A β-normal Tychonoff space which is not normal

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Abstract. α-normality and β-normality are properties generalizing normality of topological spaces. They consist in separating dense subsets of closed disjoint sets. We construct an example of a Tychonoff β-normal non-normal space and an example of a Hausdorff α-normal non-regular space.

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The notions of α and β-normality have been introduced recently by A.V. Arhangel'skii and L. Ludwig in [AL].

Definition. A topological space X is called *α-normal* (*β-normal*) whenever for each pair of closed disjoint subsets $A, B \subset X$ there are open sets $U, V \subset X$ such that $\overline{A \cap U} = A$, $\overline{B \cap V} = B$ and $U \cap V = \emptyset$ ($\overline{U} \cap \overline{V} = \emptyset$, respectively).

Arhangel'skii and Ludwig presented several examples of Tychonoff not α-normal and of α-normal not β-normal spaces. They made observations we recall in Propositions 1 and 2.

Proposition 1. *A space X is β-normal if and only if for every A closed and U open, $A \subset U \subset X$, there exists an open $V \subset X$ such that $\overline{A \cap V} = A$ and $\overline{V} \subset U$.*

Proposition 2.

(a) *Every α-normal T_1-space is Hausdorff.*
(b) *Every β-normal T_1-space is regular.*

This leads naturally to the following questions:

1. Does there exist a Hausdorff α-normal non-regular space? ([AL], Question 1).
2. Does there exist a regular β-normal non-Tychonoff space? We shall return to this topic in Remark 4.
3. Does there exist a Tychonoff β-normal not normal space? ([AL], Question 4).

The aim of the present paper is to answer them in the affirmative. Another ZFC example of a Hausdorff α-normal non-regular space was found independently

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by D. Burke. And there is a consistent example of a Tychonoff β-normal not normal space due to Ludwig and Szeptycki [LS].

Let us observe that a positive answer to the third question yields a positive answer to the first one:

Proposition 3. *Suppose X is a regular β-normal not normal space. Let A, B be closed disjoint subsets of X that cannot be separated by open neighborhoods. Then the quotient space $Y = X/A$ obtained by identifying A with a point is Hausdorff, α-normal and not regular.*

Proof: Let $q : X \to Y$ be the canonical quotient map, $\{a\} = q[A]$. It is easy to see that Y is a Hausdorff space and that $q[B]$ and a cannot be separated.

Choose closed disjoint subsets F, H of Y. If $a \notin F \cup H$ then dense parts of F and H can be separated using α-normality of X. Assume that $a \in F$, i.e. $A \subset q^{-1}[F]$ and $A \cap q^{-1}[H] = \emptyset$. The space X is β-normal, hence there is O open in X such that $\overline{q^{-1}[H] \cap O} = q^{-1}[H]$ and $q^{-1}[F] \cap \overline{O} = \emptyset$. Put $U = q[O]$, $V = Y \setminus \overline{U}$. As $\overline{O} \cap A = \emptyset$, U is an open subset of Y and $\overline{U} = q[\overline{O}]$. But $\overline{H \cap U} = H$, $V \supset F$. □

Note that the previous argument is not valid, if α-normality is assumed instead of β-normality.

We give an affirmative answer to the third question in ZFC now.

Example 1. A β-normal Tychonoff space which is not normal.

We shall keep the notation $S = \{\alpha < \omega_2 : \ \mathrm{cf}\, \alpha = \omega_1\}$.

Consider the set

$$X = \big\{(\alpha, \beta) : \ \beta \le \alpha \le \omega_2 \ \& \ (\alpha, \beta) \ne (\omega_2, \omega_2)\big\}$$

and its partition into

$$\begin{aligned} A &= \big\{(\alpha, \alpha) : \ \alpha < \omega_2\big\}, \\ B &= \big\{(\omega_2, \beta) : \ \beta < \omega_2\big\}, \\ D &= \big\{(\alpha, \beta) : \ \beta < \alpha < \omega_2\big\}. \end{aligned}$$

Topologize X in this way: let each $(\alpha, \beta) \in D$ be isolated, let an open base in $(\alpha, \alpha) \in A$ consist of all sets of the type

$$\big\{(\gamma, \gamma) : \ \alpha_0 < \gamma \le \alpha\big\} \cup \bigcup \big\{\{\gamma\} \times C_\gamma : \ \alpha_0 < \gamma \le \alpha \ \& \ \gamma \in S\big\},$$

where $\alpha_0 < \alpha$ and every C_γ is a closed unbounded (club) subset of γ, and let the base in $(\omega_2, \beta) \in B$ consist of all the sets

$$\big\{(\alpha, \gamma) : \ \beta_0 < \gamma \le \beta \ \& \ \alpha_\gamma < \alpha \le \omega_2\big\},$$

where $\beta_0 < \beta$ and $\beta \le \alpha_\gamma < \omega_2$ for each γ.

All above defined basic open neighborhoods are closed, i.e. X is a zero-dimensional (in particular, Tychonoff) space.

Let us disprove normality of X. Consider closed disjoint sets A, B and an open O containing A. Then $\overline{O} \cap B \neq \emptyset$. Indeed, S is a stationary subset of ω_2 and as $\{(\alpha,\alpha) : \alpha \in S\} \subset A \subset O$, for each $\alpha \in S$ there is a C_α club in α such that $\{\alpha\} \times C_\alpha \subset O$. But by Fodor's Pressing Down Lemma, there is an unbounded $R \subset S$ and a $\beta \in \omega_2$ such that $(\forall \alpha \in R)\ \beta \in C_\alpha$. It follows that $(\omega_2, \beta) \in \overline{O}$.

Using Proposition 1 we shall prove β-normality of X.

Assume that $F \subset O \subset X$, F is closed, O open. We are looking for an open U such that $\overline{F \cap U} = F$ and $\overline{U} \subset O$. Note that the only interesting cases are when $F \subset A$ or $F \subset B$. Indeed, for a general F, denote $F_A = F \cap A$, $F_B = F \cap B$, $F_D = F \cap D$ (F_A and F_B are closed). If there are open sets U_1, U_2 such that

$$\overline{F_A \cap U_1} = F_A \ \ \& \ \ \overline{U_1} \subset O,$$
$$\overline{F_B \cap U_2} = F_B \ \ \& \ \ \overline{U_2} \subset O,$$

then $U = U_1 \cup U_2 \cup F_D$ is open, $\overline{F \cap U} \supset \overline{(F_A \cap U_1)} \cup \overline{(F_B \cap U_2)} \cup \overline{F_D} = F$, $\overline{U} = \overline{U_1} \cup \overline{U_2} \cup \overline{F_D} \subset O \cup F = O$.

As $A \simeq B \simeq \omega_2$, $A \cup D$ and $B \cup D$ are (open) normal subspaces of X. Therefore we can assume that $O = X \setminus B$ in case of $F \subset A$ and $O = X \setminus A$ if $F \subset B$. Moreover, for each $\beta_0 < \omega_2$

$$\big\{(\alpha,\beta) \in X :\ \beta \le \beta_0\big\}$$

is a clopen normal subspace. It follows that if $F \subset A$ ($F \subset B$, respectively) and $|F| \le \aleph_1$, we are done.

Let $F \subset A$ (or $F \subset B$) be unbounded (i.e. $|F| = \aleph_2$). The set $F^{is} = F \setminus F'$ is open dense in F, so there is an open subset V of A (of B, respectively) such that $F \cap V = F^{is}$. As $F' \cap V = \emptyset$, V is isomorphic to a nonstationary set in ω_2.

Suppose that $F \subset A$. The set $N = \{\alpha \in S :\ (\alpha,\alpha) \in V\}$ is nonstationary. Hence there is a nondecreasing regressive function $f : N \to \omega_2$ such that

$$\forall_{\beta<\omega_2}\ \big|\{\alpha \in N :\ f(\alpha) = \beta\}\big| \le \aleph_1.$$

For each $\alpha \in N$ put $C_\alpha = \{\gamma :\ f(\alpha) < \gamma < \alpha\}$. Thus

$$U = V \cup \bigcup \big\{\{\alpha\} \times C_\alpha : \alpha \in N\big\}$$

is an open subset of X, $\overline{F \cap U} = \overline{F \cap V} = \overline{F^{is}} = F$ and by the choice of f, $\overline{U} = \overline{V} \cup \bigcup \{\{\alpha\} \times C_\alpha :\ \alpha \in N\}$ does not intersect B, i.e. $\overline{U} \subset O$.

Now, assume that $F \subset B$. The set $C = \omega_2 \setminus \{\beta : \ (\omega_2, \beta) \in V\}$ is club in ω_2, $M = S \setminus C' = \{\alpha \in S : \ \sup(C \cap \alpha) < \alpha\}$ is nonstationary. Fix a regressive nondecreasing function $f : M \to \omega_2$ such that

$$\forall_{\beta<\omega_2} \left|\{\alpha \in M : \ f(\alpha) = \beta\}\right| \leq \aleph_1.$$

For $\alpha \in S$ define

$$C_\alpha = \begin{cases} C \cap \alpha, & \text{if } \ \alpha \in C', \\ \{\gamma : \ f(\alpha) < \gamma < \alpha\}, & \text{otherwise (i.e. if } \ \alpha \in M). \end{cases}$$

Thus, $G = A \cup \bigcup \big\{\{\alpha\} \times C_\alpha : \ \alpha \in S\big\}$ is an open neighborhood of A and $\overline{G} \cap V = \emptyset$. Indeed, for any $(\omega_2, \beta) \in V$, $\{\alpha \in S : \ \beta \in C_\alpha\}$ is bounded: the set is equal to

$$\big\{\alpha \in S \cap C' : \ \beta \in C_\alpha\big\} \cup \big\{\alpha \in M : \ f(\alpha) < \beta\big\},$$

while the first summand is empty and the second one is bounded.

Put $U = X \setminus \overline{G}$. Clearly, U is open and $F = \overline{F \cap V} \subset \overline{F \setminus \overline{G}} = \overline{F \cap U}$. Moreover, $\overline{U} \cap A = \overline{(X \setminus \overline{G})} \cap A = \emptyset$. Hence $\overline{U} \subset O$. □

Remark 4. A construction of F.B. Jones that produces a non-Tychonoff space Y starting from a non-normal space X (see [Jo]) was used by Arhangel'skii and Ludwig to find, under CH, an example of a non-Tychonoff α-normal regular space ([AL, Example 3.3]). It is not hard to prove that if X is regular and β-normal, then so is Y. Hence, starting with X from Example 1, Y is a T_3, non-$T_{3\frac{1}{2}}$ β-normal space.

Let us turn attention to the first question in a particular case.

Proposition 5. *Every first countable Hausdorff α-normal space is regular.*

Proof: We shall assume that X is a first countable Hausdorff non-regular space and prove that it is not α-normal.

There is an $x \in X$ and a closed $F \subset X$ such that $x \notin F$ but x, F cannot be separated by disjoint open sets. Let $\{O_n : \ n \in \omega\}$ be an open base in x, $O_n \supset O_{n+1}$ $(\forall n)$.

Pick, inductively, $x_n \in \overline{O_n} \cap F$; as X is Hausdorff, at each step we can assume that $x_n \notin \overline{O_{n+1}}$. Note that, then,

$$x_n \in \overline{O_m} \Leftrightarrow m \leq n.$$

Define $A = \{x_n : \ n \in \omega\}$. Then $A' = \emptyset$. Indeed, for a $y \neq x$ there is an open set U and $m \in \omega$ such that $y \in U$ and $U \cap \overline{O_m} = \emptyset$. Therefore $U \cap A \subset \{x_0, \ldots, x_{m-1}\}$ and $y \notin A'$.

We conclude that A is closed and has no proper dense subset, while $x \notin A$. As $\{x\}$ and A cannot be separated, they refute α-normality of X. □

The space from the following example is first countable at each point except one. In fact, cardinality of local basis in $\omega_1 \in X$ is equal to the character of Club filter on ω_1. In particular, $\aleph_2 \leq \mathfrak{b}_{\omega_1} \leq \chi(\omega_1, X) \leq \mathfrak{d}_{\omega_1}$, see, e.g., [BS] (5.23–5.25).

Example 2. An α-normal Hausdorff space which is not regular.

Let X denote the space with underlying set $\omega_1 + 1$ and topology τ such that: ω_1 with ordinal topology is an open subspace and a base in the point ω_1 consists of all sets of the type

$$O_C = \{\omega_1\} \cup \{\alpha + 1 : \ \alpha \in C\},$$

where $C \subset \omega_1$ is closed unbounded.

The topology τ is stronger than the order topology on $\omega_1 + 1$, in particular, X is Hausdorff.

The space is not regular: consider club $C \subset \{\alpha < \omega_1 : \ \alpha \text{ is a limit ordinal}\}$. Then $O_C \cap C = \emptyset$, therefore $\omega_1 \notin \overline{C}$ and C is a closed subset of X. But for every club D,

$$\overline{O_D} \cap C \supset D' \cap C \neq \emptyset,$$

hence ω_1 and C cannot be separated.

It remains to prove that X is α-normal. Pick closed disjoint $A, B \subset X$. If $A, B \subset \omega_1$ then they can be separated in the open normal subspace ω_1. Let $B = C \cup \{\omega_1\}$ with $C \subset \omega_1$. It is easy to separate A and B, if A is bounded in ω_1.

Suppose A is unbounded. This implies that C is bounded, in particular $\omega_1 \notin \overline{C}$. Fix open disjoint $U_1, V_1 \subset \omega_1$ such that $A \subset U_1$, $C \subset V_1$. We shall separate ω_1 from $A^{is} = A \setminus A'$.

Let $A = \{a_\alpha : \ \alpha < \omega_1\}$ be an increasing enumeration. In this notation $A^{is} = \{a_0\} \cup \{a_{\alpha+1} : \ \alpha < \omega_1\}$ and it is dense in A. Put

$$\begin{aligned} G_{-1} &= \langle 0, \ a_0 \rangle, \\ G_\alpha &= \begin{cases} \{a_{\alpha+1}\}, & \text{if } \ a_{\alpha+1} = a_\alpha + 1, \\ (a_\alpha + 1, \ a_{\alpha+1}\rangle, & \text{if } \ a_{\alpha+1} > a_\alpha + 1, \end{cases} \\ G &= \bigcup_{-1 \leq \alpha < \omega_1} G_\alpha. \end{aligned}$$

Then G is an open neighborhood of A^{is}.

As $\omega_1 \notin \overline{A} = A$, there is a club D such that $O_D \cap A = \emptyset$, or, equivalently, $\beta \in D \Rightarrow \beta + 1 \notin A$. Moreover, note that $a_\alpha + 1 \in G$ iff $a_{\alpha+1} = a_\alpha + 1$. It is now easy to check that

$$\begin{aligned} G \cap O_{D \cap A} &= \{a_\alpha + 1 : \ \alpha < \omega_1 \ \ \& \ \ a_\alpha \in D \ \ \& \ \ a_{\alpha+1} = a_\alpha + 1\} \\ &= \{\beta + 1 : \ \beta \in D \cap A \ \ \& \ \ \beta + 1 \in A\} = \emptyset. \end{aligned}$$

Put $U = U_1 \cap G$, $V = V_1 \cup O_{D\cap A}$. We have shown that $U \cap V = \emptyset$, $B \subset V$ and $\overline{U \cap A} = A$. $\square$

Referring to Proposition 2, we have constructed an example of an α-normal, not β-normal space.

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