.1 to the boundary $\partial\Omega$ we first consider the case of a flat boundary portion.

If $y_o \equiv (y_{o1}, \dots, y_{o\,n-1}, 0)$, we set

$$\begin{aligned} B_R^+ &= (B(y_o, R))^+ = B(y_o, R) \cap \mathbb{R}^n_+ \\ &= B(y_o, R) \cap \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}. \end{aligned}$$

Proposition 3.2. *Let $u \in W^{2,p}(B_1^+)$, $u = 0$ on $B_1 \cap \partial\mathbb{R}^n_+$, such that $\Delta u \in N^{p,\lambda}(B_1^+)$. Then, for every $R \in]0,1[$, $H(u) \in N^{p,\lambda}(B_R^+)$ and there exists a constant $C = C(n, p, \lambda) > 0$ such that*

$$\|H(u)\|_{N^{p,\lambda}(B_R^+)} \le C\left[\|u\|_{N^{p,\lambda}(B_1^+)} + \|\Delta\, u\|_{N^{p,\lambda}(B_1^+)}\right]. \tag{16}$$

Proof: We extend u and the weight $\nu_{x_o}(x) = |x - x_o|^{-\lambda}$, $x_o \in B_1^+$, to all of B_1 (see [2, Lemma IX.2]) by setting

$$\tilde{\nu}_{x_o}(x', x_n) = \begin{cases} \nu_{x_o}(x', x_n) & \text{for } (x', x_n) \in B_1^+ \\ \nu_{x_o}(x', -x_n) & \text{for } (x', -x_n) \in B_1 \setminus B_1^+, \end{cases}$$

$$\tilde{u}(x', x_n) = \begin{cases} u(x', x_n) & \text{for } (x', x_n) \in B_1^+ \\ 0 & \text{for } (x', x_n) \in B_1 \cap \partial\mathbb{R}^n_+ \\ u(x', -x_n) & \text{for } (x', -x_n) \in B_1^+. \end{cases}$$

It can be readily checked that the function $\tilde{u} \in W^{2,p}(B_1)$ and moreover

$$\|\Delta\, \tilde{u}\|_{N^{p,\lambda}(B_1)} \le C\|\Delta\, u\|_{N^{p,\lambda}(B_1^+)} < +\infty.$$

Arguing as in the previous theorem, for $R \in]0, 1[$, let us introduce a cutoff function $\eta \in C_o^2(B_1)$ satisfying

$$\begin{aligned} &0 \le \eta \le 1, \quad \forall\, x \in B_1 \\ &\eta = 1 \ \text{ in } B_R \\ &\eta = 0 \ \text{ for } |x - y_o| \ge R', \ \ R' = \frac{1+R}{2} \\ &|D\eta| \le \frac{4}{(1-R)}, \quad |H(\eta)| \le \frac{16}{(1-R)^2} \end{aligned}$$

and consider the function $v = \eta\tilde{u} \in W_o^{2,p}(B_1)$.

Then, since $\Delta v \in N^{p,\lambda}(B_1)$, we have $H(\tilde{u}) \in N^{p,\lambda}(B_R)$ and

$$\|H(\tilde{u})\|_{N^{p,\lambda}(B_R)} \leq C \left[\|\tilde{u}\|_{N^{p,\lambda}(B_1)} + \|\Delta \tilde{u}\|_{N^{p,\lambda}(B_1)} \right].$$

The estimate (16) follows now in the standard way:

$$\begin{aligned} \|H(u)\|_{N^{p,\lambda}(B_R^+)} &\leq \|H(\tilde{u})\|_{N^{p,\lambda}(B_R)} \\ &\leq C \left[\|\tilde{u}\|_{N^{p,\lambda}(B_1)} + \|\Delta \tilde{u}\|_{N^{p,\lambda}(B_1)} \right] \\ &\leq C \left[\|u\|_{N^{p,\lambda}(B_1^+)} + \|\Delta u\|_{N^{p,\lambda}(B_1^+)} \right]. \end{aligned}$$

□

With the aid of the previous propositions we derive a global estimate.

Proposition 3.3. *Let $u \in W^{2,p} \cap W_o^{1,p}(\Omega)$ such that $\Delta u \in N^{p,\lambda}(\Omega)$. Then $H(u) \in N^{p,\lambda}(\Omega)$ and there exists a constant $C = C(n,p,\lambda,\partial\Omega) > 0$ such that*

$$\|H(u)\|_{N^{p,\lambda}(\Omega)} \leq C \left(\|u\|_{N^{p,\lambda}(\Omega)} + \|\Delta u\|_{N^{p,\lambda}(\Omega)} \right). \tag{17}$$

Proof: Since $\partial\Omega \in C^2$, for each point $y_o \in \partial\Omega$ there is a neighborhood $\mathcal{N} = \mathcal{N}_{y_o}$ and a corresponding diffeomorphism $\psi = \psi_{y_o}$ from $\mathcal{N}$ onto the unit ball $B = B(0,1)$ in $\mathbb{R}^n$ such that

(i) $\psi \in C^2(\mathcal{N})$, $\psi^{-1} \in C^2(B)$,
(ii) $\psi(\mathcal{N} \cap \Omega) = B^+$,
(iii) $\psi(\mathcal{N} \cap \partial\Omega) = B \cap \partial\mathbb{R}^n_+$.

Writing

$$\tilde{u}(x) = u(\psi(x)), \quad x \in \mathcal{N}$$

we have $\tilde{u} \in W^{2,p}(B^+)$, $\Delta \tilde{u} \in N^{p,\lambda}(B^+)$ and $\tilde{u} = 0$ on $B \cap \partial\mathbb{R}^n_+$.

By Proposition 3.2 we thus obtain the estimate

$$\|H(\tilde{u})\|_{N^{p,\lambda}(B_R^+)} \leq C \left[\|\tilde{u}\|_{N^{p,\lambda}(B_1)} + \|\Delta\tilde{u}\|_{N^{p,\lambda}(B_1)} \right], \quad R \in]0,1[\ .$$

Taking $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}_{y_o} = \psi^{-1}(B_{R/2})$ and returning back to our original coordinates, we obtain

$$\|H(u)\|_{N^{p,\lambda}(\tilde{\mathcal{N}})} \leq C \left[\|u\|_{N^{p,\lambda}(\mathcal{N})} + \|\Delta u\|_{N^{p,\lambda}(\mathcal{N})} \right].$$

Finally, by covering $\partial\Omega$ with a finite number of such neighborhoods $\tilde{\mathcal{N}}$ and using also the interior estimate (14) we obtain the thesis. □

The following inequality of Miranda-Talenti type holds (see Talenti [35], Grisvard [18, Section 2.3] and also Gilbarg, Trudinger [15, Chapter 9]).

Theorem 3.2. *There exists a constant* $C_{MT} = C_{MT}(n, p, \lambda, \partial\Omega) > 0$ *such that, for any* $u \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)$ (5), *we have*

$$\|u\|_{W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)} \leq C_{MT} \|\Delta u\|_{N^{p,\lambda}(\Omega)} . \tag{18}$$

Proof: Since $N^{p,\lambda}(\Omega) \subset L^p(\Omega)$ (6) the Laplace operator

$$\Delta : W^{2,p} \cap W_o^{1,p}(\Omega) \to N^{p,\lambda}(\Omega)$$

is a bijection. Moreover, by virtue of Proposition 3.3

$$\Delta : W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \to N^{p,\lambda}(\Omega),$$

is also a bijection.

On the other hand, being

$$\|\Delta u\|_{N^{p,\lambda}(\Omega)} \leq \|u\|_{W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)} ,$$

it follows that

$$\Delta : W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \to N^{p,\lambda}(\Omega)$$

is continuous and thus, by the “open mapping” Theorem, also Δ^{-1} is continuous, i.e.

$$\|\Delta^{-1}(\Delta u)\|_{W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)} \leq C_{MT} \|\Delta u\|_{N^{p,\lambda}(\Omega)} .$$

□

4. Applications to elliptic equations

Let us now consider the question of existence and uniqueness in $W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)$ of the solution to the Dirichlet problem:

$$\begin{cases} E(u) \equiv \displaystyle\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

The structural hypotheses on the operator E (see Cordes [10], [11], [12], Talenti [35], Giusti [16], Campanato, Cannarsa [8], Campanato [6] and also Guglielmino [19], Nicolosi [28]) are:

5 Due to inequality (11) we can equip $W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)$ with the norm (3).

6 See Proposition 2.1.

(a) $a_{ij}(x) \in L^\infty(\Omega)$, $a_{ij}(x) = a_{ji}(x) \quad i,j = 1,2,\dots,n$;
(b) (Strong ellipticity condition) there exists a constant $\nu > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \geq \nu|\xi|^2 \quad \text{a.a. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n; \tag{20}$$

(c) (Cordes-type condition) there exists a constant $K \in [0,1[$ such that

$$\frac{\left(\sum_{i=1}^{n} a_{ii}(x)\right)^2}{\sum_{i,j=1}^{n}(a_{ij}(x))^2} \geq n - \frac{K^2}{C_{MT}^2} \quad \text{a.a. } x \in \Omega. \tag{21}$$

Existence-uniqueness of the solution in the space $W^{2,2} \cap W_o^{1,2}(\Omega)$ and regularity of its second derivatives in the classical Morrey space $L^{2,\lambda}(\Omega)$ for such a class of elliptic equations have been studied respectively by Talenti [35] and by Talenti [36], Giusti [16], [17]; while in the case of a generic $p \in]1,+\infty[$, as far as the author is aware, until now only existence-uniqueness of the solution in the space $W^{2,p} \cap W_o^{1,p}(\Omega)$ have been studied by Pucci [29] and Campanato [4], [5], [6] (see also Pucci, Talenti [30]).

It is our aim to prove global regularity in $N^{p,\lambda}(\Omega)$ of the second derivatives of the solution to the problem (19).

Before proving the above stated result we will premise some remarks.

Remark 4.1. Hypothesis (20) implies that

$$\sum_{i=1}^{n} a_{ii}(x) \geq n\,\nu. \tag{22}$$

Moreover, by Cauchy-Schwartz inequality we infer

$$\sum_{i=1}^{n} a_{ii}(x) = \sum_{i,j=1}^{n} a_{ij}(x)\,\delta_{ij} \leq \sqrt{n}\Big(\sum_{i,j=1}^{n}(a_{ij}(x))^2\Big)^{1/2}. \tag{23}$$

The above two inequalities yield

$$\sum_{i,j=1}^{n}(a_{ij}(x))^2 \geq n\,\nu^2. \tag{24}$$

From (a), (22), (23) and (24) we deduce that the function

$$a(x) = \frac{\sum_{i=1}^{n} a_{ii}(x)}{\sum_{i,j=1}^{n}(a_{ij}(x))^2} \tag{25}$$

is measurable, strictly positive and bounded a.e. in Ω ([7]) (see also Giusti [16, p. 368] and Campanato, Cannarsa [8, pp. 1378–1379]).

Now, using the Lax-Milgram type Theorem of [21] (see also Campanato [5], [6], [7]) we prove the following theorem:

Theorem 4.1. *Let* $f \in N^{p,\lambda}(\Omega)$ *and let conditions* (a),(b),(c) *be satisfied. Then there exists a unique solution* u *of the problem*

$$\left\{ \begin{array}{l} u \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \\ \sum_{i,j=1}^n a_{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x). \end{array} \right. \tag{26}$$

Moreover we have the estimate

$$\|u\|_{W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)} \leq \frac{C_{MT}}{\nu(1-K)} \|f\|_{N^{p,\lambda}(\Omega)} . \tag{27}$$

Proof: Fixed $f \in N^{p,\lambda}(\Omega)$, let us observe that, by virtue of Remark 4.1, problem (26) is equivalent to problem

$$\left\{ \begin{array}{l} u \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \\ A(u) \equiv \sum_{i,j=1}^n a(x)\, a_{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} = a(x)\, f(x). \end{array} \right. \tag{28}$$

We will prove that the operator A is “near” by the Laplace operator

$$\Delta : W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \to N^{p,\lambda}(\Omega).$$

[7] By (23) and (24) we get

$$a(x) = \frac{\sum_{i=1}^n a_{ii}(x)}{(\sum_{i,j=1}^n (a_{ij}(x))^2)^{1/2}} \frac{1}{(\sum_{i,j=1}^n (a_{ij}(x))^2)^{1/2}} \leq \frac{1}{\nu} .$$

In fact, for any $u \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)$, we have

(29)
$$
\begin{aligned}
&\Big\|\Delta u - \sum_{i,j=1}^{n} a(x)\, a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}\Big\|_{N^{p,\lambda}(\Omega)} \\
&= \Big\|\sum_{i,j=1}^{n} (\delta_{ij} - a(x)\, a_{ij}(x)) \frac{\partial^2 u}{\partial x_i \partial x_j}\Big\|_{N^{p,\lambda}(\Omega)} \\
&\le \Big\|\Big[\sum_{i,j=1}^{n} (\delta_{ij} - a(x)\, a_{ij}(x))^2\Big]^{1/2} \Big[\sum_{i,j=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2\Big]^{1/2}\Big\|_{N^{p,\lambda}(\Omega)} \\
&= \Big\|\Big[n - 2\sum_{i=1}^{n} a(x)\, a_{ii}(x) + \sum_{i,j=1}^{n} (a(x)\, a_{ij}(x))^2\Big]^{1/2} \Big[\sum_{i,j=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2\Big]^{1/2}\Big\|_{N^{p,\lambda}(\Omega)} \\
&= \Big\|\Big[n - \frac{\Big(\sum_{i=1}^{n} a_{ii}(x)\Big)^2}{\sum_{i,j=1}^{n} (a_{ij}(x))^2}\Big]^{1/2} \Big[\sum_{i,j=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2\Big]^{1/2}\Big\|_{N^{p,\lambda}(\Omega)} \\
&\le \frac{K}{C_{MT}} \|H(u)\|_{N^{p,\lambda}(\Omega)}
\end{aligned}
$$

where we have exploited Cauchy-Schwartz inequality, the definition of $a(x)$ and hypotheses (a), (21).

From (29) and (18) we deduce

$$
\|\Delta u - A(u)\|_{N^{p,\lambda}(\Omega)} \le K \,\|\Delta u\|_{N^{p,\lambda}(\Omega)} \, . \tag{30}
$$

Thus from the Theorem in [21] it follows that there exists a unique $u \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega)$ which satisfies equation (26).

To prove the required estimate for the solution u we will argue in the following way:

$$
\begin{aligned}
\|\Delta u\|_{N^{p,\lambda}(\Omega)} \le\ & \|\Delta u - A(u)\|_{N^{p,\lambda}(\Omega)} \\
& + \Big\|\sum_{i,j=1}^{n} a(x)\, a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}\Big\|_{N^{p,\lambda}(\Omega)} \\
\le\ & K \,\|\Delta u\|_{N^{p,\lambda}(\Omega)} + 1/\nu \ \|f\|_{N^{p,\lambda}(\Omega)}
\end{aligned} \tag{31}
$$

from which it follows

$$
\|\Delta u\|_{N^{p,\lambda}(\Omega)} \le \frac{1}{\nu\,(1-K)} \|f\|_{N^{p,\lambda}(\Omega)} \, . \tag{32}
$$

Combining together (18) and (32) we get (27). □

Corollary 4.1. *Let the hypotheses of Theorem* 4.1 *be satisfied.*

(i) *If* $1 < p \leq n$, $n - p < \lambda < n$, *then* $Du \in C^{0,\mu}(\bar{\Omega})$ *with* $\mu = 1 - \frac{n-\lambda}{p}$;

(ii) *if* $p > n$, $0 \leq \lambda < n$, *then* $Du \in C^{0,\mu}(\bar{\Omega})$ *with* $\mu = 1 - \frac{n}{p}$.

Remark 4.2. Given a function $\psi \in W^{2,p,(\lambda)}(\Omega)$, the result of Theorem 4.1 can be readily extended to the nonhomogeneous Dirichlet problem

$$\begin{cases} u \in W^{2,p,(\lambda)}(\Omega) \\ \sum_{i,j=1}^{n} a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} = f(x) \in N^{p,\lambda}(\Omega) \\ u - \psi \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \end{cases}$$

just observing that the previous problem is equivalent to the following one

$$\begin{cases} w \in W^{2,p,(\lambda)} \cap W_o^{1,p}(\Omega) \\ \sum_{i,j=1}^{n} a_{ij}(x)\dfrac{\partial^2 w}{\partial x_i \partial x_j} = f(x) - \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\dfrac{\partial^2 \psi}{\partial x_i \partial x_j}\,. \end{cases}$$

Remark 4.3. Let us consider the fully nonlinear second order elliptic operator of "quasi-basic" type

$$A(u) = a(x, H(u))$$

where

$$x \in \Omega, \;\; u : \Omega \to \mathbb{R}^N \quad (N \in \mathbb{N}),$$

$a(x,\xi)$ is a vector of $\mathbb{R}^N$, measurable in x and continuous in ξ such that $a(x,0) = 0$, elliptic in the sense of the definition (A_q) of Campanato [6], i.e. there exist three constants α, γ, δ, with $\gamma + \delta < 1$, such that $\forall\, x \in \Omega$ and $\forall\, \xi, \tau \in \mathbb{R}^{n^2 N}$

$$\left\| \sum_i \xi_{ii} - \frac{\alpha}{C(q)}[a(x,\xi+\tau) - a(x,\tau)] \right\|_{\mathbb{R}^N} \leq \frac{\gamma^{\frac{q+1}{q}}}{C(q)}\|\xi\| + \delta^{\frac{q+1}{q}} \left\| \sum_i \xi_{ii} \right\| \quad (^8)$$

With a few formal adjustments the above result can as well be extended to quasi-basic operators just substituting the constant $C(q)$ by the constant C_{MT} from Theorem 3.2.

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[8] $C(q)$ is the constant of the unweighted (i.e. $\lambda = 0$) Miranda-Talenti inequality.

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