A counterexample to the smoothness of the solution to an equation arising in fluid mechanics

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Abstract. We analyze the equation coming from the Eulerian-Lagrangian description of fluids. We discuss a couple of ways to extend this notion to viscous fluids. The main focus of this paper is to discuss the first way, due to Constantin. We show that this description can only work for short times, after which the "back to coordinates map" may have no smooth inverse. Then we briefly discuss a second way that uses Brownian motion. We use this to provide a plausibility argument for the global regularity for the Navier-Stokes equations.

Keywords: Navier-Stokes equations, Euler equations, regularity of systems of PDE's, Eulerian-Lagrangian description of viscous fluids

Classification: Primary 35Q35, 76D05; Secondary 55M25, 60H30

1. Introduction

Recently there has been interest in some new variables describing the solutions to the Navier-Stokes and Euler equations. These variables go under various names, for example, *the magnetization variables*, *impulse variables*, *velicity* or *Kuzmin-Oseledets variables*.

Let us start by considering the incompressible Euler equations in the entire three-dimensional space, that is,

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\times(0,T) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\mathbf{u}$ and p are given functions, $\mathbf{u} : \mathbb{R}^3 \times (0,T) \mapsto \mathbb{R}^3$ and $p : \mathbb{R}^3 \times (0,T) \mapsto \mathbb{R}$, $0 < T \le \infty$ (see Section 1 for further explanation).

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The question of global existence of even only weak solutions to system (1.1) is an open question and only the existence of either measure-valued solutions (see [3]) or dissipative solutions (see [7]) is known. Nevertheless, a common approach to try to prove the global existence of smooth solutions is to use local existence results, and thus reduce the problem to proving a priori estimates. So we will assume that we have a smooth solution to the equations.

In that case, we can rewrite the Euler equations as the following system of equations (see for example [1]):

$$\left.\begin{aligned} \frac{\partial \mathbf{m}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{m} + \mathbf{m}\cdot(\nabla\mathbf{u})^T = \mathbf{f} \\ \mathbf{u} = \mathbf{m} - \nabla\eta \\ \operatorname{div}\mathbf{u} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3\times(0,T) \tag{1.2}$$
$$\mathbf{m}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3 .$$

Here $\mathbf{m} : \mathbb{R}^3 \times (0,T) \mapsto \mathbb{R}^3$ is called the magnetization variable. This new formulation has several advantages to the usual one, in particular the solution can be written rather nicely in the following way. Suppose that the initial value for $\mathbf{m}$ may be written as

$$\mathbf{m}(\mathbf{x},0) = \sum_{i=1}^{R} \beta_i(\mathbf{x},0)\nabla\alpha_i(\mathbf{x},0), \tag{1.3}$$

and suppose that α and β satisfy the transport equations, that is

$$\frac{\partial \alpha_i}{\partial t} + \mathbf{u}\cdot\nabla\alpha_i = 0$$
$$\frac{\partial \beta_i}{\partial t} + \mathbf{u}\cdot\nabla\beta_i = \sum_{j=1}^{R} Q_{j,i} f_j$$

where $\mathbf{Q} = (Q_{i,j})$ is the matrix inverse of the matrix whose entries are $\frac{\partial\alpha_i}{\partial x_j}$. (There is some difficulty to suppose that this inverse exists unless $R = 3$ — see below. But generally this will not be a problem if $\mathbf{f} = 0$.) Then

$$\mathbf{m}(\mathbf{x},t) = \sum_{i=1}^{R} \beta_i(\mathbf{x},t)\nabla\alpha_i(\mathbf{x},t)$$

is the solution to system (1.2). That is to say, at least in the case that $\mathbf{f} = 0$, the magnetization variable may be thought of as a “1-form” acting naturally under a change of basis induced by the flow of the fluid.

The advantage of the magnetization variable is that it is local in that its support never gets larger, it is simply pushed around by the flow. It is only at the end, after one has calculated the final value of $\mathbf{m}$, that one needs to take the Leray projection to compute the velocity field $\mathbf{u}$.

Indeed one very explicit way to write $\mathbf{m}$ according to equation (1.3) is to set $\alpha_i(\mathbf{x}, 0)$ equal to the ith unit vector, and $\beta_i(\mathbf{x}, 0) = u_i(\mathbf{x}, 0)$, for $1 \le i \le R = 3$. In that case let us denote $A_i^E(\mathbf{x}, t) = \alpha_i(\mathbf{x}, t)$ and $v_i(\mathbf{x}, t) = \beta_i(\mathbf{x}, t)$. In that case we see that $\mathbf{A}^E(\mathbf{x}, t)$ is actually the back to coordinates map, that is, it denotes the initial position of the particle of fluid that is at $\mathbf{x}$ at time t (see for example [1]). Furthermore in the case that $\mathbf{f} = 0$, we see that $\mathbf{v}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{A}^E(\mathbf{x}))$. Furthermore, it is well known if $\mathbf{u}$ is smooth, that $\mathbf{A}^E(\cdot, t)$ is smoothly invertible, and that the determinant of the Jacobian of $\mathbf{A}^E$ is identically equal to 1 (because $\operatorname{div} \mathbf{u} = 0$). Hence the matrix $\mathbf{Q}$ exists. For definiteness, we write the explicit equation for $\mathbf{m}$:

$$m_i(\mathbf{x}, t) = \frac{\partial \mathbf{A}^E(\mathbf{x}, t)}{\partial x_i} \cdot \mathbf{v}^E(\mathbf{x}, t). \tag{1.4}$$

The desire, then, is to try to extend this notion to the Navier-Stokes equations

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0, T) \tag{1.5}$$
$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3 .$$

(Only local-in-time existence of smooth solutions to the Navier-Stokes equations is known — see for example [4]; globally in time, only existence of weak solutions is known, see [6].)

Again, these can be rewritten into the magnetic variables formulation as follows:

$$\left.\begin{aligned} \frac{\partial \mathbf{m}}{\partial t} - \nu \Delta \mathbf{m} + \mathbf{u} \cdot \nabla \mathbf{m} + \mathbf{m} \cdot (\nabla \mathbf{u})^T = \mathbf{f} \\ \mathbf{u} = \mathbf{m} - \nabla \eta \\ \operatorname{div} \mathbf{u} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^3 \times (0, T) \tag{1.6}$$
$$\mathbf{m}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3 .$$

The problem is to find the analogue of equation (1.4). The difficulty comes from the term $\nu \Delta \mathbf{m}$. There are two ways known to the authors — one is to use probabilistic techniques. Since this technique seems to be not as widely known as it should be, we will include a short (non-rigorous) description of this method at the end of the paper. We will also include a short plausibility argument for the global regularity for the Navier-Stokes equations.

Another approach was developed by Peter Constantin (see [2]). He used new quantities $\mathbf{A}^N$ and $\mathbf{v}^N$ obeying the following equations. Let us represent $\mathbf{u}$ in a form similar to (1.2)

$$u_i(\mathbf{x},t) = \frac{\partial \mathbf{A}^N(\mathbf{x},t)}{\partial x_i} \cdot \mathbf{v}^N(\mathbf{x},t) - \frac{\partial n(\mathbf{x},t)}{\partial x_i}\,,$$

where

$$\begin{aligned} \Gamma(\mathbf{A}^N) &= \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times (0,T) \\ \mathbf{A}^N(\mathbf{x},0) &= \mathbf{x} \qquad \text{in } \mathbb{R}^3 \\ \Gamma &= \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla - \nu\Delta\,, \end{aligned} \tag{1.7}$$

and $\mathbf{v}^N$ obeys a rather complicated equation

$$\Gamma(v_i^N) = 2\nu C_{m,k;i}\frac{\partial v_m}{\partial x_k} + Q_{j,i}f_j\,,$$

where $\mathbf{Q}$ is the inverse matrix to $\nabla\mathbf{A}^N$ and $\Gamma^m_{i,j} = -Q_{k,j}C_{m,k;i}$ denotes the Christoffel coefficients. In order for the equation for $\mathbf{v}$ to make sense, it is necessary for the map $\mathbf{A}^N$ to have a smooth inverse. An approach to proving such a result is to consider the system of PDE's

$$\begin{aligned} \Gamma(\mathbf{Q}) &= (\nabla\mathbf{u})\mathbf{Q} + 2\nu\mathbf{Q}\partial_k(\nabla\mathbf{A}^N)\partial_k\mathbf{Q} \qquad \text{in } \mathbb{R}^3\times(0,T) \\ \mathbf{Q}(\mathbf{x},0) &= \mathbf{I} \qquad \text{in } \mathbb{R}^3. \end{aligned} \tag{1.8}$$

If the above equations have smooth solutions, then it can easily be shown that $\mathbf{Q}$ is the inverse to $\nabla\mathbf{A}^N$. However, the problem is that while it is easy and standard to show that equation (1.8) has local smooth solutions, it is not clear that it has global solutions in any sense at all.

The purpose of this note is to show that indeed global smooth solutions do not exist. As Peter Constantin pointed out to us, this does not invalidate his method, but it does mean that to make his method work for a large time period that one has to break that interval into shorter pieces, and apply the method to each small interval.

The main result is summarized in the following theorem.

Theorem 1.1. *There exists* $\mathbf{u} \in C_0^\infty(\mathbb{R}^3 \times [0,\infty))$*, divergence free such that if* $\mathbf{A}^N$ *is a smooth solution to* (1.7) *then there exists* $t > 0$ *such that*

(a) $\mathbf{A}^N(\mathbf{0},t)$ *does not have a smooth inverse,*

(b) $\limsup_{\tau\to t^-} \|\mathbf{Q}\|_\infty(\tau) = \infty$,

where $\mathbf{Q}$ *is a solution to* (1.8) *corresponding to* $\mathbf{u}$ *and* $\mathbf{A}^N$.

2. Outline of the proof of Theorem 1.1

In this section we will give the plan for the proof of Theorem 1.1. The idea of the proof is really quite simple. We will in fact construct a family of divergence free, smooth solutions $\mathbf{u}_s$ to (1.8) parameterized by a number $s \in [0, 2\pi]$. We will use simple ideas from algebraic topology to show that there exists $s_0 \in [0, 2\pi]$ such that u_{s_0} provides an example to prove Theorem 1.1.

In fact all of the solutions we construct will be axisymmetric, indeed, when written in cylindrical coordinates, they have the form: $\mathbf{u}_s = (0, u_\theta(r, z, t), 0)$. We will prove that there exists $s_0 \in [0, 2\pi]$ and $t_0 > 0$ such that the Jacobian $\nabla \mathbf{A}^N_{s_0}(0, t_0)$ is non-invertible. To this end we have the following representation result.

Lemma 2.1. *For any $t > 0$, $\nabla \mathbf{A}^N_s(\mathbf{0}, t)$ can be uniquely written as*

$$\begin{pmatrix} a_s \cos b_s, & -a_s \sin b_s, & 0 \\ a_s \sin b_s, & a_s \cos b_s, & 0 \\ 0, & 0, & 1 \end{pmatrix}$$

for some $a_s(t) \in \mathbb{R}^+$ and $b_s(t) \in [0, 2\pi)$.

The proof of Theorem 1.1 will proceed as follows. For each $s \in [0, 2\pi]$, we will construct $\mathbf{u}_s$. The Theorem will be proved if we can show the existence of $s_0 \in [0, 2\pi]$ and $t_0 > 0$ such that $a_{s_0}(t_0) = 0$. We will assume the opposite, and give a proof by contradiction.

We will need some simple facts from algebraic topology. We refer the reader to [8] for more details. Let us consider the collection of continuous functions $[0, \infty] \to \mathbb{R}^2 - \{(0, 0)\}$ which map 0 and ∞ to $(1, 0)$. We will say two such functions f and g are *homotopic with base point* $(1, 0)$ (or simply *homotopic*) if there exists a jointly continuous function $F : [0, \infty] \times [0, 2\pi] \to \mathbb{R}^2 - \{(0, 0)\}$ such that $F(\cdot, 0) = f$, $F(\cdot, 2\pi) = g$ and $F(0, \cdot) = F(\infty, \cdot) = (1, 0)$. We will call the function F a homotopy. Clearly being homotopic is an equivalence relation.

It is well known that a constant map $f(t) = (1, 0)$, and a map with "winding number 1", for example, $g(t) = (\cos(2\pi t/(1 + t)), \sin(2\pi t/(1 + t)))$ are not homotopic. (Since $\mathbb{R}^2 - \{(0, 0)\}$ is homotopy equivalent to the unit circle, this is basically saying that the fundamental group of the unit circle is non-trivial.)

In order to provide our contradiction we will prove the following result.

Lemma 2.2. *If $\mathbf{u}_s$ is constructed as described in the next section, with the various parameters chosen appropriately, then the function*

$$F(t, s) = (a_s(t) \cos b_s(t), a_s(t) \sin b_s(t))$$

provides a homotopy between the function f and a function homotopic to g.

3. Properties of the operator Γ

We will not prove the smoothness of solution to (1.7); it can be done in a very standard way, using the estimates to parabolic equations given for example in [5]. Let us only summarize the main result here. This will show that the function F described in Lemma 2.2 is continuous on any compact subset of $[0,\infty) \times [0, 2\pi]$.

Lemma 3.1. *Let* $\mathbf{u} \in C_0^\infty([0,T) \times \mathbb{R}^3)$ *for some* $T > 0$. *Then, in the class of functions* $V_k = \{\mathbf{v} \in L^2((0,T); L^2_{loc}(\mathbb{R}^3)); \mathbf{v} - \mathbf{x} \in L^2((0,T); W^{k,2}(\mathbb{R}^3)); \frac{\partial \mathbf{v}}{\partial t} \in L^2((0,T); W^{k-2,2}(\mathbb{R}^3))\}$, $k \geq 2$, *there exists exactly one solution to* (1.7). *Moreover, this solution is smooth, that is, in* $C^\infty((0,T] \times \mathbb{R}^3) \cap C([0,T] \times \mathbb{R}^3)$, *and* $\mathbf{A}^N - \mathbf{x} \in L^2((0,T); W^{k,2}(\mathbb{R}^3))$ *for any* $k \geq 0$. *Furthermore the solution depends smoothly upon the choice of* $\mathbf{u}$.

Remark 3.1. Note that if $\mathbf{u}$ belongs to $L^\infty((0,T); \vee 2)$ $\cap L^2((0,T); W^{1,2}(\mathbb{R}^3))$ $\cap L^1((0,T); \vee\infty)$ (the usual information about a weak solution to the Navier-Stokes equations), then $\mathbf{A}^N - \mathbf{x} \in L^\infty((0,T); W^{1,2}(\mathbb{R}^3)) \cap L^2((0,T); W^{2,2}(\mathbb{R}^3)) \cap L^\infty((0,T); \vee\infty)$ and $\frac{\partial \mathbf{A}^N}{\partial t} \in L^2((0,T); \vee 2)$. The proof is essentially the same as the proof of Lemma 3.1 using [5] and is left as an exercise.

Lemma 3.2. *There exists an interval* $(0,t)$ *such that for* $\mathbf{u}$ *and* $\mathbf{A}^N$ *smooth as in Lemma* 3.1, $\mathbf{Q}$ *is a smooth solution to* (1.8).

Proof: The existence of the solution can be shown using the Galerkin method combined with standard a priori estimates. We leave the details of the proof to the reader as an exercise. □

Now, on the time interval from Lemma 3.2 we see that

$$\mathbf{Z} = (\nabla \mathbf{A}^N)\mathbf{Q} - \mathbf{I}$$

obeys the equation (see [2])

$$\Gamma \mathbf{Z} = 2\nu \mathbf{Z} \partial_k (\nabla \mathbf{A}^N) \partial_k \mathbf{Q} \tag{3.1}$$

in $\mathbb{R}^3 \times (0,T)$ with the initial condition $\mathbf{Z}(\mathbf{x}, 0) = \mathbf{0}$. Since, for $\nabla^2 \mathbf{A}^N$ and $\nabla \mathbf{Q}$ bounded, there exists the unique solution to (3.1), we have $\mathbf{Z} \equiv \mathbf{0}$ and thus $\mathbf{Q} = (\nabla \mathbf{A}^N)^{-1}$ pointwise.

Also, we are now in a position to prove Lemma 2.1. Since (1.7) are uniquely solvable, it follows that the solution is axisymmetric and hence we can apply the following result.

Lemma 3.3. *Let* $\mathbf{F} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *be a vector field which is of the class* C^1 *on some neighborhood of the origin and, written in polar coordinates,* F_r *and* F_ϑ *are independent of* ϑ. *Then*

$$\frac{\partial F_x(\mathbf{0})}{\partial x} = \frac{\partial F_y(\mathbf{0})}{\partial y}, \qquad \frac{\partial F_x(\mathbf{0})}{\partial y} = -\frac{\partial F_y(\mathbf{0})}{\partial x}.$$

PROOF: Denote $F_r = f(r)$ and $F_\vartheta = g(r)$. Then we get

$$\frac{\partial F_x}{\partial x} = f' \cos^2\vartheta + \frac{f}{r}\sin^2\vartheta - g'\sin\vartheta\cos\vartheta + \frac{g}{r}\sin\vartheta\cos\vartheta.$$

Since $\lim_{r\to 0} \frac{\partial F_x}{\partial x}$ exists, necessarily

$$\lim_{r\to 0}\left(f'(r) - \frac{f(r)}{r}\right) = 0 \qquad \text{and} \qquad \lim_{r\to 0}\left(g'(r) - \frac{g(r)}{r}\right) = 0.$$

Thus $\frac{\partial F_x(\mathbf{0})}{\partial x} = f'(0)$. Next

$$\frac{\partial F_y}{\partial y} = f' \sin^2\vartheta + \frac{f}{r}\cos^2\vartheta + g'\sin\vartheta\cos\vartheta - \frac{g}{r}\sin\vartheta\cos\vartheta$$

and also $\frac{\partial F_y(\mathbf{0})}{\partial y} = f'(0)$. Analogously we get that $\frac{\partial F_x(\mathbf{0})}{\partial y} = -g'(0)$ and $\frac{\partial F_y(\mathbf{0})}{\partial x} = g'(0)$. The lemma is proved. □

4. Construction of the fluid flow

We will consider the following vector field in cylindrical coordinates:

$$\mathbf{u}_s = \mathbf{u} = (0, u_\theta(r,z,t), 0) \tag{4.1}$$

with $u_\theta(r,z,t) = \alpha(r)\beta(|z|)\gamma_s(t)r$, where

$$\begin{aligned}
&\alpha(r) = 0 \ \text{ for } r \ge R^o, \quad \alpha(r) = 1 \ \text{ for } r \le R^i, \quad \alpha \in C_0^\infty([0,\infty)),\ 0 \le \alpha(r) \le 1\\
&\beta(|z|) = 0 \ \text{ for } |z| \ge Z^o,\ \beta(|z|) = 1 \ \text{ for } |z| \le Z^i,\ \beta \in C_0^\infty(\mathbb{R}), \qquad 0 \le \beta(r) \le 1\\
&\gamma_s(t) = 0 \ \text{ for } t \ge t_0, \quad \gamma_s(0) = 0, \qquad \gamma_s(t) \ge 0\\
&\int_0^\infty \gamma_s(\tau)d\tau = \int_0^{t_0} \gamma_s(\tau)\,d\tau = s \in [0, 2\pi].
\end{aligned}$$

The vector field $\mathbf{u}$ from (4.1) is divergence free and smooth (in Cartesian coordinates (x,y,z)). Evidently, there exist $\mathbf{f}^E$ and $\mathbf{f}^N$, smooth axially symmetric vector fields such that $\mathbf{u}$ satisfies (with constant pressure) the Euler equations and the Navier-Stokes equations, respectively.

In the cylinder $|z| \le Z_i$, $r \le R_i$ it corresponds to the rotation by the angle s during the time interval $[0, t_0]$ and outside of the cylinder $|z| \le Z^o$, $r \le R^o$ the fluid does not move at all.

Let us start by analyzing $\mathbf{A}^E$. This is actually quite easy to compute explicitly. Writing the input vector in cylindrical coordinates, and the output in Cartesian coordinates, we have

$$\mathbf{A}^E(r,z,\vartheta,t_0) = \Big(r\cos[\vartheta - s\alpha(r)\beta(|z|)], r\sin[\vartheta - s\alpha(r)\beta(|z|)], z\Big)$$

that is,

$$\begin{aligned}\mathbf{A}^E(x,y,z,t_0) = \quad & \Big(x\cos[s\alpha(\sqrt{x^2+y^2})\beta(|z|)] + y\sin[s\alpha(\sqrt{x^2+y^2})\beta(|z|)],\\ & -x\sin[s\alpha(\sqrt{x^2+y^2})\beta(|z|)] + y\cos[s\alpha(\sqrt{x^2+y^2})\beta(|z|)], z\Big).\end{aligned}$$

Inside the inner cylinder we have

$$\nabla\mathbf{A}^E = \begin{pmatrix} \cos s, & \sin s, & 0\\ -\sin s, & \cos s, & 0\\ 0, & 0, & 1\end{pmatrix};$$

outside the outer cylinder

$$\nabla\mathbf{A}^E = \begin{pmatrix} 1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & 1\end{pmatrix};$$

for $|z| \le Z_i$, $R_i \le r \le R^o$

$$\nabla\mathbf{A}^E = \begin{pmatrix} \cos[s\alpha(r)], & \sin[s\alpha(r)], & 0\\ -\sin[s\alpha(r)], & \cos[s\alpha(r)], & 0\\ 0, & 0, & 1\end{pmatrix} + \mathbf{M}_1 + \mathbf{M}_2$$

with

$$\mathbf{M}_1 = \begin{pmatrix} -s\frac{x^2}{r}\sin[s\alpha(r)]\alpha'(r), & s\frac{y^2}{r}\cos[s\alpha(r)]\alpha'(r), & 0\\ -s\frac{x^2}{r}\cos[s\alpha(r)]\alpha'(r), & -s\frac{y^2}{r}\sin[s\alpha(r)]\alpha'(r), & 0\\ 0, & 0, & 0\end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} s\frac{xy}{r}\cos[s\alpha(r)]\alpha'(r), & -s\frac{xy}{r}\sin[s\alpha(r)]\alpha'(r), & 0\\ -s\frac{xy}{r}\sin[s\alpha(r)]\alpha'(r), & -s\frac{xy}{r}\cos[s\alpha(r)]\alpha'(r), & 0\\ 0, & 0, & 0\end{pmatrix};$$

for $Z_i \le |z| \le Z^o$, $r \le R_i$

$$\nabla \mathbf{A}^E = \begin{pmatrix} 1, & 0, & -sx\ \mathrm{sign}(z)\sin[s\beta(|z|)]\beta'(|z|) + sy\ \mathrm{sign}(z)\cos[s\beta(|z|)]\beta'(|z|) \\ 0, & 1, & -sy\ \mathrm{sign}(z)\sin[s\beta(|z|)]\beta'(|z|) - sx\ \mathrm{sign}(z)\cos[s\beta(|z|)]\beta'(|z|) \\ 0, & 0, & 1 \end{pmatrix};$$

and finally for $Z_i \le |z| \le Z^o$, $R_i \le r \le R^o$ we get a combination of the last two cases. We will use the structure of $\nabla \mathbf{A}^E$ later.

Let us now look at the difference between $\mathbf{A}^N$ and $\mathbf{A}^E$, our goal being inequality (4.3) below. We have

$$\frac{\partial}{\partial t}(\mathbf{A}^N - \mathbf{A}^E) + \mathbf{u}\cdot\nabla(\mathbf{A}^N - \mathbf{A}^E) = \nu\Delta\mathbf{A}^N$$
$$(\mathbf{A}^N - \mathbf{A}^E)(\mathbf{x}, 0) = \mathbf{0}.$$

Taking the spatial gradient we get

$$\frac{\partial}{\partial t}[\nabla(\mathbf{A}^N - \mathbf{A}^E)] + \mathbf{u}\cdot\nabla[\nabla(\mathbf{A}^N - \mathbf{A}^E)] = \nu\Delta\nabla\mathbf{A}^N - (\nabla(\mathbf{A}^N - \mathbf{A}^E))\nabla\mathbf{u}. \tag{4.2}$$

Now, since

$$\sup_{t\in[0,t_0]} \|\nabla^3\mathbf{A}^N\|_p \le C(\|\nabla\mathbf{u}\|_{k,p})$$

for some k sufficiently large, we have, after testing equation (4.2) by $|\nabla(\mathbf{A}^N - \mathbf{A}^E)|^{p-2}\nabla(\mathbf{A}^N - \mathbf{A}^E)$

$$\frac{d}{dt}\|\nabla(\mathbf{A}^N - \mathbf{A}^E)\|_p \le \nu\|\nabla^3\mathbf{A}^N\|_p + \|\nabla\mathbf{u}\|_\infty\|\nabla(\mathbf{A}^N - \mathbf{A}^E)\|_p\,.$$

Thus, as $\nabla(\mathbf{A}^N - \mathbf{A}^E)(\mathbf{x}, 0) = \mathbf{0}$, we get

$$\sup_{t\in[0,t_0]} \|\nabla(\mathbf{A}^N - \mathbf{A}^E)\|_p \le \nu C(\|\nabla\mathbf{u}\|_{k,2}, t_0) \tag{4.3}$$

for all $p \in (1, \infty]$.

5. The decay of $\nabla\mathbf{A}^N - \mathbf{I}$

Let us now look in particular at $\nabla\mathbf{A}^N$ for $t > t_0$. We have that $\nabla\mathbf{A}^N$ satisfies the heat equation

$$\frac{\partial}{\partial t}(\nabla\mathbf{A}^N) - \nu\Delta(\nabla\mathbf{A}^N) = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times (t_0, \infty)$$
$$\nabla\mathbf{A}^N(t_0) \quad \text{given.}$$

Therefore also

$$\frac{\partial}{\partial t}(\nabla \mathbf{A}^N - \mathbf{I}) - \nu \Delta(\nabla \mathbf{A}^N - \mathbf{I}) = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times (t_0, \infty)$$

and, especially, at $\mathbf{x} = \mathbf{0}$,

$$\nabla \mathbf{A}^N(\mathbf{0}, t) - \mathbf{I} = \frac{C}{(t-t_0)^{\frac{3}{2}} \nu^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathrm{e}^{-\frac{|\mathbf{p}|^2}{4\nu(t-t_0)}} (\nabla \mathbf{A}^N - \mathbf{I})(\mathbf{p}, t_0)\, d\mathbf{p}. \tag{5.1}$$

Our first goal will be to show that the function F in Lemma 2.2 satisfies $F(t,s) \to (1,0)$ as $t \to \infty$ uniformly in $s \in [0, 2\pi]$. This will complete the proof that F is continuous on $[0,\infty] \times [0,2\pi]$, and that $F(\infty, s) = (1,0)$, so that F is indeed a homotopy.

We have that $\nabla \mathbf{A}^N - \mathbf{I} = (\nabla \mathbf{A}^N - \nabla \mathbf{A}^E) + (\nabla \mathbf{A}^E - \mathbf{I})$. Using the fact that $\nabla \mathbf{A}^E(\mathbf{p}, t_0) - \mathbf{I}$ has bounded support and $\|\nabla(\mathbf{A}^N - \mathbf{A}^E)\|_p(t_0) \le C$, we use the Hölder inequality and end up with

$$|\nabla \mathbf{A}^N(\mathbf{0}, t) - \mathbf{I}| \le \frac{C}{(t-t_0)^a}$$

with some positive power a (C may depend on any constants which appeared above, but is independent of the time).

Since it is clear that $F(\cdot, 0)$ is the constant function, the proof of Lemma 2.2 will be complete when we have shown that $F(\cdot, 2\pi)$ is homotopic to the function whose winding number is 1, at least if ν, $R^o - R_i$, and $Z^o - Z_i$ are small enough.

It is clear that the representation of $\nabla \mathbf{A}^E_{2\pi}(\cdot, 0)$ has this property. So let us put $s = 2\pi$. We need to show that $\nabla \mathbf{A}^E_{2\pi}(\cdot, 0) - \nabla \mathbf{A}^N_{2\pi}(\cdot, 0)$ is small enough to construct a linear homotopy between the representation of $\mathbf{A}^E_{2\pi}(\cdot, 0)$ and $F(\cdot, 2\pi)$ that does not pass through $(0,0)$. We have already shown this property for $t \le t_0$ in equation (4.3), at least when ν is sufficiently small. So all that remains is to show the following result.

Lemma 5.1. *There exist ε_1 and $\varepsilon_2 > 0$ such that if* $\max\{R^o - R_i, Z^o - Z_i\} \le \varepsilon_1$ *and $\nu \le \varepsilon_2(\varepsilon_1)$ then for $s = 2\pi$, $|\nabla \mathbf{A}^N - \mathbf{I})(\mathbf{0}, t)| \le \frac{1}{10}$ for any $t \ge t_0$.*

Proof: We denote by I_1 the part of integral (5.1) with $(\nabla \mathbf{A}^N - \mathbf{I})(\mathbf{p}, t_0)$ replaced by $(\nabla \mathbf{A}^N - \nabla \mathbf{A}^E)(\mathbf{p}, t_0)$, and by I_2–I_5 the parts of integral (5.1) with $(\nabla \mathbf{A}^N - \mathbf{I})(\mathbf{p}, t_0)$ replaced by $(\nabla \mathbf{A}^E - \mathbf{I})(\mathbf{p}, t_0)$; namely by I_2 the integral over the inner cylinder, by I_3 over the cylinder $C(R^o, Z_i)$ without the inner cylinder, by I_4 the integral over the outer cylinder $C(R^o, Z^o)$ minus the cylinder $C(R^o, Z_i)$ and finally by I_5 over the complement of the outer cylinder.

Evidently, $I_2 = I_5 = 0$ since $s = 2\pi$. Let us now consider I_3. If we rewrite $\nabla A^E(\mathbf{0}, t_0) - \mathbf{I}$ (in Cartesian components) into the cylindrical coordinates, we get that it is equal to $\mathbf{M}_0 + \mathbf{M}_1 + \mathbf{M}_2$ with

$$\mathbf{M}_0 = \begin{pmatrix} \cos[2\pi\alpha(r)] - 1, & \sin[2\pi\alpha(r)], & 0 \\ -\sin[2\pi\alpha(r)], & \cos[2\pi\alpha(r)] - 1, & 0 \\ 0, & 0, & 0 \end{pmatrix},$$

$$\mathbf{M}_1 = \begin{pmatrix} -2\pi r \sin[2\pi\alpha(r)]\alpha'(r)\cos^2\vartheta, & 2\pi r \cos[2\pi\alpha(r)]\alpha'(r)\sin^2\vartheta, & 0 \\ -2\pi r \cos[2\pi\alpha(r)]\alpha'(r)\cos^2\vartheta, & -2\pi r \sin[2\pi\alpha(r)]\alpha'(r)\sin^2\vartheta, & 0 \\ 0, & 0, & 0 \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} 2\pi r \cos[2\pi\alpha(r)]\alpha'(r)\sin\vartheta\cos\vartheta, & -2\pi r \sin[2\pi\alpha(r)]\alpha'(r)\sin\vartheta\cos\vartheta, & 0 \\ -2\pi r \sin[2\pi\alpha(r)]\alpha'(r)\sin\vartheta\cos\vartheta, & -2\pi r \cos[2\pi\alpha(r)]\alpha'(r)\sin\vartheta\cos\vartheta, & 0 \\ 0, & 0, & 0 \end{pmatrix}.$$

The heat kernel is independent of the angle ϑ; after integration over it the matrix $\mathbf{M}_2$ disappears and from $\mathbf{M}_1$ we are left with integrals of the type

$$\frac{C}{(t-t_0)^{\frac{3}{2}}\nu^{\frac{3}{2}}} \int_{\substack{R_i \le r \le R^o \\ |z| \le Z_i}} \mathrm{e}^{-\frac{r^2+z^2}{4\nu(t-t_0)}} r^2 \sin[2\pi\alpha(r)]\alpha'(r)\, dr\, dz$$

(in some terms, sin is replaced by cos). Using the standard change of variables and integrating over the z variable we end up with

$$C \int_{\frac{R_i}{\sqrt{\nu(t-t_0)}} \le u \le \frac{R^o}{\sqrt{\nu(t-t_0)}}} \mathrm{e}^{-\frac{u^2}{4}} u^2 \sin\left(2\pi\alpha(u\sqrt{\nu(t-t_0)})\right)\alpha'(u\sqrt{\nu(t-t_0)})\, du.$$

Now the application of the Taylor theorem on the function $\mathrm{e}^{-\frac{u^2}{4}} u^2$ yields

$$\mathrm{e}^{-\frac{u^2}{4}} u^2 = \mathrm{e}^{-\frac{R_i^2}{4\nu(t-t_0)}} \frac{R_i^2}{\nu(t-t_0)} + \mathrm{e}^{-\frac{\xi^2}{4}} \left(2\xi - \frac{\xi^3}{2}\right)\left(u - \frac{R_i}{\sqrt{\nu(t-t_0)}}\right),$$

where $\xi \in (\frac{R_i}{\sqrt{\nu(t-t_0)}}, \frac{R^o}{\sqrt{\nu(t-t_0)}})$. Moreover

$$\int_{\frac{R_i}{\sqrt{\nu(t-t_0)}} \le u \le \frac{R^o}{\sqrt{\nu(t-t_0)}}} \sin\left(2\pi\alpha(u\sqrt{\nu(t-t_0)})\right)\alpha'(u\sqrt{\nu(t-t_0)})\, du = 0.$$

A similar argument can be applied also on terms coming from $\mathbf{M}_0$. Thus we have

$$|I_3| \le C \mathrm{e}^{-\frac{R_i^2}{4\nu(t-t_0)}} \Big(\frac{2R^o}{\sqrt{\nu(t-t_0)}} + \frac{(R^o)^3}{\big(\sqrt{\nu(t-t_0)}\big)^3} \Big)$$

$$\max_{u \in (\frac{R_i}{\sqrt{\nu(t-t_0)}}, \frac{R^o}{\sqrt{\nu(t-t_0)}})} |\alpha'(u\sqrt{\nu(t-t_0)})| \frac{(R^o - R_i)^2}{\nu(t-t_0)}$$

$$+ \mathrm{e}^{-\frac{R_i^2}{4\nu(t-t_0)}} (R^o - R_i) \frac{R^o}{\sqrt{\nu(t-t_0)}} \, .$$

We can choose $\alpha(r)$ in such a way that $\alpha'(r) \le \frac{C}{R^o - R_i}$ and as

$$\mathrm{e}^{-\frac{R_i^2}{4\nu(t-t_0)}} \Big(\sqrt{\nu(t-t_0)}\Big)^a \le C(a, R_i)$$

for any $a \in \mathbb{R}$, we finally get

$$|I_3| \le C(R^o - R_i)$$

with the constant in particular independent of ν and t. Therefore for the "boundary layer" sufficiently thin, this term can be done arbitrarily small, independently of the viscosity and the time.

Similarly we can estimate I_4; here $(\nabla \mathbf{A}^E)_{i,3}^{i=1,2}(\mathbf{p}, t_0)$ are odd functions in z and thus we get zero after the integration of the z variable. For the components i,j; $i,j = 1,2$ proceed similarly as above and end up with the following integral

$$\int_\Omega \mathrm{e}^{-\frac{u^2+v^2}{4}} u^2 \sin\left(2\pi\alpha\Big[u\sqrt{\nu(t-t_0)}\Big]\beta\Big[|v|\sqrt{\nu(t-t_0)}\Big]\right) \alpha'\Big(u\sqrt{\nu(t-t_0)}\Big) \, du \, dv$$

with $\Omega = \{(u,v); \frac{R_i}{\sqrt{\nu(t-t_0)}} \le u \le \frac{R^o}{\sqrt{\nu(t-t_0)}}, \frac{Z_i}{\sqrt{\nu(t-t_0)}} \le |v| \le \frac{Z^o}{\sqrt{\nu(t-t_0)}}\}$. But now $\beta \neq 1$ and we cannot proceed as above. Nevertheless, we get that the integral above is bounded by

$$C(Z^o - Z_i)(\nu(t-t_0))^a \mathrm{e}^{-\frac{R_i^2 + Z_i^2}{4\nu(t-t_0)}}$$

with the constant independent of ν and t. Thus, if $Z^o - Z_i$ is small, we get that also I_4 is small.

Finally,

$$|I_1| \le \frac{C}{\big(\nu(t-t_0)\big)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathrm{e}^{-\frac{|\mathbf{p}|^2}{4\nu(t-t_0)}} |\nabla \mathbf{A}^N - \nabla \mathbf{A}^E|(\mathbf{p}, t_0) \, d\mathbf{p}$$

$$\le C \|\nabla \mathbf{A}^N - \nabla \mathbf{A}^E\|_\infty(t_0) \le \nu C(R^o - R_i, Z^o - Z_i, t_0).$$

$\square$

Remark 5.1. We have shown that $\mathbf{A}^N$ may have no smooth inverse. However it would be more interesting to provide an example in which it can be shown that $\mathbf{A}^N$ has no inverse at all. Looking at the representation of $\nabla \mathbf{A}_s^E(\mathbf{0}, t_0)$ it is not difficult to see that $\nabla^2 \mathbf{A}_s^E(\mathbf{0}, t_0)$ is odd in x and y and therefore, since the same holds also for $\nabla^2 \mathbf{A}_s^N(\mathbf{0}, t_0)$, we get that $\nabla^2 \mathbf{A}_s^N(\mathbf{0}, t) = \mathbf{0}$ for any $t > t_0$ and any $s \in [0, 2\pi]$ and thus $\mathbf{A}^N$ is in fact invertible with a non-smooth inverse.

6. The probabilistic approach

Here we will describe a probabilistic approach to solving equation (1.6). For simplicity let us consider the case when the forcing term $\mathbf{f} = 0$. We will not be rigorous.

We will suppose that we have found $\mathbf{u}$ using equation (1.5). Now let $\mathbf{b}_t$ be a Brownian motion in 3 dimensions, starting at the origin. Define $\tilde{\mathbf{u}}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x} + 2\nu \mathbf{b}_t, t)$. Let $\widetilde{\mathbf{m}}$ be a random vector field that satisfies the equations

$$
\begin{aligned}
&\frac{\partial \widetilde{\mathbf{m}}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{m}} + \widetilde{\mathbf{m}} \cdot (\nabla \tilde{\mathbf{u}})^T = 0 \quad \text{in } \mathbb{R}^3 \times (0, T) \\
&\widetilde{\mathbf{m}}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3.
\end{aligned} \tag{6.1}
$$

Now let $\overline{\mathbf{m}}(\mathbf{x}, t) = \mathbf{m}(x - 2\nu \mathbf{b}_t, t)$. Then $\mathbf{m}(\mathbf{x}, t) = E(\overline{\mathbf{m}}(\mathbf{x}, t))$ satisfies equation (1.6). (Here $E(\cdot)$ represents the expected value.)

The reason why this works is because of the Itô formula. We have that

$$\frac{\partial \overline{\mathbf{m}}}{\partial t} + \mathbf{u} \cdot \nabla \overline{\mathbf{m}} + \overline{\mathbf{m}} \cdot (\nabla \mathbf{u})^T = \nu \Delta \overline{\mathbf{m}} + 2\nu \frac{\partial \mathbf{b}}{\partial t} \cdot \nabla \overline{\mathbf{m}},$$

and taking expectations the result follows.

The solution to equation (6.1) can be computed as follows. Suppose that the initial value of $\mathbf{m}$ satisfies equation (1.3). Then if

$$
\begin{aligned}
&\frac{\partial \alpha_i}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \alpha_i = 0 \\
&\frac{\partial \beta_i}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \beta_i = 0
\end{aligned}
$$

then

$$\widetilde{\mathbf{m}}(\mathbf{x}, t) = \sum_{i=1}^{R} \beta_i(\mathbf{x}, t) \nabla \alpha_i(\mathbf{x}, t)$$

is the solution to system (6.1). But the transport equations are easily solved by $\alpha_i(\mathbf{x}, t) = \alpha_i(\tilde{\mathbf{A}}(\mathbf{x}, t), 0)$ and $\beta_i(\mathbf{x}, t) = \beta_i(\tilde{\mathbf{A}}(\mathbf{x}, t), 0)$, where $\tilde{\mathbf{A}}$ is the back to coordinates map induced by the flow $\tilde{\mathbf{u}}$.

This can be used to obtain the following plausibility argument for the regularity of the Navier-Stokes equations. Let $W^{-1,BMO}$ denote the space of functions from $\mathbb{R}^3$ for which minus one derivative is in the space of functions of bounded mean oscillation. It is known that the space $L^\infty(I; W^{-1,BMO})$ is a critical space for proving regularity for the Navier-Stokes equations (see below). That is, if one can show that the solution to the Navier-Stokes equations is uniformly in time in any space better than $W^{-1,BMO}$ (such as $W^{-1+\epsilon,BMO}$ for any $\epsilon > 0$), then the solution is regular.

Now if the initial data are very nice, then by using some partition of unity argument, we may suppose that indeed the initial value of $\mathbf{m}$ does satisfy equation (1.3) for some finite value of R, where the initial values of α_i and β_i are compactly supported smooth functions. Then it is easy to see that the solutions for α_i and β_i provided by the transport equations stay uniformly in L^∞. Thus it follows that $\nabla\alpha_i$ is uniformly in the space $W^{-1,BMO}$.

Thus $\widetilde{\mathbf{m}}$ is a finite sum of a product of functions uniformly in L^∞ and functions uniformly in $W^{-1,BMO}$. Thus it might seem that we are close to showing that $\mathbf{u}$ (which is the Leray projection of an average of translations of $\tilde{\mathbf{m}}$) is in a space that is critical for proving regularity.

There are some large, probably insurmountable problems with this approach. The lesser problem is that we need a space that is better than critical. The bigger problem is that the space created by taking the convex closure of products of bounded functions and functions in $W^{-1,BMO}$ is not really a well defined space, in that it encompasses every function.

Criticality of $L^\infty(I; W^{-1,BMO})$: Let us present a formal proof of this fact, in the case of the Cauchy problem with zero right-hand side. Let $\mathbf{u}$ be the solution to the Navier-Stokes equations which belongs to the space $L^\infty(I; W^{-1,BMO})$. Multiply equation $(1.5)_1$ by $\Delta\mathbf{u}$ and integrate over $\mathbb{R}^3$. Notice also that

$$\left|\int \Delta\mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\right| = \left|\int \frac{\partial}{\partial x_i}\mathbf{u}\cdot\left(\frac{\partial}{\partial x_i}\mathbf{u}\cdot\nabla\right)\mathbf{u}\right| \le \|\nabla\mathbf{u}\|_3^3.$$

Then

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2 + \nu\|\nabla^2\mathbf{u}\|_2^2 \le \|\nabla\mathbf{u}\|_3^3.$$

Using the inequality

$$\|\nabla\mathbf{u}\|_3 \le C\|\mathbf{u}\|_{-1,BMO}^{\frac{1}{3}}\|\nabla^2\mathbf{u}\|_2^{\frac{2}{3}}$$

(see [9]) we get

$$\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2 + \nu\|\nabla^2\mathbf{u}\|_2^2 \le C\|\mathbf{u}\|_{-1,BMO}\|\nabla^2\mathbf{u}\|_2^2 \tag{6.2}$$

and if $\|\mathbf{u}\|_{-1,BMO}$ is sufficiently small, the solution is smooth. The proof can be done rigorously using the fact that for smooth initial condition there exists a local smooth solution; on this interval we obtain estimate (6.2) and therefore the solution cannot blow up.

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