Duality properties and Riesz representation theorem in the Besicovitch-Orlicz space of almost periodic functions

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Abstract. In [6], the classical Riesz representation theorem is extended to the class of Besicovitch space of almost periodic functions B^q a.p., $q \in]1, +\infty[$. It is also shown that this space is reflexive. We shall consider here such results in the context of Orlicz spaces, namely in the class of Besicovitch-Orlicz space of almost periodic functions B^ϕ a.p., where ϕ is an Orlicz function.

Keywords: Besicovitch-Orlicz space, almost periodic function, reflexivity, duality

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1. Introduction

The class B^q a.p., $q \geq 1$ of Besicovitch almost periodic functions was introduced and developed in [2]. In [1], [4], this class was extended to the context of Orlicz spaces, namely, the authors introduced the Besicovitch-Orlicz space of almost periodic functions B^ϕ a.p.

The paper [4] is a large investigation on structure and topological properties of this space. Some duality properties are also stated using an identification argument based on the Bohr compactification of the real line.

In [8], we characterized the fundamental metric properties of this space, giving necessary and sufficient conditions for the strict and uniform convexity.

In this paper, we shall give first, necessary and sufficient conditions for the reflexivity of B^ϕ a.p. and then state a Riesz type representation theorem in this space.

2. Preliminaries and notations

2.1 Orlicz functions. In all what follows, the notation ϕ is used for an Orlicz function, i.e., a function $\phi : \mathbb{R} \to \mathbb{R}^+$ which is even, convex and satisfies $\phi(0) = 0$, $\phi(u) > 0$ if $u \neq 0$, moreover $\lim_{u\to 0} \frac{\phi(u)}{u} = 0$, $\lim_{u\to\infty} \frac{\phi(u)}{u} = +\infty$.

This function is called of Δ_2-type if there exist constants $K > 2$ and $u_0 \geq 0$ for which $\phi(2u) \leq K\phi(u)$, $\forall\, u \geq u_0$.

We recall here some useful results concerning Orlicz functions (cf. [3], [7], [12]).

An Orlicz function admits a derivative ϕ' unless on a denumerable set of points. It satisfies $\phi'(0) = 0$, $\phi'(u) > 0$ if $u \neq 0$, and $\lim_{|u|\to\infty} \phi'(|u|) = +\infty$ so that it is strictly increasing to infinity.

The derivative satisfies the inequality:

$$u\phi'(u) \leq \phi(2u) \leq 2u\phi'(2u), \ \forall\, u \geq 0. \tag{2.1}$$

From [3], we know that if ϕ is an Orlicz function then, for every $\varepsilon > 0$ there exists an Orlicz function ϕ_ε with continuous derivative and satisfying

$$\phi_\varepsilon(x) \leq \phi(x) \leq (1+\varepsilon)\phi_\varepsilon(x), \qquad \forall\, x \in \mathbb{R}.$$

In view of this, we may assume in the following ϕ' to be continuous.

The function $\psi(y) = \sup\{x|y| - \phi(x), x \geq 0\}$ is called conjugate to ϕ. It is an Orlicz function when ϕ is. The pair (ϕ, ψ) satisfies the Young's inequality:

$$xy \leq \phi(x) + \psi(y), \ x \in \mathbb{R}, \ y \in \mathbb{R}.$$

Let us note that equality holds in the Young's inequality iff $x = \psi'(y)$ or $y = \phi'(x)$.

With each pair (ϕ, ψ) of conjugate Orlicz functions, we may associate an equivalent normalized pair $(\widetilde{\phi}, \widetilde{\psi})$, i.e. such that ϕ is equivalent to $\widetilde{\phi}$ (respectively ψ is equivalent to $\widetilde{\psi}$) and $\widetilde{\phi}(1) + \widetilde{\psi}(1) = 1$. The functions ϕ and $\widetilde{\phi}$ (respectively ψ and $\widetilde{\psi}$) define the same Orlicz space and the corresponding norms are equivalent (cf. [12]).

With all these considerations, we may use in the following, without any restriction, normalized pairs of conjugate functions.

2.2 The Besicovitch-Orlicz space of almost periodic functions. Let $M(\mathbb{R})$ be the set of all real Lebesgue measurable functions. The functional

$$\rho_\phi : M(\mathbb{R}) \to [0, +\infty], \rho_\phi(f) = \overline{\lim_{T\to+\infty}} \ \frac{1}{2T} \int_{-T}^{+T} \phi(|f(t)|)\, dt$$

is a pseudomodular (cf. [4], [8]).

The associated modular space,

$$\begin{aligned} B^\phi(\mathbb{R}) &= \Big\{ f \in M(\mathbb{R}), \lim_{\alpha\to 0} \rho_\phi(\alpha f) = 0 \Big\} \\ &= \Big\{ f \in M(\mathbb{R}), \rho_\phi(\lambda f) < +\infty, \quad \text{for some} \quad \lambda > 0 \Big\} \end{aligned}$$

is called the Besicovitch-Orlicz space.

This space is endowed with the Luxemburg pseudonorm (cf. [4], [8]),

$$\|f\|_{\phi} = \inf\left\{k > 0, \rho_{\phi}\left(\frac{f}{k}\right) \leq \phi(1)\right\}.$$

Let $\mathcal{P}$ be the set of all generalized trigonometric polynomials, i.e.;

$$\mathcal{P} = \left\{P(t) = \sum_{j=1}^{n} \alpha_j \exp(i\lambda_j t),\ \lambda_j \in \mathbb{R},\ \alpha_j \in \mathbb{C},\ j \in \mathbb{N},\ n \in \mathbb{N}\right\}.$$

The Besicovitch-Orlicz space of almost periodic functions denoted by B^{ϕ} a.p. (respectively $\widetilde{B}^{\phi}$ a.p.) is the closure of the linear set $\mathcal{P}$ in $B^{\phi}(\mathbb{R})$, with respect to the pseudonorm $\|.\|_{\phi}$ (respectively to the modular convergence), more exactly:

$$\begin{aligned}
B^{\phi}\,\text{a.p.} &= \left\{f \in B^{\phi}(\mathbb{R}), \exists p_n \in \mathcal{P}, n = 1, 2, \ldots; \text{s.t. } \lim_{n\to\infty} \|f - p_n\|_{\phi} = 0\right\} \\
&= \left\{f \in B^{\phi}(\mathbb{R}), \exists p_n \in \mathcal{P}, n = 1, 2, \ldots; \text{s.t. } \forall k > 0, \lim_{n\to\infty} \rho_{\phi}(k(f - p_n)) = 0\right\}, \\
\widetilde{B}^{\phi}\,\text{a.p.} &= \left\{f \in B^{\phi}(\mathbb{R}), \exists p_n \in \mathcal{P}, n = 1, 2, \ldots; \text{s.t. } \exists k > 0, \lim_{n\to\infty} \rho_{\phi}(k(f - p_n)) = 0\right\},
\end{aligned}$$

clearly B^{ϕ} a.p. $\subset \widetilde{B}^{\phi}$ a.p. and the equality holds when $\phi \in \Delta_2$. Some structural and topological properties of this spaces are considered in [1], [4].

From [4], [8], we know that $\phi(|f|) \in B^1$ a.p. if $f \in B^{\phi}$ a.p. and then (cf. [2]) the limit exists in the expression of $\rho_{\phi}(f)$, i.e.:

$$\rho_{\phi}(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} \phi(|f(t)|)\, dt,\ f \in B^{\phi}\,\text{a.p.}$$

This fact will be very useful in our computations.

Let us denote by {u.a.p.} the classical algebra of Bohr's almost periodic functions, or what is the same, the uniform closure of the linear set $\mathcal{P}$. It is known that if $f \in$ {u.a.p.} then

$$\phi(|f|) \in \{\text{u.a.p.}\}, \quad (\text{cf. [2]}). \tag{2.2}$$

In [2], the following property is stated:

$$\text{if } f \in \{\text{u.a.p.}\},\ f \neq 0, \quad \text{then} \quad M(|f|) > 0, \tag{2.3}$$

where $M(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)\, dt$.

Notice that from (2.2) and (2.3), we deduce that $\|.\|_\phi$ is in fact a norm on $\{$u.a.p.$\}$.

From now on, B^ϕ a.p. will denote the quotient space obtained when identifying the elements of the subspace $\{f \in B^\phi \text{ a.p.}; \|f\|_\phi = 0\}$.

With each $f \in B^\phi$ a.p., we can associate a formal Fourier series, more precisely: define the Bohr's transform of f,

$$a(\lambda, f) = M(f \exp(i\lambda t)),\ \ \lambda \in \mathbb{R}.$$

There is at most a denumerable set of scalars $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ for which $a(\lambda, f) \neq 0$ (these are called the Fourier-Bohr's exponents). The associated coefficients $\{a(\lambda_i, f)\}_{i\geq 1}$ are the Fourier-Bohr's coefficients.

Questions concerning the convergence of the associated formal Fourier series

$$S(f)(x) = \sum_{n\geq 1} a(\lambda_n, f) \exp(i\lambda_n x)$$

are not trivial and only partial results are available.

The Bochner's approximation result will be of importance here:

Let $f \in B^\phi$ a.p. and $S_n(f)(x) = \sum_{k=1}^{n} a(\lambda_k, f) \exp(i\lambda_k x)$ be the partial sums of its formal Fourier series. Then there exists a sequence

$$\sigma_m(f)(x) = \sum_{k=1}^{m} \mu_{m_k} a(\lambda_k, f) \exp(i\lambda_k x), \qquad \text{(cf. [4])}$$

where the convergence factors $\{\mu_{m_k}\}$ depend only on the sequence of characteristic exponents $\{\lambda_k\}$ of the function f and satisfy $0 < \mu_{m_k} \leq 1$.

The sequence $\{\sigma_m(f)\}$ has the following approximation properties (cf. [4]):

1. $\|\sigma_m(f)\|_\phi \leq \|f\|_\phi$, $m = 1, 2, \ldots$ $\qquad (\rho_\phi(\sigma_m(f)) \leq \rho_\phi(f))$.
2. $\|\sigma_m(f) - f\|_\phi \to 0$ when $m \to \infty$ $\qquad (\forall\, \alpha > 0, \rho_\phi(\alpha(\sigma_m(f) - f)) \to 0$ when $m \to \infty)$.

To end this section, we define an Orlicz pseudonorm in the B^ϕ a.p. space as usual,

$$|||f|||_\phi = \sup\Big\{M(|fg|),\ g \in B^\psi \text{ a.p.},\ \rho_\psi(g) \leq 1\Big\}.$$

3. Convergence results in the B^ϕ a.p. space

A sequence $\{f_k\}_{k\geq 1}$ from $B^\phi(\mathbb{R})$ is called modular convergent to some $f \in B^\phi(\mathbb{R})$, when $\lim_{k\to\infty} \rho_\phi(f_k - f) = 0$.

Let $P(\mathbb{R})$ be the family of subsets of $\mathbb{R}$ and $\Sigma(\mathbb{R})$ be the Σ-algebra of its Lebesgue measurable sets. We define the set function

$$\overline{\mu}(A) = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{+T} \chi_A(t)\,dt = \overline{\lim_{T\to\infty}} \frac{1}{2T}\mu([-T,+T]\cap A).$$

Clearly, $\overline{\mu}$ is null on sets with μ-finite measure and is not σ-additive.

As usual, a sequence of Σ-measurable functions $\{f_k\}_{k\geq 1}$ will be called $\overline{\mu}$-convergent to f when, for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \overline{\mu}\,\{t\in\mathbb{R}, |f_k(t)-f(t)|\geq\varepsilon\} = 0.$$

We now state some fundamental convergence results that will be used below:

Proposition 1. *Let $\{f_k\}_{k\geq 1}$ be a sequence of functions from $B^\phi(\mathbb{R})$. We have the following* (cf. [8], [9], [10]):

(i) *Suppose there exist $f\in B^\phi(\mathbb{R})$ such that $\lim_{k\to\infty}\rho_\phi(f_k-f)=0$ and $g\in B^\phi$ a.p. for which $\max(|f_k|,|f|)\leq g$. Then $\lim_{k\to\infty}\rho_\phi(f_k)=\rho_\phi(f)$.*

(ii) *If $f\in B^\phi$ a.p. and $\{P_n\}$ is the associated sequence of Bochner-Fejèr's polynomials, we have $\lim_{n\to\infty}\rho_\phi(P_n)=\rho_\phi(f)$.*

(iii) *If $f\in B^\phi$ a.p. is such that $\lim_{n\to\infty}\rho_\phi(f_n-f)=0$, then*

(a) $\underline{\lim}_{n\to\infty}\rho_\phi(f_n)\geq\rho_\phi(f)$.

(b) *$\{f_n\}_{n\geq 1}$ is $\overline{\mu}$-convergent to f.*

4. Auxiliary results

Lemma 1. *Let $f\in B^\phi$ a.p., $f\neq 0$ and $\{f_n\}_{n\geq 1}$ be modular convergent to f. Then there exist constants α_1, β_1, θ_1, with $\theta_1\in\,]0,1[$, $0<\alpha_1<\beta_1$, $n_1\in\mathbb{N}$ and $G_n=\{t\in\mathbb{R},\ \alpha_1\leq|f_n(t)|\leq\beta_1\}$ such that $\overline{\mu}(G_n)\geq\theta_1,\ \forall n\geq n_1$.*

Proof: By [8], there exist α, β and θ with $\theta\in\,]0,1[$, $0<\alpha<\beta$ and $G=\{t\in\mathbb{R},\ \alpha\leq|f(t)|\leq\beta\}$ such that $\overline{\mu}(G)\geq\theta$. Take $\alpha_1=\dfrac{\alpha}{2}$, $\beta_1=\dfrac{\alpha}{2}+\beta$ and $\theta_1=\dfrac{\theta}{2}$. Then, since $\{f_n\}_{n\geq 1}$ is modular convergent to f, it is also $\overline{\mu}$-convergent to f (cf. Proposition 1(iii)) and so,

$$\overline{\mu}\left\{t\in\mathbb{R},\ |f_n(t)-f(t)|\geq\frac{\alpha}{2}\right\}<\frac{\theta}{2},\qquad \text{for }\ n\geq n_1.$$

Let $G'_n=\{t\in\mathbb{R},\ |f_n(t)-f(t)|\geq\frac{\alpha}{2}\}$, then $G\backslash G'_n\subset G_n$, $\forall n\geq n_1$. Indeed, if $t\in G\backslash G'_n$ we have $\alpha\leq|f(t)|\leq\beta$ and $|f_n(t)-f(t)|\leq\frac{\alpha}{2}$. It follows that $\alpha_1\leq|f_n(t)|\leq\beta_1$, $\forall n\geq n_1$, i.e. $t\in G_n$, $\forall n\geq n_1$. Finally, we get

$$\overline{\mu}(G_n)\geq\overline{\mu}(G\backslash G'_n)\geq\overline{\mu}(G)-\overline{\mu}(G'_n)\geq\theta-\frac{\theta}{2}=\theta_1,\ \forall n\geq n_1. \qquad \square$$

Lemma 2. (i) *Let $\{a_n\}_{n\geq 1}$, $a_n > 0$, be a sequence of real numbers. With every $n \geq 1$, we associate a measurable set $A_n \subset [0,1]$ such that:*

(a) *$A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{n\geq 1} A_n \subset [0,\alpha]$, where $0 < \alpha < 1$;*

(b) *$\sum_{n\geq 1} \phi(a_n)\mu(A_n) < +\infty$.*

Consider the function $f = \sum_{n\geq 1} a_n \chi_{A_n}$ on $[0;1]$ and let $\widetilde{f}$ be its periodic extension to the whole $\mathbb{R}$, with period $\tau = 1$. Then, $\widetilde{f} \in \widetilde{B}^\phi\,$a.p.

(ii) *If ϕ satisfies the Δ_2-condition then the mapping*

$$\begin{array}{rcl} i : (L^\phi[0,1], \|.\|_\phi) & \longrightarrow & \widetilde{B}^\phi\,\text{a.p.} \\ f & \mapsto & \widetilde{f} \end{array}$$

where $\widetilde{f}$ is the periodic extension of f, is an isometry (and also a modular isometry). $L^\phi[0,1]$ is the classical Orlicz space on $[0,1]$.
The result remains true if we take an interval $[a,b]$, $a, b \in \mathbb{R}$.

Proof: The part (i) of the lemma is in [8].

(ii) We have only to prove that $\widetilde{f} \in \widetilde{B}^\phi\,$a.p. Indeed, if $f \in L^\phi[0,1]$, then, for every $\varepsilon > 0$ there exists a step function $f_\varepsilon = \sum_{i=1}^n a_i \chi_{Ai}$ defined on $[0,1]$ and such that $\|f - f_\varepsilon\|_\phi \leq \varepsilon$ (cf. [12]). Let $\widetilde{f}$ and $\widetilde{f_\varepsilon}$ be the respective periodic extension (with the same period $\tau = 1$) of f and f_ε to the whole $\mathbb{R}$. From (i), we know that $f_\varepsilon \in B^\phi\,$a.p. and then, there exists a trigonometric polynomial P_ε such that $\|\widetilde{f_\varepsilon} - P_\varepsilon\|_\phi \leq \varepsilon$, hence

$$\begin{aligned} \left\|\widetilde{f} - P_\varepsilon\right\|_\phi &\leq \left\|\widetilde{f} - \widetilde{f_\varepsilon}\right\|_\phi + \left\|\widetilde{f_\varepsilon} - P_\varepsilon\right\|_\phi \\ &\leq \left\|f - f_\varepsilon\right\|_{L^\phi} + \left\|\widetilde{f_\varepsilon} - P_\varepsilon\right\|_\phi \\ &\leq \varepsilon + \varepsilon, \end{aligned}$$

i.e. $\widetilde{f} \in \widetilde{B}^\phi\,$a.p. □

Lemma 3. *Let (ϕ, ψ) be a complementary pair of normalized Orlicz functions. Then:*

(i) *if $f \in B^\phi\,$a.p. and $\|f\|_\phi \neq 0$ we have $\rho_\phi\left(\frac{f}{\|f\|_\phi}\right) = \phi(1)$. Moreover if $f \in \{\text{u.a.p.}\}$, then*

$$\rho_\phi(f) = \phi(1) \Longleftrightarrow \|f\|_\phi = 1;$$

(ii) *if $f \in B^\phi\,$a.p. and $g \in B^\psi\,$a.p., we have $f \cdot g \in B^1\,$a.p., and,*

$$M(|fg|) \leq \|f\|_\phi \cdot \|g\|_\psi \qquad \text{(Hölder's inequality).} \tag{4.1}$$

Proof: (i) Let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty} \varepsilon_n = 0$. Then $\rho_\phi\big(\frac{f}{\|f\|_\phi+\varepsilon_n}\big) \le \phi(1)$. Moreover $f_n = \frac{f}{\|f\|_\phi+\varepsilon_n}$ is modular convergent to $\frac{f}{\|f\|_\phi}$ since

$$\rho_\phi\bigg(\frac{f}{\|f\|_\phi+\varepsilon_n} - \frac{f}{\|f\|_\phi}\bigg) = \rho_\phi\bigg(\frac{\varepsilon_n f}{\|f\|_\phi(\|f\|_\phi+\varepsilon_n)}\bigg) \le \varepsilon_n \rho_\phi\bigg(\frac{f}{\|f\|_\phi^2}\bigg) \to 0 \text{ as } n\to\infty.$$

It follows from Proposition 1 that $\lim_{n\to\infty} \rho_\phi(f_n) = \rho_\phi\big(\frac{f}{\|f\|_\phi}\big) \le \phi(1)$. On the other hand, since $\rho_\phi\big(\frac{f}{\|f\|_\phi-\varepsilon_n}\big) \ge \phi(1)$, using the same argument we get $\rho_\phi\big(\frac{f}{\|f\|_\phi}\big) \ge \phi(1)$ and finally $\rho_\phi\big(\frac{f}{\|f\|_\phi}\big) = \phi(1)$.

Suppose now that $f \in \{\text{u.a.p.}\}$ and $\rho_\phi\big(\frac{f}{a}\big) = \phi(1)$ for some $a > 0$. The function $\phi\big(\frac{f}{a}\big)$ being also u.a.p., from (2.3) we get $\phi\big(\frac{f}{a}\big) = \phi\big(\frac{f}{\|f\|_\phi}\big)$ and then, since ϕ is strictly increasing, $\|f\|_\phi = a$.

(ii) If $\|f\|_\phi \neq 0$ and $\|g\|_\psi \neq 0$, it follows directly from Young's inequality and (i) that

$$M(|fg|) \le \|f\|_\phi \cdot \|g\|_\psi.$$

This inequality remains true whenever $\|f\|_\phi = 0$ or (and) $\|g\|_\psi = 0$.

Let (P_n) and (Q_n) be the sequences of Bochner-Fejèr's polynomials that converge respectively to f and g in the respective norms. We have

$$\begin{aligned} M(|fg - P_nQ_n|) &\le M(|f|\cdot|g-Q_n|) + M(|Q_n|\cdot|f-P_n|) \\ &\le \|f\|_\phi \cdot \|g-Q_n\|_\psi + \|g\|_\psi \cdot \|f-P_n\|_\phi, \end{aligned}$$

so that $M(|fg - P_nQ_n|) \to 0$ if $n\to\infty$, i.e. $fg \in B^1$ a.p. □

Lemma 4. *For $f \in B^\phi$ a.p., we have:*

(i) $|||f|||_\phi = \inf\big\{\frac{1}{k}(1+\rho_\phi(kf)), k>0\big\}$.
Moreover, $|||f|||_\phi = \frac{1}{k_0}(1+\rho_\phi(k_0 f))$ *for some* $k_0 > 0$.

(ii) $\rho_\phi\big(\frac{f}{|||f|||_\phi}\big) \le 1, \qquad |||f|||_\phi \neq 0.$

(iii) $\phi(1)\|f\|_\phi \le |||f|||_\phi \le \frac{1}{\psi(1)}\|f\|_\phi$.

Proof: (i) The proof will be done in several steps.

(a) From easy computations we get

$$\begin{aligned} |||f|||_\phi &= \sup\Big\{M(|fg|), g\in B^\psi \text{ a.p.}, \rho_\psi(g)\le 1\Big\} \\ &\le \frac{1}{\psi(1)} \sup\Big\{M(|fh|), h\in B^\psi \text{ a.p.}, \rho_\psi(h)\le \psi(1)\Big\}. \end{aligned}$$

Now, since $\rho_\psi(h) \leq \psi(1)$ implies $\|h\|_\psi \leq 1$, using Hölder's inequality it follows,

$$\||f\||_\phi \leq \frac{1}{\psi(1)}\|f\|_\phi. \tag{4.2}$$

(b) First, we suppose $f = P$ is a trigonometric polynomial.

Consider the function $F : [0,+\infty[\to [0,+\infty[$, $F(k) = \rho_\psi[\phi'(k|P(t)|)]$. If $P \neq 0$, by Lemma 1, there exist α, β, θ, with $0 < \alpha < \beta$ and $\theta \in (0,1)$ such that $\overline{\mu}(G) \geq \theta$, where $G = \{t \in \mathbb{R},\ \alpha \leq |P(t)| \leq \beta\}$.

It follows then,

$$\rho_\psi\left[\phi'(k|P|)\right] \geq \overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{[-T,+T]\cap G} \psi(\phi'(k|P(x)|))\,dx \geq \theta\cdot\psi\left[\phi'(k\alpha)\right].$$

Now, since an Orlicz function increases to infinity with its derivative (cf. [3], [11]) we get $\lim_{k\to\infty} F(k) = +\infty$. Let us show that F is continuous. For, let $k_0 \in]0,+\infty[$ and $\{k_n\}$ be a sequence of scalars converging to k_0. A trigonometric polynomial being uniformly bounded, we put $\|P\|_\infty = M$. Using the uniform continuity of ϕ' on the interval $\left[\frac{k_0M}{2}, \frac{3k_0M}{2}\right]$, we have: $\forall\,\varepsilon > 0$, $\exists\, n_0$ such that

$$n \geq n_0 \Rightarrow \left|\phi'(k_n|P|) - \phi'(k_0|P|)\right| \leq \psi^{-1}(\varepsilon);$$

hence,

$$\rho_\psi\left[\phi'(k_n|P|) - \phi'(k_0|P|)\right] \leq \varepsilon. \tag{4.3}$$

Let us put $f_n = \phi'(k_n|P|)$ and $f = \phi'(k_0|P|)$. Then, clearly $f_n \in \{\text{u.a.p.}\}$ and $f \in \{\text{u.a.p.}\}$. Since ϕ' is increasing, we have moreover $f_n \leq \phi'(2k_0|P|)$. Now, from (4.3) we have $\lim_{n\to\infty}\rho_\psi(f_n - f) = 0$. Finally in view of Proposition 1.(i) we get $\lim_{n\to\infty}\rho_\psi(f_n) = \rho_\psi(f)$ and then F is continuous at k_0. Consequently, since $F(0) = 0$ and $\lim_{k\to\infty} F(k) = +\infty$, there exists $k_0 \in]0,+\infty[$ for which $\rho_\psi[\phi'(k_0|P|)] = 1$. To end the proof of (b), consider the following inequalities:

$$M(|Pg|) = \frac{1}{k}M(|kPg|) \leq \frac{1}{k}\left[\rho_\phi(kP) + \rho_\psi(g)\right], \qquad k > 0.$$

We get immediately

$$\||P\||_\phi \leq \inf_{k>0}\left\{\frac{1}{k}(1 + \rho_\phi(kP))\right\}.$$

Now, considering the case of equality in the Young's inequality, it follows

$$\begin{aligned}\||P\||_\phi &\geq \frac{1}{k_0}M(|k_0P|\cdot\phi'(k_0|P|))\\ &\geq \frac{1}{k_0}(\rho_\phi(k_0P) + \rho_\psi[\phi'(k_0|P|)]) \geq \frac{1}{k_0}(\rho_\phi(k_0P) + 1)\end{aligned}$$

and, finally,

$$|||P|||_{\phi} = \inf_{k>0} \left\{ \frac{1}{k}(\rho_{\phi}(kP) + 1) \right\} = \frac{1}{k_0}(\rho_{\phi}(k_0 P) + 1).$$

Notice that we have also $|||P|||_{\phi} = M(|P(x)| \cdot \phi'(k_0|P(x)|))$.

(c) We now show that the result of (b) remains true for $f \in B^{\phi}$ a.p. For, let $\{P_n\}$ be the sequence of Bochner-Fejèr's polynomials of the approximation of f. From (b) we know that

$$\forall n \geq 1, \exists k_n \in]0, +\infty[\text{ such that } |||P_n|||_{\phi} = \left\{ \frac{1}{k_n}(1 + \rho_{\phi}(k_n P_n)) \right\}. \tag{4.4}$$

Hence, from (4.2) and the properties of the Bochner-Fejèr's polynomials (see 2.2.1.), we get

$$\frac{1}{k_n} \leq |||P_n|||_{\phi} \leq \frac{1}{\psi(1)} \|P_n\|_{\phi} \leq \frac{1}{\psi(1)} \|f\|_{\phi}$$

and thus $k_n \geq \frac{\psi(1)}{\|f\|_{\phi}} = c_1 > 0$. Let us show that $k_n \leq c_2$, $\forall n \geq 0$, for some constant c_2. Indeed, if this were not the case, there would exist a subsequence denoted by $\{k_n\}$ increasing to infinity and then

$$\begin{aligned}
1 = \rho_{\psi}\left[\phi'(k_n|P_n|)\right] &\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{-T}^{+T} \psi(\phi'(k_n|P_n(x)|))\, dx \\
&\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{G_n} \psi(\phi'(k_n|P_n(x)|))\, dx \\
&\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{G_n} \psi(\phi'(k_n\alpha_1))\, dt \geq \theta_1 \cdot \psi\left[\phi'(k_n\alpha_1)\right] \to \infty, \quad \text{as } \ n \to \infty,
\end{aligned}$$

where G_n, θ_1, α_1 are defined in Lemma 1. A contradiction. Now the sequence $\{k_n\}$ being bounded, there exists a subsequence denoted by $\{k_n\}$ that converges to some k_0 with $0 < k_0 < +\infty$. We assert that $\lim_{n\to\infty} \rho_{\phi}(k_n P_n) = \rho_{\phi}(k_0 f)$. Indeed, we have by 1. and 2. of 2.2.,

$$\begin{aligned}
\rho_{\phi}(k_n P_n - k_0 f) &\leq \frac{1}{2}\rho_{\phi}(2(k_n - k_0)P_n) + \frac{1}{2}\rho_{\phi}(2k_0(P_n - f)) \\
&\leq |k_n - k_0|\rho_{\phi}(f) + \frac{1}{2}\rho_{\phi}(2k_0(P_n - f))
\end{aligned}$$

and then $\lim_{n\to\infty} \rho_{\phi}(k_n P_n - k_0 f) = 0$. Now, in view of Proposition 1.(iii) it follows that $\underline{\lim}_{n\to\infty} \rho_{\phi}(k_n P_n) \geq \rho_{\phi}(k_0 f)$. On the other hand, from the inequality $\rho_{\phi}(k_n P_n) \leq \rho_{\phi}(k_n f)$ (cf. 2.2.1.), we have

$$\overline{\lim_{n\to\infty}} \rho_{\phi}(k_n P_n) \leq \overline{\lim_{n\to\infty}} \rho_{\phi}(k_n f) = \lim_{n\to\infty} \rho_{\phi}(k_n f) = \rho_{\phi}(k_0 f).$$

Hence,

$$\overline{\lim_{n\to\infty}}\,\rho_\phi(k_nP_n) \le \rho_\phi(k_0f) \le \underline{\lim_{n\to\infty}}\,\rho_\phi(k_nP), \quad \text{i.e.} \quad \lim_{n\to\infty}\rho_\phi(k_nP_n) = \rho_\phi(k_0f).$$

Finally, letting $n \to \infty$ in (4.4) we get,

$$|||f|||_\phi = \frac{1}{k_0}(\rho_\phi(k_0f)+1).$$

(ii) Suppose first that $f \in \{\text{u.a.p.}\}$, $f \neq 0$. Let $g \in B^\psi$ a.p. Then:

(a) If $\rho_\psi(g) \le 1$, we have $M(|fg|) \le |||f|||_\phi$.

(b) If $\rho_\psi(g) > 1$, we have $\rho_\psi\big(\frac{g}{\rho_\psi(g)}\big) \le \frac{1}{\rho_\psi(g)} \cdot \rho_\psi(g) = 1$ and then, $M\big(\big|f\frac{g}{\rho_\psi(g)}\big|\big) \le |||f|||_\phi$.

It follows that in all cases, we have $M(|fg|) \le \max(1,\rho_\psi(g)) \cdot |||f|||_\phi$.
Suppose now that $g = \phi'(\frac{f}{|||f|||_\phi})$, hence $g \in \{\text{u.a.p.}\}$. Using the case of equality in the Young's inequality and the fact that in this case the limits exist, it follows:

$$\rho_\phi\left(\frac{f}{|||f|||_\phi}\right) + \rho_\psi(g) = M\left(\left|\frac{f}{|||f|||_\phi}\cdot g\right|\right) \le \max(1,\rho_\psi(g))$$

so that we get $\rho_\phi(\frac{f}{|||f|||_\phi}) \le 1$.
Consider now the case of $f \in B^\phi$ a.p. Let $\{P_n\}$ be the sequence of Bochner-Fejèr polynomials of the approximation of f. We have:

$$\rho_\phi\left(\frac{P_n}{|||P_n|||_\phi}\right) \le 1, \quad \forall n \ge 1.$$

But, in view of Lemma 4(i) and 1. of 2.2., we can write:

$$|||P_n|||_\phi = \inf_{k>0}\left\{\frac{1}{k}(1+\rho_\phi(kP_n))\right\} \le \inf_{k>0}\left\{\frac{1}{k}(1+\rho_\phi(kf))\right\} = |||f|||_\phi,$$

so that,

$$\rho_\phi\left(\frac{P_n}{|||f|||_\phi}\right) \le \rho_\phi\left(\frac{P_n}{|||P_n|||_\phi}\right) \le 1,$$

and then by (ii) of Proposition 1, $\rho_\phi(\frac{f}{|||f|||_\phi}) \le 1$.

(iii) From the inequality $\rho_\phi(\frac{\phi(1)f}{|||f|||_\phi}) \le \phi(1)\cdot\rho_\phi(\frac{f}{|||f|||_\phi}) \le \phi(1)$ it follows $\|f\|_\phi \le \frac{1}{\phi(1)}|||f|||_\phi$. Now, in view of (4.2), we get $\phi(1)\cdot\|f\|_\phi \le |||f|||_\phi \le \frac{1}{\psi(1)}\|f\|_\phi$. $\square$

Lemma 5. *If* $f \in B^{\psi}\,\text{a.p.}$ *then*

$$\begin{aligned}|||f|||_{\psi} &= \sup\left\{|M(fg)|,\ g \in B^{\phi}\,\text{a.p.},\ \rho_{\phi}(g) \le 1\right\} \\ &= \sup\left\{|M(fQ)|,\ Q \in \mathcal{P},\ \rho_{\phi}(Q) \le 1\right\}.\end{aligned}$$

Proof: We consider first the case when $f = P \in \mathcal{P}$. In view of (4.1) we have $|M(PQ)| \le |||P|||_{\psi}, \forall\, Q \in \{\text{u.a.p.}\}, \rho_{\phi}(Q) \le 1$. Moreover, from the proof of Lemma 4.(i), there exists $0 < k_0 < +\infty$ such that $\rho_{\psi}[\phi'(k_0|P|)] = 1$ and

$$|||P|||_{\psi} = M\left(|P| \cdot \psi'(k_0|P(x)|)\right) = M\left(P(x) \cdot \operatorname{sign} P(x) \cdot \psi'(k_0|P(x)|)\right).$$

Now, since $\operatorname{sign} P(x) \cdot \psi'(k_0|P(x)|) \in \{\text{u.a.p.}\}$, it follows that

$$|||P|||_{\psi} = \sup\left\{|M(PQ)|,\ Q \in \{\text{u.a.p.}\},\ \rho_{\phi}(Q) \le 1\right\}. \tag{4.5}$$

(To see that $\operatorname{sign} P(x) \cdot \psi'(k_0|P(x)|) \in \{\text{u.a.p.}\}$, note that the function $F(u) = u \cdot \frac{\psi'(k_0|u|)}{|u|}$ if $u \ne 0$ and $F(0) = 0$ is continuous so that $F(P) \in \{\text{u.a.p.}\}$ if $P \in \{\text{u.a.p.}\}$.) In fact, we have also $|||P|||_{\psi} = \sup\{|M(PQ)|,\ Q \in \mathcal{P},\ \rho_{\phi}(Q) \le 1\}$. Indeed, from (4.5) we may write: $\forall\, \varepsilon > 0, \exists\, Q_{\varepsilon} \in \{\text{u.a.p.}\}$ such that $\rho_{\phi}(Q_{\varepsilon}) \le 1$ and $|||P|||_{\psi} \le |M(PQ_{\varepsilon})| + \varepsilon$. But, for $Q_{\varepsilon} \in \{\text{u.a.p.}\}$, we may find $\widetilde{Q}_{\varepsilon} \in \mathcal{P}$ with $\rho_{\phi}(\widetilde{Q}_{\varepsilon}) \le \rho_{\phi}(Q_{\varepsilon}) \le 1$ and $\|\widetilde{Q}_{\varepsilon} - Q_{\varepsilon}\|_{\phi} \le \frac{\varepsilon}{\|P\|_{\psi}}$. It follows then,

$$\begin{aligned}|M(PQ_{\varepsilon})| &\le |M(P\widetilde{Q}_{\varepsilon})| + M(|P| \cdot |Q_{\varepsilon} - \widetilde{Q}_{\varepsilon}|) \\ &\le |M(P\widetilde{Q}_{\varepsilon})| + \|P\|_{\psi} \cdot \|Q_{\varepsilon} - \widetilde{Q}_{\varepsilon}\|_{\phi} \le |M(P\widetilde{Q}_{\varepsilon})| + \varepsilon\end{aligned}$$

and then $|||P|||_{\psi} \le |M(P\widetilde{Q}_{\varepsilon})| + 2\varepsilon$. Finally, $|||P|||_{\psi} = \sup\{|M(PQ)|,\ Q \in \mathcal{P},\ \rho_{\phi}(Q) \le 1\}$. Consider now the general case of $f \in B^{\psi}\,\text{a.p.}$ Let $\{P_n\}$ be the sequence of Bochner-Fejèr's polynomials that converge to f in $B^{\psi}\,\text{a.p.}$ Put

$$I(f) = \sup\{|M(fQ)|, Q \in \mathcal{P}, \rho_{\phi}(Q) \le 1\}.$$

Then $I(f) \le \sup\{M(|fQ|),\ Q \in \mathcal{P},\ \rho_{\phi}(Q) \le 1\} \le |||f|||_{\psi}$. Moreover, $\forall\, \varepsilon > 0$, $\exists\, n_0$ such that $\forall\, n \ge n_0$ one has, $|||f - P_n|||_{\psi} \le \varepsilon$ and $|||f|||_{\psi} \le |||P_n|||_{\psi} + \varepsilon$. Then, using the particular case and Hölder's inequality,

$$\begin{aligned}|||f|||_{\psi} - \varepsilon \le |||P_n|||_{\psi} &= \sup\{|M(P_nQ)|,\ Q \in \mathcal{P},\ \rho_{\phi}(Q) \le 1\} \\ &\le \sup\{\|f - P_n\|_{\psi} \cdot \|Q\|_{\phi},\ Q \in \mathcal{P},\ \rho_{\phi}(Q) \le 1\} \\ &\quad + \sup\{|M(fQ)|,\ Q \in \mathcal{P},\ \rho_{\phi}(Q) \le 1\} \\ &\le I(f) + \varepsilon.\end{aligned}$$

Finally, $I(f) \le |||f|||_{\psi} \le I(f) + 2\varepsilon$. Now, since $\varepsilon > 0$ is arbitrary, we get $I(f) = |||f|||_{\psi}$. This is the desired result. $\square$

5. Reflexivity of the space $\widetilde{B}^{\phi}$ a.p.

Theorem 1. *The space $\widetilde{B}^{\phi}$ a.p. is reflexive iff $\phi \in \Delta_2 \cap \nabla_2$ (i.e. ϕ and its conjugate ψ satisfy the Δ_2-condition).*

Proof: **Sufficiency.** If $\phi \in \Delta_2 \cap \nabla_2$ then (cf. [5]), there exists an Orlicz function ϕ_1 equivalent to ϕ and satisfying the Δ_2-condition (with its conjugate function) on $\mathbb{R}^+ = [0, +\infty[$. We also know that we may associate with ϕ_1 an equivalent function ϕ_2 which is uniformly convex on $\mathbb{R}^+$. Clearly ϕ_2 satisfies the Δ_2-condition and is also strictly convex on $\mathbb{R}^+$. From [8] the space $\widetilde{B}^{\phi}$ a.p. is then uniformly convex and hence also reflexive.

Necessity. Suppose that ϕ does not satisfy the Δ_2-condition. We will show that $\widetilde{B}^{\phi}$ a.p. contains an isometric copy of C_0, the classical Banach space of sequences converging to 0. This leads to a contradiction with the reflexivity of $\widetilde{B}^{\phi}$ a.p. Let S_n, $n \geq 1$, be a family of disjoints subsets of $[0,1[$ such that $0 < \mu(S_n) \leq \frac{\phi(1)}{2^n}$ and $\bigcup_{n\geq 1} S_n \subset [0,1[$. Let $a_{n,1}$, $n \geq 1$ be a sequence of real numbers with $\phi(a_{n,1}) > \frac{1}{\mu(S_n)} \geq \frac{2^n}{\phi(1)}$. Since ϕ does not satisfy the Δ_2-condition, for every fixed $n \geq 1$, we may find a sequence $(a_{n,k})_{k\geq 1}$ increasing to infinity and such that $\phi((1+\frac{1}{k})a_{n,k}) > 2^k \phi(a_{n,k})$, $k \geq 1$. For each $n \geq 1$, let $\{S_{n,k}\}$ be a family of disjoint subsets of S_n with $\mu(S_{n,k}) = \frac{\phi(1)}{2^{n+k}\phi(a_{n,k})}$, and set $f_n = \sum_{k\geq 0} a_{n,k}\chi_{S_{n,k}}$ on $[0,1[$. Let $\widetilde{f}_n$ be the periodic extension of f_n to the whole $\mathbb{R}$, with period $\tau = 1$. By Lemma 2 we know that $\widetilde{f}_n \in \widetilde{B}^{\phi}$ a.p., $n = 1, 2, \ldots$. Moreover,

$$\begin{aligned}
\rho_\phi(\widetilde{f}_n) &= \int_0^1 \phi(|f_n(t)|)\,dt = \sum_{k\geq 1} \phi(a_{n,k})\mu(S_{n,k}) \\
&= \sum_{k\geq 1} \frac{\phi(1)}{2^{n+k}} = \frac{\phi(1)}{2^n} \leq \phi(1), \ n = 1, 2, \ldots \ .
\end{aligned}$$

For $\alpha > 1$ there exists k_0 such that $1 + \frac{1}{k} < \alpha$, $\forall\, k \geq k_0$ and thus,

$$\begin{aligned}
\rho_\phi(\alpha\widetilde{f}_n) &= \int_0^1 \phi(\alpha|f_n(t)|)\,dt = \sum_{k\geq 1} \phi(\alpha a_{n,k})\mu(S_{n,k}) \\
&\geq \sum_{k\geq 1} \phi\left(\left(1+\frac{1}{k}\right)a_{n,k}\right)\mu(S_{n,k}) \qquad (5.1)\\
&\geq \sum_{k\geq k_0} 2^k \phi(a_{n,k})\mu(S_{n,k}) \geq \sum_{k\geq k_0} \frac{\phi(1)}{2^n} = +\infty, \forall\, n \geq 1.
\end{aligned}$$

We get finally $\|\widetilde{f}_n\|_\phi = 1$.

With each $c = (c_n)_{n\geq 1} \in C_0$ we associate the function $\widetilde{f_c} = \sum_{n\geq 1} c_n \widetilde{f_n}$. We assert that $\widetilde{f_c} \in \widetilde{B}^\phi$ a.p. Indeed, if $\widetilde{f_N} = \sum_{n=1}^{N} c_n \widetilde{f_n}$, we have clearly $\widetilde{f_N} \in B^\phi$ a.p. and then it is sufficient to show that the sequence $\{\widetilde{f_N}\}$ is norm convergent to $\widetilde{f_c}$ or that for each $\lambda > 0$, $\lim_{N\to\infty} \rho_\phi(\lambda(\widetilde{f_c} - \widetilde{f_N})) = 0$.

$$\begin{aligned}
\rho_\phi(\lambda(\widetilde{f_c} - \widetilde{f_N})) &= \rho_\phi\Big(\sum_{n\geq N} \lambda c_n \widetilde{f_n}\Big) = \int_0^1 \phi\Big(\sum_{n\geq N} \lambda c_n f_n\Big)\,dt \\
&= \sum_{n\geq N} \int_0^1 \phi(\lambda c_n f_n)\,dt.
\end{aligned} \tag{5.2}$$

Take N such that $\lambda c_n < 1$ for $n \geq N$, we will have,

$$\begin{aligned}
(5.2) &= \sum_{n\geq N}\sum_{k\geq 1} \phi(\lambda c_n a_{n,k})\mu(S_{n,k}) \leq \sum_{n\geq N}\sum_{k\geq 1} \phi(a_{n,k})\mu(S_{n,k}) \\
&\leq \sum_{n\geq N}\Big(\sum_{k\geq 1} \frac{\phi(1)}{2^{n+k}}\Big) \leq \frac{\phi(1)}{2^N}
\end{aligned}$$

which tends to zero when $N \to \infty$. Hence, the space being complete (cf. [4]), we get $\widetilde{f_c} \in \widetilde{B}^\phi$ a.p. Now consider the mapping,

$$\begin{array}{rccc}
P: & C_0 & \longrightarrow & \widetilde{B}^\phi \text{ a.p.} \\
& C = (c_n)_{n\geq 1} & \dashrightarrow & \widetilde{f_c}.
\end{array}$$

We will show that P is an isometry. For $\lambda < \|c\|_\infty$ there exists an n_0 such that $\frac{c_{n_0}}{\lambda} = \alpha_0 > 1$. Consequently:

$$\rho_\phi\Big(\frac{\widetilde{f_c}}{\lambda}\Big) = \int_0^1 \phi\Big(\sum_{n\geq 1} \frac{c_n}{\lambda} f_n\Big)\,dt \geq \int_0^1 \phi(\alpha_{n_0} f_{n_0})\,dt = +\infty, \qquad \text{(see (5.1))}.$$

If $\lambda \geq \|c\|_\infty$, we will have,

$$\rho_\phi\Big(\frac{\widetilde{f_c}}{\lambda}\Big) = \int_0^1 \phi\Big(\sum_{n\geq 1} \frac{c_n}{\lambda} f_n\Big)\,dt \leq \sum_{n\geq 1} \int_0^1 \phi(f_n)\,dt \leq \sum_{n\geq 1}\Big(\sum_{k\geq 1} \frac{\phi(1)}{2^{n+k}}\Big) \leq \phi(1).$$

Finally, $\|\widetilde{f_c}\|_\phi = \|c\|_\infty$ and P is an isometry.

Necessity of the Δ_2-condition for ψ. Since $\phi \in \Delta_2$, from Lemma 2.(ii), $\widetilde{B}^\phi$ a.p. contains an isometric copy of $L^\phi([a,b])$ for each $a, b \in \mathbb{R}$. Now, the Δ_2-condition for ψ is necessary for the reflexivity of the Orlicz spaces $L^\phi([a,b])$, $a, b \in \mathbb{R}$, and hence also for the reflexivity of $\widetilde{B}^\phi$ a.p. $\square$

6. Riesz representation theorem in B^ϕ a.p.

Theorem 2 (Riesz representation theorem)**.** *If $\phi \in \Delta_2 \cap \nabla_2$ then $[B^\phi\,\text{a.p.}]^*$ is isomorphically isometric to B^ψ a.p. More precisely: If G is a continuous linear functional on B^ϕ a.p. then there exists a unique $g \in B^\psi$ a.p. such that:*

(i) $G(f) = M(fg), \qquad \forall\, f \in B^\phi\,\text{a.p.};$

(ii) $\|G\| = |||g|||_\psi.$

Conversely, the condition $\phi \in \Delta_2 \cap \nabla_2$ is necessary for this identification.

Proof: Let us consider the linear mapping

$$A: \begin{array}{ccc} B^\psi\,\text{a.p.} & \rightarrow & [B^\phi\,\text{a.p.}]^* \\ g & \dashrightarrow & A(g) \end{array}, \qquad A(g)(f) = M(fg).$$

A is well defined, and in fact it is an isometry since by Lemma 5:

$$\|A(g)\| = \sup_{\rho_\phi(f) \le 1} |M(fg)| = |||g|||_\psi.$$

It remains only to show that A is surjective. Let $E = A(B^\psi\,\text{a.p.})$. Then E is a complete subspace of $[B^\phi\,\text{a.p.}]^*$. From Banach's classical results, it is sufficient to show that for each $F \in [B^\phi\,\text{a.p.}]^{**}$ such that $F(A(g)) = 0$, $\forall\, g \in B^\psi$ a.p., we have also $F \equiv 0$ i.e. $F(h) = 0$, $\forall\, h \in [B^\phi\,\text{a.p.}]^*$. For, let $F \in [B^\phi\,\text{a.p.}]^{**}$ be such that $F(A(g)) = 0$, $\forall\, g \in B^\psi$ a.p. Since B^ϕ a.p. is reflexive, there exists $f \in B^\phi$ a.p. such that $\pi(f) = F$, (where π is the canonical isometry), i.e.

$$\pi(f)(A(g)) = A(g)(f) = M(fg) = 0,\ \forall\, g \in B^\psi\,\text{a.p.}$$

It follows immediately (see Lemma 5) that $|||f|||_\phi = 0$ and then $\|F\| = 0$.

Conversely, if the identification $[B^\phi\,\text{a.p.}]^* = B^\psi$ a.p. holds, we will also have

$$[B^\phi\,\text{a.p.}]^{**} = [B^\psi\,\text{a.p.}]^* = B^\phi\,\text{a.p.},$$

so that B^ϕ a.p. is reflexive and, consequently, $\phi \in \Delta_2 \cap \nabla_2$. □

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