Disasters in metric topology without choice

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Abstract. We show that it is consistent with ZF that there is a dense-in-itself compact metric space (X, d) which has the countable chain condition (ccc), but X is neither separable nor second countable. It is also shown that X has an open dense subspace which is not paracompact and that in ZF the Principle of Dependent Choice, DC, does not imply *the disjoint union of metrizable spaces is normal.*

Keywords: Axiom of Choice, Axiom of Multiple Choice, Principle of Dependent Choice, Ordering Principle, metric spaces, separable metric spaces, second countable metric spaces, paracompact spaces, compact T_2 spaces, ccc spaces.

Classification: 03E25, 54A35, 54D20, 54E35, 54E45, 54F05

1. Introduction

In [14, Theorem 11], we constructed a Fraenkel-Mostowski permutation model $\mathcal{N}$ such that every infinite set $X \in \mathcal{N}$ is Dedekind-infinite, i.e. X *has a countably infinite subset*, and yet there is a dense-in-itself (i.e. there are no isolated points) compact metric space $(A, d) \in \mathcal{N}$ which fails to be separable.

In this paper we strengthen this result by constructing a symmetric model of ZF (the forcing version of $\mathcal{N}$) in which there is a dense-in-itself compact ccc (i.e. every pairwise disjoint family of open sets is countable) metric space (X, d) which is neither separable nor second countable. In fact we will show that (X, d) has no dense subset D which can be expressed as a countable union of finite sets. We also show that X has an open dense subset which is not paracompact.

We recall here a result of [14] which established that the two non-constructive properties, separability and second countability, of compact metric spaces coincide.

Theorem 1.1 ([14]). *The following are equivalent:*

- (i) *the set of all non-empty closed subsets of a compact metric space has a choice function;*
- (ii) *compact metric spaces are separable;*
- (iii) *compact metric spaces are second countable.*

We also construct a second symmetric model of ZF in which DC holds and yet there exists a compact T_2 space having an open dense metrizable subspace which is not paracompact. Thus, DC does not imply Stone's theorem, i.e. metric spaces are paracompact, a fact first established in [4], see the remark at the end of the proof of Theorem 3 in [4].

In the realm of metric spaces and various choice principles a great amount of work has been accomplished by several researchers. Below we give a list, by no means complete, which surveys results that demonstrate the non-constructive character of properties shared by metric spaces. Some of these results demonstrate the exact portion of choice needed for their establishment. For any undefined notion the reader is referred to [11] and [17].

Theorem 1.2 ([4]).

(1) *It is consistent relative to ZF that there is a (locally connected, locally compact) metric space that is not paracompact.*
(2) *It is consistent with ZF that there is a zero-dimensional metric space that is not paracompact.*

Theorem 1.3 ([8]).

(1) *Each one of the following statements implies those beneath it.*
  (a) *MC (the axiom of multiple choice).*
  (b) *Every metric space has a σ-locally finite base.*
  (c) *Metric spaces are paracompact.*
(2) *Metric spaces are paracompact iff they are metacompact.*
(3) *Metric spaces are paracompact does not imply MC in ZF^0-Zermelo-Fraenkel set theory minus the axiom of regularity (basic Fraenkel model).*

Theorem 1.4 ([3]).

(1) *The statement "Lindelöf metric spaces are separable" is not provable in ZF (second Cohen model, [1]).*
(2) *The statement "Second countable metric spaces are Lindelöf " is not provable in ZF (basic Cohen model, [1]).*

Theorem 1.5 ([6]). *In ZF the following conditions are equivalent:*

(1) $\mathbb{N}$ *is Lindelöf.*
(2) $\mathbb{Q}$ *is Lindelöf.*
(3) $\mathbb{R}$ *is Lindelöf.*
(4) *Every subspace of* $\mathbb{R}$ *is separable.*
(5) *In* $\mathbb{R}$*, a point x is in the closure of a set A iff there exists a sequence in A converging to x.*
(6) *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point x iff f is sequentially continuous at x.*
(7) *In* $\mathbb{R}$*, every unbounded subset contains a countable, unbounded set.*
(8) *The axiom of countable choice holds for subsets of* $\mathbb{R}$*.*

Theorem 1.6 ([5]). *The axiom of countable choice CAC is equivalent to the statement: functions between metric spaces are continuous iff they are sequentially continuous.*

Theorem 1.7 ([13]). *The following are equivalent:*

(1) *a metric space is separable iff it has a countable base;*
(2) *the axiom of countable choice for subsets of* $\mathbb{R}$.

Theorem 1.8.
([12]) (1) *The countable multiple axiom of choice, CMC, implies the countable product of metrizable spaces is metrizable.*

(2) *The countable product of metrizable spaces is metrizable implies* CMC_ω (*CMC restricted to families of countable sets*).

([7]) *The multiple choice axiom MC iff "the disjoint union of metrizable spaces is metrizable"* $+\omega$*-MC* (= *for every family* $\mathcal{A} = \{A_i : i \in k\}$ *of non-empty pairwise disjoint sets there exists a family* $\mathcal{F} = \{F_i : i \in k\}$ *of countable non-empty sets such that for every* $i \in k$, $F_i \subseteq A_i$).

([15]) *In all Fraenkel-Mostowski models, Lindelöf metric spaces are separable iff they are second countable iff they have size at most* 2^ω.

([9]) *The axiom of countable choice for subsets of* $\mathbb{R}$ (= $\mathbb{R}$ *is Lindelöf, Theorem* 1.5) *implies each one of the following statements:*

(1) $\aleph_1$ *is a regular cardinal;*
(2) *the countable union of countable subsets of* $\mathbb{R}$ *is countable;*
(3) *the countable union of meager subsets of* $\mathbb{R}$ *is meager.*

2. Main results

In what follows, the notation used is the one established in [16]. For any undefined notion the reader is referred to [16] or [11].

Theorem 2.1. *It is consistent with ZF that there exists a dense-in-itself compact ccc metric space* (X, d) *having no countable dense subset.*

Proof: First we construct a symmetric extension $(\mathcal{N}, \in)$ of a countable transitive model $\mathcal{M}$ satisfying ZF + V = L and such that $\mathcal{N} \subseteq \mathcal{M}[G]$, where $\mathcal{M}[G]$ is a generic extension of $\mathcal{M}$. Then we will show that in $\mathcal{N}$ there exists a dense-in-itself compact metric space (X, d) having no countable dense subset. Let $P = Fn(\omega^+ \times \mathbb{R}^2 \times \omega_1 \times \omega_1, 2, \omega_1)$, $\omega^+ = \omega \backslash 1$, the set of partial functions p with $|p| < \omega_1$, $\mathrm{dom}(p) \subseteq \omega^+ \times \mathbb{R}^2 \times \omega_1 \times \omega_1$ and $\mathrm{ran}(p) \subseteq 2 = \{0, 1\}$. Order P by reverse inclusion, i.e. $p \leq q$ if and only if $p \supseteq q$. Then $(P, \leq)$ is an ω_1-closed poset (partially ordered set) having the empty function $\emptyset$ as its largest element $\mathbf{1}$. Let G be a P-generic set over $\mathcal{M}$ and let $\mathcal{M}[G]$ be the corresponding generic extension of $\mathcal{M}$. In $\mathcal{M}[G]$ define the following sets:

$$x_{nri} = \{j \in \omega_1 : \exists p \in G, p(n,r,i,j) = 1\}, \tag{1}$$

$$X_{nr} = \{x_{nri} : i \in \omega_1\}, \tag{2}$$

$$A_n = \{X_{nr} : r \in B(0,1/n)\}, \tag{3}$$

where $B(0,1/n)$ is the set of points on the circle of radius $1/n$ centered at 0, and

$$A = \{A_n : n \in \omega^+\}. \tag{4}$$

The group of permutations $\mathcal{G}$ is the set of all permutations π on $\omega^+ \times \mathbb{R}^2 \times \omega_1$ satisfying $\pi(n,r,i) = (n,\rho(r),j)$, where $\pi(n,r,.)$ is a permutation on ω_1 for fixed n, r and $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation of $B(0,1/k)$ by an angle $\vartheta_k \in \mathbb{R}$, and ρ is the identity on $\mathbb{R}^2 \setminus \bigcup_{k\in\omega^+} B(0,1/k)$. The effect of $\pi \in \mathcal{G}$ on an element $p \in P$ is as follows: $\pi p(\pi(n,r,i),j) = p(n,r,i,j)$. That is, πp is produced by p by changing the first three coordinates of the five tuples of p according to the dictates of π. The effect of π on an element of $\mathcal{M}$ is defined by $\in$-recursion: $\pi(\emptyset) = \emptyset$ and $\pi(x) = \{\pi(y) : y \in x\}$.

Let $\mathcal{F}$ be the (normal) filter generated by $\{\mathrm{fix}(e) : e \in [\omega^+ \times \mathbb{R}^2 \times \omega_1]^{<\omega}\}$, where $\mathrm{fix}(x) = \{\pi \in \mathcal{G} : \forall\, y \in x,\ \pi(y) = y\}$. A set $s \in \mathcal{M}$ is called *symmetric* if and only if $\mathrm{sym}(x) \in \mathcal{F}$, where $\mathrm{sym}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\}$. s is called *hereditarily symmetric* if and only if every element $z \in \mathrm{TC}(\{s\})$ is symmetric, where $\mathrm{TC}(x)$ is the transitive closure of x. Let HS be the set of all hereditarily symmetric names of $\mathcal{M}$ and let $\mathcal{N} = \{\tau_G : \tau \in HS\}$, where τ_G is the value of the name τ given in Definition 2.7, p. 189, from [16]. As in Theorem 5.14 from [11], it can be verified that $\mathcal{N}$ is an almost universal, transitive class of $\mathcal{M}[G]$ closed under the eight Gödel operations. Hence, $(\mathcal{N}, \in)$ is a model of ZF. Furthermore, according to Theorem 6.14 of [16] we have that $\mathbb{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{N}} = \mathbb{R}^{\mathcal{M}[G]}$, and so $(\mathbb{R}^2)^{\mathcal{M}} = (\mathbb{R}^2)^{\mathcal{N}} = (\mathbb{R}^2)^{\mathcal{M}[G]}$.

Claim 1. The sets x_{nri}, X_{nr}, A_n, A defined by (1), (2), (3) and (4) respectively, belong to $\mathcal{N}$.

Proof of Claim 1: Define the following names:
$t_{nri} = \{(\check{j}, p) : j \in \omega_1 \wedge p \in P \wedge p((n,r,i,j)) = 1\}$,
$T_{nr} = \{(t_{nri}, \mathbf{1}) : i \in \omega_1\}$,
$S_n = \{(T_{nr}, \mathbf{1}) : r \in B(0,1/n)\}$,
$S = \{(S_n, \mathbf{1}) : n \in \omega^+\}$.

Clearly t_{nri}, T_{nr}, S_n, S are names for x_{nri}, X_{nr}, A_n and A respectively. Since their elements are hereditarily symmetric, it suffices to show that these names are symmetric. It is straightforward to verify that $\mathrm{fix}(\{(n,r,i)\}) \subseteq \mathrm{sym}(t_{nri})$, so t_{nri} is symmetric. Let $i_0 \in \omega_1$. Then $\mathrm{fix}(\{(n,r,i_0)\}) \subseteq \mathrm{sym}(T_{nr})$ and so T_{nr} is also symmetric. Finally, it is evident that $\mathrm{sym}(S_n) = \mathrm{sym}(S) = \mathcal{G}$. This completes the proof of Claim 1. □

Claim 2. The family A given by (4) is countable in $\mathcal{N}$.

Proof of Claim 2: $\underline{A} = \{(\mathrm{op}(\check{n}, S_n), \mathbf{1}) : n \in \omega^+\}$, where $\mathrm{op}(\sigma,\tau)$ is the name given in Definition 2.16, p. 191 in [16], is a hereditarily symmetric name ($\mathrm{sym}(\underline{A}) = \mathcal{G}$) for the enumeration $\{(n, A_n) : n \in \omega^+\}$ of A. □

Claim 3. The family $A = \{A_n : n \in \omega^+\}$ has no multiple choice function in $\mathcal{N}$.

Proof of Claim 3: Assume the contrary and let $f \in \mathcal{N}$ be a multiple choice function for A. Let F be a hereditarily symmetric name for f and let $e \in [\omega^+ \times \mathbb{R}^2 \times \omega_1]^{<\omega}$ such that $\mathrm{fix}(e) \subseteq \mathrm{sym}(F)$. Then there exists $p_0 \in G$ such that

$$p_0 \Vdash (F \text{ is a function}) \wedge$$
$$[(\forall n)(n \in (\omega^+)\check{} \rightarrow F(n) \text{ is a non-empty finite subset of } S_n)].$$

Since e is finite, fix $n \in \omega^+$ such that $e \cap (\{n\} \times \mathbb{R}^2 \times \omega_1) = \emptyset$. Since A_n is infinite and $f(n)$ is finite, there are $r, s \in B(0, 1/n)$ and $p \leq p_0$ such that

$$p \Vdash T_{nr} \in F(\check{n}) \wedge T_{ns} \notin F(\check{n}).$$

As $|p| < \omega_1$, there is an $i \in \omega_1$ so that for all $k \geq i$ and all $t \in \mathbb{R}^2$ and $j \in \omega_1$, $(n, t, k, j) \notin \mathrm{dom}(p)$. Let ϑ be the central angle corresponding to the arc $\widehat{rs}$ and let $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the permutation on $\mathbb{R}^2$ which is the identity map on $\mathbb{R}^2 \backslash B(0, 1/n)$ and $\sigma = \rho|_{B(0,1/n)}$ is a rotation of $B(0, 1/n)$ by the angle ϑ. Also let $\varphi : [0, i] \rightarrow [i+1, 2i]$ be an order preserving bijection. Let π be the permutation on $\omega^+ \times \mathbb{R}^2 \times \omega_1$ defined by

$$\pi(m,t,k) = \begin{cases} (m,t,k),\ m \neq n, \\ (n,t,\varphi(k)),\ m = n,\ t \in \mathbb{R}^2 \backslash B(0,1/n),\ k \in [0,i], \\ (n,t,\varphi^{-1}(k)),\ m = n,\ t \in \mathbb{R}^2 \backslash B(0,1/n),\ k \in [i+1,2i], \\ (n,\sigma(t),\varphi(k)),\ m = n,\ t \in B(0,1/n),\ k \in [0,i], \\ (n,\sigma(t),\varphi^{-1}(k)),\ m = n,\ t \in B(0,1/n),\ k \in [i+1,2i], \\ (n,t,k),\ m = n,\ t \in \mathbb{R}^2 \backslash B(0,1/n),\ k > 2i, \\ (n,\sigma(t),k),\ m = n,\ t \in B(0,1/n),\ k > 2i. \end{cases}$$

By the definition of π we have that $\pi \in \mathrm{fix}(e)$, so $\pi(F) = F$. Furthermore, $\pi(T_{nr}) = T_{n\sigma(r)} = T_{ns}$. Next we show that p and $\pi(p)$ are compatible. It suffices to show that $p(x) = \pi p(x)$ for every $x \in \mathrm{dom}(p) \cap \mathrm{dom}(\pi p)$. Suppose that $(m, t, k, j) \in \mathrm{dom}(p) \cap \mathrm{dom}(\pi p)$ for $m \neq n$. Then by the definition of π we have that $p(m, t, k, j) = \pi p(m, t, k, j)$. On the other hand since $(n, t, k, j) \notin \mathrm{dom}(p)$ for all $k \geq i$, it is easily seen by the definition of π that it is impossible to have $(n, t, k, j) \in \mathrm{dom}(p) \cap \mathrm{dom}(\pi p)$.

Thus, $g = p \cup \pi p$ is a well defined extension of p. Now we have that $g \Vdash \pi T_{nr} \in \pi F(\check{n}) \wedge \pi T_{ns} \notin \pi F(\check{n})$, and so $g \Vdash T_{ns} \in F(\check{n})$ and since $g \leq p$ we also have that $g \Vdash T_{ns} \notin F(\check{n})$. This contradiction completes the proof of Claim 3. $\square$

Mimicking the proof of Claim 3 we readily have that the Kinna-Wagner selection principle also fails for the family A. In $\mathcal{M}[G]$ define the following sets:

$$d_n = \big\{(X_{nr}, X_{ns}, \rho(r,s)) : r, s \in B(0, 1/n)\big\},$$
$$Z = \big\{(X_{mr}, X_{ns}, \max\{1/n, 1/m\}) : m \neq n, r \in B(0,1/m), s \in B(0,1/n)\big\},$$

where ρ is the Euclidean metric. It is easy to see that $d_n, Z \in \mathcal{N}$. For instance, $\underline{d_n} = \{(\mathrm{op}(\mathrm{op}(T_{nr}, T_{ns}), (\rho(r,s))\check{}\ ), \mathbf{1}) : r, s \in B(0, 1/n)\}$ is a hereditarily symmetric name for d_n. Thus, $d = Z \cup \bigcup_{n \in \omega^+} d_n \in \mathcal{N}$ and clearly d is a metric on $Y = \bigcup A$.

Claim 4. For every $n \in \omega^+$, (A_n, d) is compact.

Proof of Claim 4: Fix $n \in \omega^+$ and let $\mathcal{U}$ be an open cover of A_n in $\mathcal{N}$. As each $U \in \mathcal{U}$ is expressible as a union of open discs we may assume without loss of generality that each member of $\mathcal{U}$ is an open disc. Let f be the bijection $X_{nx} \mapsto x$, $x \in B(0, 1/n)$. It can be easily verified that $f \in \mathcal{N}$. Then $f(\mathcal{U})$ is an open cover of $B(0,1/n)$ and since $B(0,1/n)$ is compact, $f(\mathcal{U})$ has a finite subcover $\mathcal{V}$. Clearly $f^{-1}(\mathcal{V})$ is an open cover of A_n and $f^{-1}(\mathcal{V}) \in \mathcal{N}$. $\square$

Let $X = \{*\} \cup Y$ and define a function $d^* : X \times X \to \mathbb{R}$ by requiring:

$$d^*(x,y) = d^*(y,x) = \begin{cases} d(x,y) & \text{if } x \in A_n,\ y \in A_m, \\ 1/n & \text{if } x \in A_n \text{ and } y = *, \\ 0 & \text{if } x = y = *. \end{cases}$$

Clearly (X, d^*) is a dense-in-itself compact metric space in $\mathcal{N}$.

Claim 5. X has the ccc.

Proof of Claim 5: Let $\mathcal{U}$ be a disjoint family of non-empty open sets in X. Since every neighborhood of $*$ contains all but finitely many A_n and each A_n has the ccc, we may assume without loss of generality (wlog) that $* \notin U$ for all $U \in \mathcal{U}$. For each $n \in \omega^+$, put $\mathcal{U}_n = \{U \in \mathcal{U} : U \cap A_n \neq \emptyset\}$. Wlog assume that $\mathcal{U}_n \neq \emptyset$ for all $n \in \omega^+$. Since every infinite subfamily of A has no Kinna-Wagner selection function, there is an $N \in \omega^+$ such that for all $n \geq N$, $\bigcup \mathcal{U}_n = A_n$. Furthermore, as each A_n is connected, we have that for all $n \geq N$, $|\mathcal{U}_n| = 1$. Namely $\mathcal{U}_n = \{A_n\}$ for $n \geq N$. Thus, $\mathcal{U}$, being a finite union of countable sets, is countable. This completes the proof of Claim 5. $\square$

We assert now that X has no dense set D which is expressible as $\bigcup\{D_n : n \in \omega\}$, where each D_n is a non-empty finite set. Assume the contrary and let $D = \bigcup\{D_n : n \in \omega\}$ be a dense subset of X. Since each A_n is clearly an open set of X, we have that $D \cap A_n \neq \emptyset$ for all $n \in \omega^+$. On the basis of $\{D_n : n \in \omega\}$ one can now readily define a multiple choice function on $\{A_n : n \in \omega^+\}$. This contradicts Claim 3. Finally, by Theorem 1.1 we have that X is not second countable either. This completes the proof of Theorem 2.1. $\square$

Remark. Working as in the proof of Theorem 2 of [4] one can verify that the open, dense metric subspace (Y, d) of X defined in the proof of Theorem 2.1 is not paracompact. Furthermore, by Claim 2 and Claim 3 we see that the countable axiom of choice CAC fails in $\mathcal{N}$. Since DC implies CAC, it follows that DC also fails in $\mathcal{N}$.

Similarly to the model of Theorem 1 in [4] one expects that the axiom of choice for families of pairs fails also in the model $\mathcal{N}$ of Theorem 2.1. Indeed this is the case as the next theorem clarifies. In fact we show something more, namely, $\mathrm{AC}_n, n \geq 2$, i.e. the axiom of choice for families of n-element sets, fails in $\mathcal{N}$. Thus the Ordering Principle, OP, and consequently the Boolean Prime Ideal Theorem, BPI, fail in the model $\mathcal{N}$.

Theorem 2.2. *Let $\mathcal{N}$ be the model defined in the proof of Theorem 2.1. Then for all $n \geq 2$, AC_n fails in $\mathcal{N}$.*

Proof: Fix an integer $n \geq 2$ and define $\mathcal{A} = \{X : (\exists m \in \omega^+)\ X \subseteq A_m$ and $|X| = n\}$. $\mathcal{A} \in \mathcal{N}$ because $\mathrm{sym}(\underline{\mathcal{A}}) = \mathcal{G}$. We show that $\mathcal{A}$ has no choice function in $\mathcal{N}$. Assume the contrary and let f be such a function with symmetric name F. Let $E \in [\omega^+ \times \mathbb{R}^2 \times \omega_1]^{<\omega}$ so that $\mathrm{fix}(E) \subseteq \mathrm{sym}(F)$. Fix $n_0 \in \omega^+$ so that $E \cap (\{n_0\} \times \mathbb{R}^2 \times \omega_1) = \emptyset$ and let $Z_{n_0} = \{X_{n_0 r_1}, X_{n_0 r_2}, \ldots, X_{n_0 r_n}\}, r_i \in B(0, 1/n_0)$ for $i \leq n$, be such that $\widehat{r_1 r_2} = \widehat{r_2 r_3} = \cdots = \widehat{r_{n-1} r_n} = \widehat{r_n r_1} = \theta$. Wlog assume that $f(Z_{n_0}) = X_{n_0 r_1}$. Then there exists $p \in G$ so that

$$p \Vdash (F \text{ is a function}) \ \wedge\ (F(\underline{Z_{n_0}}) = T_{n_0 r_1}),$$

where $\underline{Z_{n_0}} = \{(T_{n_0 r_1}, \mathbf{1}), \ldots, (T_{n_0 r_n}, \mathbf{1})\}$. Let i and φ be as in Claim 3 of the proof of Theorem 2.1. Define a permutation ψ by

$$\psi(m,r,k) = \begin{cases} (m,r,k),\ m \neq n_0, \\ (n_0, r, \varphi(k)),\ m = n_0,\ r \in \mathbb{R}^2 \backslash B(0,1/n_0),\ k \in [0,i], \\ (n_0, r, \varphi^{-1}(k)),\ m = n_0,\ r \in \mathbb{R}^2 \backslash B(0,1/n_0),\ k \in [i+1, 2i], \\ (n_0, \sigma(r), \varphi(k)),\ m = n_0,\ r \in B(0,1/n_0),\ k \in [0,i], \\ (n_0, \sigma(r), \varphi^{-1}(k)),\ m = n_0,\ r \in B(0,1/n_0),\ k \in [i+1,2i], \\ (n_0, r, k),\ m = n_0,\ r \in \mathbb{R}^2 \backslash B(0,1/n_0),\ k > 2i, \\ (n_0, \sigma(r), k),\ m = n_0,\ r \in B(0,1/n_0),\ k > 2i. \end{cases}$$

where σ is a rotation of $B(0, 1/n_0)$ by θ. As in the proof of Claim 3 in Theorem 2.1, we have that $\psi \in \mathrm{fix}(E)$ and p, ψp are compatible. Thus,

$$p \cup \psi p \Vdash \psi F(\psi \underline{Z_{n_0}}) = \psi(T_{n_0 r_1}),$$

and consequently $p \cup \psi p \Vdash F(\underline{Z_{n_0}}) = T_{n_0 r_2}$. Since $p \cup \psi p \leq p$, we also have that $p \cup \psi p \Vdash F(\underline{Z_{n_0}}) = T_{n_0 r_1}$. This contradiction completes the proof of Theorem 2.2. □

Theorem 2.3. *It is consistent with ZF + DC that there exists a compact T_2 space which has an open, dense, locally compact and non-paracompact metric subspace.*

Proof: Let $P = Fn(\mathbb{R}^+ \times \mathbb{R}^2 \times \omega_1 \times \omega_1, 2, \omega_1)$ partially ordered by reverse inclusion. The group of permutations $\mathcal{G}$ is the set of all permutations π on $\mathbb{R}^+ \times \mathbb{R}^2 \times \omega_1$ satisfying $\pi(n, r, i) = (n, \rho(r), j)$, where $\pi(n, r, .)$ is a permutation on ω_1 for fixed n, r and $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation of $B(0, k)$ by an angle $\vartheta_k \in \mathbb{R}$. Let $\mathcal{F}$ be the normal filter generated by $\{\mathrm{fix}(e) : e \in [\mathbb{R}^+ \times \mathbb{R}^2 \times \omega_1]^{<\omega_1}\}$ and let $\mathcal{N}$ be the symmetric model. As in the proof of Theorem 2.1, in $\mathcal{M}[G]$ define sets x_{nri}, X_{nr}, A_n, A. Put $Y = \bigcup A$ and let

$$d_n = \big\{(X_{nr}, X_{ns}, \frac{\rho(r,s)}{1+\rho(r,s)}) : r, s \in B(0,n)\big\},$$
$$Z = \big\{(X_{mr}, X_{ns}, 1) : m \neq n, r \in B(0,m), s \in B(0,n)\big\}.$$

Then following the proof of Theorem 2 in [4] we have that $(Y, d), d = Z \cup (\bigcup_{n \in \mathbb{R}^+} d_n)$ is a non-paracompact metric space. Taking now the one-point compactification of (Y, d) one readily obtains a compact T_2 space with the required properties. Moreover, by Lemma 8.5 in [11] we may conclude that DC holds in $\mathcal{N}$ finishing the proof of Theorem 2.3. □

Remarks. 1. In Table 1 of [10] (see also `http://www.math.purdue.edu/~jer/`) the status of the implication DC $\to$ *compact* T_2 *spaces are weakly Loeb* (= Form 116 in [10]) is indicated as unknown, where a topological space is weakly Loeb if the set of its non-empty closed subsets has a multiple choice function. Now the one-point compactification of the metric space of Theorem 2.3 is a compact T_2 space which fails to be weakly Loeb (the set $A = \{A_n : n \in \mathbb{R}^+\}$ of closed subsets has no multiple choice set). Therefore, the above implication fails in ZF.
2. In Table 1 of [10] the status of the implication DC $\to$ *metric spaces have a σ-locally finite base* (= Form 232 in [10]) is also indicated as unknown. By Theorem 1.3 of the introduction we immediately deduce that the metric space of Theorem 2.3 cannot have a σ-locally finite base.

Another implication whose status is indicated as unknown in Table 1 of [10] is DC $\to$ *the disjoint union of metrizable spaces is normal* (= Form 382 in [10]). We show next that this implication fails in ZF.

Theorem 2.4. *Let $\mathcal{M}$ be a countable transitive model of ZF + V = L. For each regular cardinal λ, there is a symmetric extension $\mathcal{N}$ of $\mathcal{M}$ satisfying $\forall\,\kappa < \lambda$ DC_κ and the negation of Form* 382.

Proof: Fix a regular cardinal λ and let P, $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{N}$ be defined as in the remark following Theorem 3 in [4]. In $\mathcal{M}[G]$ define the following sets:

$$x_{nri} = \{j \in \lambda : \exists p \in G, p(n,r,i,j) = 1\}, \tag{5}$$

$$X_{nr} = \{x_{nri} : i \in \lambda\}, \tag{6}$$

$$A_n = \{X_{nr} : r \in \mathbb{R}\}, \tag{7}$$

$$A = \{A_n : n \in \lambda\}. \tag{8}$$

Arguing as in Theorem 2.1, we have that the above sets belong to $\mathcal{N}$, the family A has no Kinna-Wagner selection function in $\mathcal{N}$ and a metric d_n can be defined on each A_n. Moreover, using the natural ordering of $\mathbb{R}$ one can readily define a linear order on each A_n which has a symmetric name, thus it belongs to $\mathcal{N}$. As in [2], for each $n \in \lambda$, let $B_n = A_n \cup \{a_n, b_n\}$, where a_n, b_n are distinct sets of $\mathcal{N}$ not belonging to A_n and extend the linear order of A_n by requiring a_n to be the least element and b_n to be the largest element. Clearly each B_n with the order topology is a metrizable space (by Corollary 4.8 in [3], Urysohn's metrization theorem, i.e. *a* T_3 *second countable space is metrizable*, holds in ZF). Let X be the disjoint topological union of the B_n. If X were normal, then for the disjoint closed subsets $B = \{a_n : n \in \lambda\}$ and $C = \{b_n : n \in \lambda\}$ there exist disjoint open sets U and V so that $B \subseteq U$ and $C \subseteq V$. But then $\{(n, A_n \backslash U) : n \in \lambda\}$ is a Kinna-Wagner selection function for A. This contradiction completes the proof of Theorem 2.4. $\square$

Remarks. 1. As the statement *the disjoint union of metrizable spaces is normal* is clearly deducible from *the disjoint union of metrizable spaces is metrizable*, we see that DC does not imply the latter statement either.

2. DC does not imply the statement "*for every metric space* (X,d)*, if every family of pairwise disjoint open sets in* X *is well ordered, then* X *has a well ordered dense subset*". Indeed, the disjoint topological union $Z = \bigcup_{n<\lambda} A_n$ of the proof of Theorem 2.4 is metrizable in $\mathcal{N}$ and every family of pairwise disjoint open sets in Z is well ordered (this can be proved following similar ideas as in Claim 5 of Theorem 2.1) but, Z has no well ordered dense subsets.

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