Sequentially compact sets in a class of generalized Orlicz spaces

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Abstract. In this paper, we will characterize sequentially compact sets in a class of generalized Orlicz spaces.

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§1. Introduction and basic results

Definition 1.1. $M : \mathbb{R} \longrightarrow \mathbb{R}$ is called an *Orlicz function* if it has the following properties:

(1) M is even, continuous, convex on $(0, \infty)$ and $M(0) = 0$,

(2) $M(u) > 0$ for all $u \neq 0$,

(3) $\lim_{u \to 0} \frac{M(u)}{u} = 0$ and $\lim_{u \to +\infty} \frac{M(u)}{u} = +\infty$.

Definition 1.2. A function $\phi : [0, \infty) \longrightarrow [0, \infty)$ is called a *φ-function*, if ϕ satisfies

(i) $\phi(0) = 0, \phi(u) > 0, u > 0$;

(ii) $\phi(u)$ is increasing, continuous;

(iii) $\lim_{x \to \infty} \phi(x) = \infty$.

A φ-function $\phi(u)$ is said to satisfy the $\triangle_2$-*condition* for small u (for all $u \geq 0$ or for large u), in symbol $\phi \in \triangle_2(0)$ ($\phi \in \triangle_2$ or $\phi \in \triangle_2(\infty)$), if there exists $u_0 > 0$ and $c > 0$ such that $\phi(2u) \leq c\phi(u)$ for $0 \leq u \leq u_0$ (for all $u \geq 0$ or for $u \geq u_0$). The generalized Orlicz class, generalized Orlicz space and subspace, respectively, generated by a φ-function ϕ are defined as follows:

$$
\begin{aligned}
\widetilde{L}^{\phi}(G) &= \big\{x(t) : x(t) \text{ is measurable on a Lebesgue measurable set } G \text{ and} \\
&\qquad \rho_{\phi}(x) = \int_G \phi(|x(t)|)\,dt < \infty\big\}, \\
L^{\phi}(G) &= \big\{x(t) : x(t) \text{ is measurable on a Lebesgue measurable set } G \text{ and} \\
&\qquad \rho_{\phi}(\lambda x) < \infty \text{ for some } \lambda > 0\big\}, \\
E^{\phi}(G) &= \big\{x(t) : x(t) \text{ is measurable on a Lebesgue measurable set } G \text{ and} \\
&\qquad \rho_{\phi}(\lambda x) < \infty \text{ for all } \lambda > 0\big\}.
\end{aligned}
$$

Now we denote by M^{-1} the inverse function to an Orlicz function M on $[0,\infty)$. It is obvious that M^{-1} is a φ-function and it is a concave function. Since $M^{-1}(2u) < 2M^{-1}(u)$ for all $u \geq 0$, we have $M^{-1} \in \triangle_2$. It follows that $\widetilde{L}^{M^{-1}}(G) = L^{M^{-1}}(G) = E^{M^{-1}}(G)$ (see [1]). So we only study one of them. The theory of generalized Orlicz spaces can be found in [2]. For convenience, we give the following definition.

Definition 1.3. Let M^{-1} be the inverse function to an Orlicz function M on $[0,\infty)$, let (Ω,Σ,μ) be a measure space. We define a class of generalized Orlicz spaces as follows:

$$L^{M^{-1}}(\Omega,\Sigma,\mu) = \{x(t) : x(t) \text{ is } \mu \text{ measurable and } \rho_{M^{-1}}(x) < \infty\}. \tag{1}$$

Let X be a vector space over $\mathbb{K}$. $\|\cdot\| : X \to \mathbb{R}$ is called an F-norm if

(i) for every $x \in X$, $\|x\| \geq 0$, and $\|x\| = 0 \Longleftrightarrow x = 0$,
(ii) $\|-x\| = \|x\|$,
(iii) $\|x+y\| \leq \|x\| + \|y\|$,
(iv) $\|x_n\| \to 0 \Longrightarrow \|\alpha x_n\| \to 0$ for $\alpha \in \mathbb{K}$ and $\alpha_n \to 0 \Longrightarrow \|\alpha_n x\| \to 0$ for $x \in X$.

As in a generalized Orlicz space, we can define the normal F-norm in $L^{M^{-1}}$ as follows:

$$\|x\|_{M^{-1}} = \inf\left\{c > 0 : \rho_{M^{-1}}\left(\frac{x}{c}\right) \leq c\right\}. \tag{2}$$

But we can define a more simple F-norm in $L^{M^{-1}}$ as in Theorem 1.1 below. The new F-norm is equivalent to the normal F-norm (2). Using the new simple F-norm, we can investigate $L^{M^{-1}}$ which we can call a *generalized Orlicz space generated by the Orlicz function M*. These generalized Orlicz spaces include L^p $(0 < p < 1)$.

Theorem 1.1. *$\|x\|_{(M^{-1})} = \rho_{M^{-1}}(x)$, $x \in L^{M^{-1}}$, is an F-norm in $L^{M^{-1}}$, and $L^{(M^{-1})} = (L^{M^{-1}}, \|\cdot\|_{(M^{-1})})$ is a linear complete space with F-norm $\|\cdot\|_{(M^{-1})}$. So it is a Fréchet space.*

Proof: By the basic properties of an Orlicz function M (see [1]), we have

$$M^{-1}(|u+v|) \le M^{-1}(|u|+|v|) \le M^{-1}(|u|) + M^{-1}(|v|), \tag{3}$$

$$M^{-1}(\lambda|u|) \le \lambda M^{-1}(|u|), \quad \lambda \ge 1, \tag{4}$$

$$M^{-1}(\lambda|u|) \le M^{-1}(|u|), \quad 0 < \lambda < 1. \tag{5}$$

These estimates imply that $L^{(M^{-1})}$ is a linear space and $\|\cdot\|_{(M^{-1})}$ is an F-norm. □

Some properties of the generalized space $L^{(M^{-1})}$ have been established in [3] by M.M. Rao and Z.D. Ren. In this paper, we will discuss the criteria for a set to be sequentially compact in $L^{(M^{-1})}$.

The three criteria for a set to be sequentially compact in $L^p[a,b]$ $(1 < p < \infty)$ were introduced by F. Riesz [4], Kolomogorov [5] and Krasnoselskii [6], respectively. The three criteria were generalized to $E^M[a,b]$, a closed subset of Orlicz space $L^M[a,b]$, by Takahashi [7], Gribanov [8] and Krasnoselskii [6], respectively. In 1951, Tsuji [9] generalized the F. Riesz criterion to the space L^p $(0 < p < 1)$.

Now we generalize the F. Riesz criterion and Krasnoselskii criterion for sequential compactness to the *generalized Orlicz space generated by the Orlicz function* $L^{(M^{-1})}(G)$.

§2. The Riesz criterion for sequential compactness in $L^{(M^{-1})}$

A set $A \subset L^{(M^{-1})}[a,b]$ is said to be *sequentially compact* if, for any $\{x_n(t)\}_{n=1}^{\infty} \subset A$, there exists a subsequence $\{x_{n_i}(t)\}_{i=1}^{\infty}$ and $x_0(t) \in L^{(M^{-1})}[a,b]$ such that $\lim_{i\to\infty} \|x_{n_i} - x_0\|_{(M^{-1})} = 0$.

Lemma 2.1. *Let $M(u) = \int_0^{|u|} p(t)\,dt$ be an Orlicz function. Then for $y > x > 0$,*

$$\frac{y-x}{p[M^{-1}(y)]} \le M^{-1}(y) - M^{-1}(x) \le \frac{y-x}{p[M^{-1}(x)]}\,. \tag{6}$$

Proof: Let $0 < t_1 < t_2$. Since

$$M(t_2) - M(t_1) = \int_{t_1}^{t_2} p(t)\,dt$$

and $p(t_1) \le p(t) \le p(t_2)$ $(t_1 \le t \le t_2)$, we have

$$p(t_1)(t_2 - t_1) \le M(t_2) - M(t_1) \le p(t_2)(t_2 - t_1)$$

or equivalently,

$$\frac{1}{p(t_2)}(M(t_2) - M(t_1)) \le t_2 - t_1 \le \frac{1}{p(t_1)}(M(t_2) - M(t_1)).$$

Since $M(t)$ is strictly increasing, $M(0) = 0$ and $M(\infty) = \infty$, we can set $x = M(t_1), y = M(t_2)$. Thus we get (6). □

Lemma 2.2. *Let $M(u) = \int_0^{|u|} p(t)\,dt$ be an Orlicz function and let $M \in \triangle_2$, then*

(i) *for nonnegative real numbers $x \ne y$,*

$$M^{-1}(|y-x|) \le \frac{\overline{B}_M \max\{p(M^{-1}(x)), p(M^{-1}(y))\}}{p[M^{-1}(|y-x|)]} |M^{-1}(|y|) - M^{-1}(|x|)|, \tag{7}$$

where

$$\overline{B}_M = \sup_{t>0} \frac{tp(t)}{M(t)}\,. \tag{8}$$

(ii) *Let $a > 1$ and $0 < y < x < ay$, then*

$$p[M^{-1}(x)] < a\overline{B}_M p[M^{-1}(y)]. \tag{9}$$

Proof: (i). Since $M \in \triangle_2$, we have $\overline{B}_M < \infty$ (see [1]). So, by (8), for nonnegative real numbers $x \ne y$, we have

$$\frac{M^{-1}(|y-x|)p[M^{-1}(|y-x|)]}{|y-x|} \le \overline{B}_M$$

or equivalently

$$|y-x| \ge \frac{1}{\overline{B}_M}\{M^{-1}(|y-x|)p[M^{-1}(|y-x|)]\}.$$

By Lemma 2.1, we get

$$\begin{aligned} |M^{-1}(|y|) - M^{-1}(|x|)| &\ge \frac{1}{\max\{p(M^{-1}(|x|)), p(M^{-1}(|y|))\}}|y-x| \\ &\ge \frac{M^{-1}(|y-x|)p[M^{-1}(|y-x|)]}{\overline{B}_M \max\{p(M^{-1}(|x|)), p(M^{-1}(|y|))\}}\,, \end{aligned}$$

i.e.,

$$M^{-1}(|y-x|) \le \frac{\overline{B}_M \max\{p(M^{-1}(|x|)), p(M^{-1}(|y|))\}}{p[M^{-1}(|y-x|)]} |M^{-1}(|y|) - M^{-1}(|x|)|.$$

(ii) Since $\frac{M^{-1}(y)p[M^{-1}(y)]}{y} > 1, y > 0$, by (8), we have

$$p[M^{-1}(ay)] \le \overline{B}_M \frac{ay}{M^{-1}(ay)} < a\overline{B}_M \frac{y}{M^{-1}(y)} < a\overline{B}_M p[M^{-1}(y)].$$

So $p[M^{-1}(x)] \le p[M^{-1}(ay)] \le a\overline{B}_M p[M^{-1}(y)]$. □

For the proof of the main theorem, we need the following result.

Lemma 2.3 (Riesz). *$F \subset L^p[a,b]$ $(1 \le p < \infty)$ is a sequentially compact set if and only if*

(1) *F is bounded: $\exists\, c > 0$, such that $\sup_{x\in F} \|x(t)\|_p \le c < \infty$;*
(2) *F is equicontinuous: $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that for $|h| < \delta$,*

$$\sup_{x\in F} \|x(t+h) - x(t)\|_p < \varepsilon.$$

The main result in this section is

Theorem 1.1. *Let $M \in \triangle_2$ be an Orlicz function. Then a set $F \subset L^{(M^{-1})}[a,b]$ is sequentially compact if and only if*

(1) *F is bounded: $\exists\, c > 0$, such that*

$$\sup_{x\in F} \int_a^b M^{-1}(|x(t)|)\, dt \le c. \tag{10}$$

(2) *F is equicontinuous: $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that for $|h| < \delta$,*

$$\sup_{x\in F} \int_a^b M^{-1}(|x(t+h) - x(t)|)\, dt < \varepsilon. \tag{11}$$

Proof: *Sufficiency.* If $x(t) \in L^{(M^{-1})}[a,b]$, then $M^{-1}(|x(t)|) \in L^1[a,b]$. Now we consider the function family $F' = \{M^{-1}(|x(t)|) : x(t) \in F\} \subset L^1[a,b]$. By condition (1), we know F' is bounded in $L^1[a,b]$, i.e, $\|M^{-1}(|x(t)|)\|_1 \le c$ for all $M^{-1}(|x(t)|) \in F'$. For every Orlicz function $M(u)$ and $x(t) \in L^{(M^{-1})}[a,b]$, by basic properties of Orlicz functions (see [1]), we have

$$|M^{-1}(|x(t+h)|) - M^{-1}(|x(t)|)| \le M^{-1}(|x(t+h) - x(t)|).$$

So by (2), $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that for $|h| < \delta$, we have

$$\sup_{M^{-1}(|x(t)|)\in F'} \int_G |M^{-1}(|x(t+h)|) - M^{-1}(|x(t)|)|\, dt < \varepsilon.$$

By Lemma 2.3 we know that $F' \subset L^1[a,b]$ is sequentially compact in $L^1[a,b]$. So for any $\{x_n\}_{n=1}^\infty \subset F$ there exists a subsequence (denoted without risk of confusion still by $\{x_n\}_{n=1}^\infty$) and $x_0 \in L^1[a,b]$, such that $\{M^{-1}(|x_n|)\}_{n=1}^\infty \xrightarrow{\|\cdot\|_1} x_0$.

Since $M \in \triangle_2$, $\forall\, 0 < \varepsilon < 6c$ (where c is introduced in condition (1)), $\exists\, K_\varepsilon > 0$, such that for $u > 0$, we have $M(\frac{u}{\frac{\varepsilon}{6c}}) \le K_\varepsilon M(u)$, or equivalently,

$$M^{-1}(\frac{v}{K_\varepsilon}) \le \frac{\varepsilon}{6c} M^{-1}(v), \quad v > 0. \tag{12}$$

Choose $\varepsilon' < \varepsilon$, such that $\sqrt{\varepsilon'} < \frac{1}{K_\varepsilon}$, $\sqrt{\varepsilon'}\mu([a,b]) < \frac{\varepsilon}{3}$ and $\overline{B}_M^2\sqrt{\varepsilon'} < \frac{\varepsilon}{3}$. Thus for any $n \in \mathbb{N}$, by (12) we get

$$\begin{aligned} \int_{[a,b]} M^{-1}(\sqrt{\varepsilon'}|x_n(t)|)\,dt &< \int_{[a,b]} M^{-1}\left(\frac{|x_n(t)|}{K_\varepsilon}\right)\,dt \\ &\le \frac{\varepsilon}{6c}\int_{[a,b]} M^{-1}(|x_n(t)|)\,dt \le \frac{\varepsilon}{6c}\cdot c = \frac{\varepsilon}{6}\,. \end{aligned} \tag{13}$$

Since $\{M^{-1}(|x_n|)\}_{n=1}^{\infty} \xrightarrow{\|\cdot\|_1} x_0 \in L^1[a,b]$, for $\varepsilon' > 0$, there exists $N(\varepsilon) > 0$ such that for any $n, m > N(\varepsilon)$, we have

$$\int_{[a,b]} |M^{-1}(|x_n(t)|) - M^{-1}(|x_m(t)|)|\,dt < \varepsilon'. \tag{14}$$

So for any $n, m > N(\varepsilon)$, we set

$$\begin{aligned} G_1 &= \{t \in [a,b] : M^{-1}(|x_n(t) - x_m(t)|) < \sqrt{\varepsilon'}\}, \\ G_2 &= \Big\{t \in [a,b] : M^{-1}(|x_n(t) - x_m(t)|) \ge \sqrt{\varepsilon'} \ \text{ and } \ \frac{|x_n(t)| + |x_m(t)|}{|x_n(t) - x_m(t)|} \\ &\quad < \frac{1}{\sqrt{\varepsilon'}}\Big\}, \\ G_3 &= \Big\{t \in [a,b] : M^{-1}[|x_n(t) - x_m(t)|] \ge \sqrt{\varepsilon'} \ \text{ and } \ \frac{|x_n(t)| + |x_m(t)|}{|x_n(t) - x_m(t)|} \\ &\quad \ge \frac{1}{\sqrt{\varepsilon'}}\Big\}. \end{aligned}$$

Obviously, G_1, G_2, G_3 are pairwise disjoint, and $[a,b] = G_1 \cup G_2 \cup G_3$. Thus

$$\int_{[a,b]} M^{-1}[|x_n(t) - x_m(t)|]\,dt = \Big(\int_{G_1} + \int_{G_2} + \int_{G_3}\Big) M^{-1}[|x_n(t) - x_m(t)|]\,dt,$$

$$\int_{G_1} M^{-1}[|x_n(t) - x_m(t)|]\,dt \le \int_{G_1} \sqrt{\varepsilon'}\,dt \le \sqrt{\varepsilon'}\mu([a,b]) < \frac{\varepsilon}{3}\,.$$

For G_2,

$$|x_n(t) - x_m(t)| \le |x_n(t)| + |x_m(t)| < \frac{1}{\sqrt{\varepsilon'}}|x_n(t) - x_m(t)|.$$

By Lemma 2.2 and (14), we get

$$\begin{aligned}
&\int_{G_2} M^{-1}[|x_n(t) - x_m(t)|]\,dt \\
&\le \int_{G_2} \frac{\overline{B}_M \max\{p(M^{-1}(|x_n(t)|)), p(M^{-1}|x_m(t)|)\}}{p(M^{-1}(|x_n(t) - x_m(t)|))} \cdot \\
&\quad \cdot |M^{-1}(|x_n(t)|) - M^{-1}(|x_m(t)|)|\,dt \\
&\le \int_{G_2} \frac{\overline{B}_M p[M^{-1}(|x_n(t)| + |x_m(t)|)]}{p[M^{-1}(|x_n(t) - x_m(t)|)]}|M^{-1}(|x_n(t)|) - M^{-1}(|x_m(t)|)|\,dt \\
&\le \int_{G_2} \overline{B}_M \frac{1}{\sqrt{\varepsilon'}} \overline{B}_M |M^{-1}(|x_n(t)|) - M^{-1}(|x_m(t)|)|\,dt \\
&\le \frac{\overline{B}_M^2}{\sqrt{\varepsilon'}} \cdot \varepsilon' = \overline{B}_M^2 \sqrt{\varepsilon'} < \frac{\varepsilon}{3}.
\end{aligned}$$

For G_3, $|x_n(t) - x_m(t)| \le \sqrt{\varepsilon'}(|x_n(t)| + |x_m(t)|)$ and we get by (13)

$$\begin{aligned}
&\int_{G_3} M^{-1}(|x_n(t) - x_m(t)|)\,dt \\
&\le \int_{G_3} M^{-1}[\sqrt{\varepsilon'}(|x_n(t)| + |x_m(t)|)]\,dt \\
&= \int_{G_3} M^{-1}(\sqrt{\varepsilon'}|x_n(t)|)\,dt + \int_{G_3} M^{-1}(\sqrt{\varepsilon'}|x_m(t)|)\,dt < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.
\end{aligned}$$

So for $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for $n, m > N(\varepsilon)$, we have

$$\int_G M^{-1}[|x_n(t) - x_m(t)|]\,dt < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

i.e., $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^{(M^{-1})}[a,b]$. Since $L^{(M^{-1})}[a,b]$ is complete, there exists $y_0 \in L^{(M^{-1})}[a,b]$, such that $\lim_{n\to\infty} \int_{[a,b]} M^{-1}[|x_n(t) - y_0(t)|]\,dt = 0$.

Necessity. Denote $G = [a,b]$. We assume that F is a sequentially compact set and condition (1) does not hold. Then there exists $\{x_n(t)\}_{n=1}^{\infty} \subset F$, such that

$$\int_G M^{-1}(|x_n(t)|)\,dt > n,$$

and there exists a subsequence of $\{x_n(t)\}_1^\infty$ (we still denote it by $\{x_n(t)\}_1^\infty$) which converges to $x_0(t) \in L^{(M^{-1})}(G)$ in $\|\cdot\|_{(M^{-1})}$. Thus for $\varepsilon_0 = 1$, there exists $N > 0$ such that, for $n > N$,

$$\int_G M^{-1}[|x_n(t) - x_0(t)|]\, dt < 1.$$

On the other hand, by $M^{-1}(|x_0|) \geq M^{-1}(|x_n|) - M^{-1}(|x_n - x_0|)$, we have

$$\begin{aligned}\int_G M^{-1}(|x_0(t)|)\, dt &\geq \int_G M^{-1}(|x_n(t)|)\, dt - \int_G M^{-1}(|x_n(t) - x_0(t)|)\, dt \\ &> n - 1 \to \infty \quad (n \to \infty),\end{aligned}$$

so $x_0 \notin L^{(M^{-1})}(G)$. This is impossible. The proof of the necessity of (1) is complete.

Since the family of continuous function $C(G)$ is dense in $L^{(M^{-1})}(G)$, and a continuous function is uniformly continuous in a bounded closed set G, for any $x(t) \in F$ and any $\varepsilon > 0$, there exists $x_c(t) \in C(G)$ such that

$$\int_G M^{-1}[x(t) - x_c(t)]\, dt < \frac{\varepsilon}{3},$$

and there exists $\delta(\varepsilon) > 0$ such that

$$\int_G M^{-1}[x_c(t+h) - x_c(t)]\, dt < \frac{\varepsilon}{3} \quad \text{for} \quad |h| < \delta(\varepsilon).$$

Thus for $|h| < \delta(\varepsilon)$,

$$\begin{aligned}&\int_G M^{-1}[|x(t+h) - x(t)|]\, dt \\ &\leq \int_G M^{-1}[|x(t+h) - x_c(t+h)|]\, dt + \int_G M^{-1}[|x_c(t+h) - x_c(t)|]\, dt \\ &\quad + \int_G M^{-1}[|x_c(t) - x(t)|]\, dt < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

The necessity of (2) is proved. □

Corollary 2.1. *For $L^p[a,b]$ $(0 < p < 1)$ with F-norm $\|x\|_p = \int_a^b |x(t)|^p\, dt$, $A \subset L^p[a,b]$ is sequentially compact if and only if*

(i) $\exists\, c < \infty$, *such that* $\sup_{x \in A} \|x\|_p \leq c$;
(ii) $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, *such that for* $|h| < \delta$,

$$\sup_{x \in A} \|x(t+h) - x(t)\|_p < \varepsilon.$$

Theorem 2.2. *Let M be an Orlicz function such that $M \in \triangle_2$. Then $F \subset L^{(M^{-1})}(-\infty,+\infty)$ is sequentially compact if and only if*

(i) *there exists $c > 0$ such that $\sup_{x\in F} \int_{-\infty}^{+\infty} M^{-1}(|x(t)|)\,dt \le c$,*
(ii) $\lim_{N\to\infty} \int_{|t|\ge N} M^{-1}(|x(t)|)\,dt = 0$ *uniformly for $x(t) \in F$,*
(iii) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $|h| < \delta$,*

$$\sup_{x\in F} \int_{(-\infty,+\infty)} M^{-1}[|x(t+h) - x(t)|]\,dt < \varepsilon.$$

Proof: *Sufficiency.* For $[-1,1]$, by (i), (iii) and Theorem 2.1, we know that the set $\{f|_{[-1,1]} : f \in F\}$ is sequentially compact. So there exists a convergent subsequence $\{x_{n_k^{(1)}}\}_{k=1}^{\infty} \subset F$ in $L^{(M^{-1})}[-1,1]$.

For $[-2,2]$, by (i) and (iii), the set $\{x_{n_k^{(1)}}|_{[-2,2]}\}_{k=1}^{\infty}$ is sequentially compact. So there exists a convergent subsequence $\{x_{n_k^{(2)}}\}_{k=1}^{\infty} \subset \{x_{n_k^{(1)}}\}_{k=1}^{\infty} \subset F$ in $L^{(M^{-1})}[-2,2]$.

Going on in this way, we get

$$\begin{array}{lll}
l_1 : & x_{n_1^{(1)}}, x_{n_2^{(1)}}, \cdots, x_{n_k^{(1)}}, \cdots & \text{convergence in } L^{(M^{-1})}[-1,1],\\
l_2 : & x_{n_1^{(2)}}, x_{n_2^{(2)}}, \cdots, x_{n_k^{(2)}}, \cdots & \text{convergence in } L^{(M^{-1})}[-2,2],\\
\cdots & \cdots \quad \cdots \quad \cdots & \\
l_m : & x_{n_1^{(m)}}, x_{n_2^{(m)}}, \cdots, x_{n_k^{(m)}}, \cdots & \text{convergence in } L^{(M^{-1})}[-m,m],\\
\cdots & \cdots \quad \cdots \quad \cdots &
\end{array}$$

satisfying $l_1 \supset l_2 \supset \cdots \supset l_m \supset \cdots$.
Choosing diagonal elements $x_{n_1^{(1)}}, x_{n_2^{(2)}}, \cdots, x_{n_k^{(k)}} \cdots$, we get a Cauchy sequence in $L^{(M^{-1})}(-\infty,+\infty)$. In fact, by (ii), for $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\int_{|t|\ge N} M^{-1}(|x_{n_k^{(k)}}(t)|)\,dt < \frac{\varepsilon}{4}, \quad k = 1, 2, \cdots.$$

For $[-N,N]$, $\{x_k^{(k)}\}_{k=N}^{\infty}$ converges in $L^{(M^{-1})}[-N,N]$ (assume it converges to $x_0 \in L^{(M^{-1})}[-N,N]$). Thus for $\varepsilon > 0$, there exists a $K_0 \in \mathbb{N}$ such that for

$k, l > K_0$, we have

$$\begin{aligned}
&\int_{[-N,N]} M^{-1}[|x_{n_k^{(k)}}(t) - x_{n_l^{(l)}}(t)|]\, dt \\
&\le \int_{[-N,N]} M^{-1}[|x_{n_k^{(k)}}(t) - x_0(t)| + |x_0(t) - x_{n_l^{(l)}}(t)|]\, dt \\
&\le \int_{[-N,N]} M^{-1}[|x_{n_k^{(k)}}(t) - x_0(t)|]\, dt + \int_{[-N,N]} M^{-1}[|x_0(t) - x_{n_l^{(l)}}(t)|]\, dt \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}\,.
\end{aligned}$$

So for $\varepsilon > 0$ and for all $k, l > K_0$, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} M^{-1}[|x_{n_k^{(k)}}(t) - x_{n_l^{(l)}}(t)|]\, dt \\
&\le \int_{[-N,N]} M^{-1}[|x_{n_k^{(k)}}(t) - x_{n_l^{(l)}}(t)|]\, dt + \int_{|t|\ge N} M^{-1}[|x_{n_k^{(k)}}(t) - x_{n_l^{(l)}}(t)|]\, dt \\
&\le \frac{\varepsilon}{2} + \int_{|t|\ge N} M^{-1}(|x_{n_k^{(k)}}(t)|)\, dt + \int_{|t|\ge N} M^{-1}[|x_{n_l^{(l)}}(t)|]\, dt \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

This means that $\{x_{n_k^{(k)}}\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^{(M^{-1})}(-\infty, +\infty)$ and, since $L^{(M^{-1})}(-\infty, +\infty)$ is complete, we know the sufficiency is true.

Necessity. The proof of (i) is similar to that of Theorem 2.1(1), now we prove (ii) and (iii). If (ii) does not hold, then there exists an $\varepsilon_0 > 0$ such that for any $n_k > 0$ there is $x_{n_k} \in F$ with

$$\int_{|t|>n_k} M^{-1}(|x_{n_k}(t)|)\, dt > \varepsilon_0.$$

Choose $n_1 = 1$. Then $\exists\, x_1 \in F$ such that

$$\int_{|t|>1} M^{-1}(|x_1(t)|)\, dt > \varepsilon_0.$$

Noting that

$$\int_{(-\infty,+\infty)} M^{-1}(|x_1(t)|)\, dt < \infty,$$

$\exists\, n_2 > 0$ such that

$$\int_{|t|>n_2} M^{-1}(|x_1(t)|)\, dt < \frac{\varepsilon_0}{2}\,.$$

For n_2, there exists $x_2 \in F$ such that $\int_{|t|>n_2} M^{-1}(|x_2(t)|)\,dt > \varepsilon_0$. Thus $\exists\, n_3 > 0$ such that

$$\int_{|t|>n_3} M^{-1}(|x_2(t)|)\,dt < \frac{\varepsilon_0}{2}\,.$$

Following these steps, we get $\{x_1, x_2, \cdots, x_k, \cdots\} \subset F$ and

$$\int_{|t|>n_k} M^{-1}(|x_k(t)|)\,dt > \varepsilon_0,$$

$$\int_{|t|>n_{k+1}} M^{-1}(|x_k(t)|)\,dt < \frac{\varepsilon_0}{2}.$$

So for any k_1, k_2 (we assume $k_1 > k_2$ without loss of generality),

$$\begin{aligned}
&\int_{-\infty}^{+\infty} M^{-1}[|x_{k_1}(t) - x_{k_2}(t)|]\,dt \\
\geq\; &\int_{-\infty}^{+\infty} |M^{-1}(|x_{k_1}(t)|) - M^{-1}(|x_{k_2}(t)|)|\,dt \\
\geq\; &\int_{|t|>n_{k_1}} |M^{-1}(|x_{k_1}(t)|) - M^{-1}(|x_{k_2}(t)|)|\,dt \\
\geq\; &\int_{|t|>n_{k_1}} M^{-1}(|x_{k_1}(t)|)\,dt - \int_{|t|>n_{k_1}} M^{-1}(|x_{k_2}(t)|)\,dt \\
>\; &\varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2},
\end{aligned}$$

i.e., $\|x_{k_1} - x_{k_2}\|_{(M^{-1})} > \frac{\varepsilon_0}{2}$, so $\{x_k\}_1^{\infty} \subset F$ has no convergent sequence in $L^{(M^{-1})}(-\infty, +\infty)$, This contradicts the sequential compactness of F.

(iii) $\forall\, \varepsilon > 0$, by (ii), $\exists\, N \in \mathbb{N}$, such that

$$\int_{|t|\geq N} M^{-1}(|x(t)|)\,dt < \frac{\varepsilon}{4}$$

for every $x(t) \in F$. So for all $x(t) \in F$ and $|h| < 1$, we have

$$\begin{aligned}
&\int_{|t|\geq N+1} M^{-1}[|x(t+h) - x(t)|]\,dt \\
\leq\; &\int_{|t|\geq N+1} M^{-1}[|x(t+h)| + |x(t)|]\,dt \\
\leq\; &\int_{|t|\geq N+1} M^{-1}[|x(t+h)|]\,dt + \int_{|t|\geq N+1} M^{-1}[|x(t)|]\,dt \\
<\; &\frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}\,.
\end{aligned}$$

For $\int_{|t|\le N+1} M^{-1}[|x(t+h)-x(t)|]\,dt$, by Theorem 2.1, we know that there exists $\delta_1 > 0$ such that for $|h| < \delta_1$,

$$\sup_{x\in F}\int_{|t|\le N+1} M^{-1}[|x(t+h)-x(t)|]\,dt < \frac{\varepsilon}{2}\,.$$

Choose $\delta = \min\{1, \delta_1\}$; then for all $x \in F$ and $|h| < \delta$,

$$\begin{aligned}
&\int_{|t|<\infty} M^{-1}[|x(t+h)-x(t)|]\,dt\\
=& \int_{|t|\le N+1} M^{-1}[|x(t+h)-x(t)|]\,dt + \int_{|t|>N+1} M^{-1}[|x(t+h)-x(t)|]\,dt\\
<& \;\frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

i.e., $\sup_{x\in F}\int_{(-\infty,+\infty)} M^{-1}[x(t+h)-x(t)]\,dt < \varepsilon$. $\square$

A natural consequence is given by:

Corollary 2.2 (Tsuji [9]). *For $L^p(-\infty,+\infty)$ $(0 < p < 1)$ with F-norm $\|x\|_p = \int_a^b |x(t)|^p\,dt$, $A \subset L^p(-\infty,+\infty)$ is sequentially compact if and only if*

(i) $\exists\, c < \infty$, *such that* $\sup_{x\in A}\int_{-\infty}^{+\infty} |x(t)|^p\,dt \le c$,

(ii) $\lim_{N\to\infty}[\sup_{x\in A}\int_{|t|\ge N} |x(t)|^p\,dt] = 0$,

(iii) $\forall\,\varepsilon > 0$, $\exists\,\delta > 0$ *such that for* $|h| < \delta$,

$$\sup_{x\in A}\int_{-\infty}^{+\infty} |x(t+h)-x(t)|^p\,dt < \varepsilon.$$

§3. Krasnoselskii criterion for sequential compactness in $L^{(M^{-1})}$

Definition 3.1. $S \subset L^{(M^{-1})}(G)$ is said to be *sequentially compact in measure*, if for any $\{f_n\}_{n=1}^{\infty} \subset S$ there exists a sequence $\{f_{n_i}\}_{i=1}^{\infty}$ and $f_0 \in L^{(M^{-1})}(G)$ such that $f_{n_i}(x) \xrightarrow{m} f_0(x)$.

Definition 3.2. $S \subset L^{(M^{-1})}(G)$ is said to be *equi-absolutely continuous in F-norm*, if $\forall\,\varepsilon > 0$, $\exists\,\delta > 0$, such that for $e \subset G$ and $\mu(e) < \delta$,

$$\sup_{f\in S}\|f\cdot\chi_e\|_{(M^{-1})} < \varepsilon.$$

Theorem 3.1. *Let M be an Orlicz function and let $\mu(G) < \infty$. Then $S \subset L^{(M^{-1})}(G)$ is sequentially compact iff*

(i) *S is sequentially compact in measure;*
(ii) *S has an equi-absolutely continuous F-norm.*

Proof: *Sufficiency.* For any $\{f_n\}_{n=1}^{\infty} \subset S$, by (i) we know there exists a subsequence, we still denote it by $\{f_n\}_{n=1}^{\infty}$, and $f_0 \in L^{(M^{-1})}(G)$, such that $f_n \xrightarrow{m} f_0$. Next we prove that the subsequence $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

By (ii), $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, for $\mu(e) < \delta$,

$$\sup_{n \geq 1} \|f_n \chi_e\|_{(M^{-1})} < \frac{\varepsilon}{4}\,.$$

Choose $\varepsilon' < \varepsilon$, such that $M^{-1}(\varepsilon')\mu(G) < \frac{\varepsilon}{2}$. Denote $G_{n,m} = \{x \in G : |f_n(x) - f_m(x)| > \varepsilon'\}$. By $f_n \xrightarrow{m} f_0$, we know $\lim_{n,m\to\infty} \mu(G_{n,m}) = 0$. So for $\delta > 0$, $\exists\, n_0 \in \mathbb{N}$, for all $n, m > n_0$, $\mu(G_{n,m}) < \delta$. Therefore

$$\begin{aligned}
&\|f_n - f_m\|_{(M^{-1})} \\
= &\int_G M^{-1}(|f_n(t) - f_m(t)|)\, dt \\
= &\int_G M^{-1}(|(f_n - f_m)(t)\chi_{G_{n,m}}(t) + (f_n - f_m)(t)\chi_{G\setminus G_{n,m}}(t)|)\, dt \\
\leq &\int_G M^{-1}(|(f_n - f_m)(t)\chi_{G_{n,m}}(t)|)\, dt + \\
&+ \int_G M^{-1}(|(f_n - f_m)(t)\chi_{G\setminus G_{n,m}}(t)|)\, dt \\
= &\|(f_n - f_m)\chi_{G_{n,m}}\|_{(M^{-1})} + \int_{G\setminus G_{n,m}} M^{-1}(|(f_n - f_m)(t)|)\, dt \\
< &\frac{\varepsilon}{2} + M^{-1}(\varepsilon')\mu(G) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

By the completeness of $L^{(M^{-1})}(G)$, we know $\{f_n\}_{n=1}^{\infty}$ converges in $L^{(M^{-1})}(G)$.

Necessity. (i). Let $S \subset L^{(M^{-1})}(G)$ be a sequentially compact set. For any $\{f_n\}_{n=1}^{\infty} \subset S$ there exists a subsequence, we still denote it by $\{f_n\}_{n=1}^{\infty}$, and $f_0 \in L^{(M^{-1})}(G)$ such that $\|f_n - f_0\|_{(M^{-1})} \to 0$ $(n \to \infty)$, i.e., $\int_G M^{-1}(|f_n(t) - f_0(t)|)\, dt \to 0 (n \to \infty)$. So $f_n \xrightarrow{m} f_0$.

(ii). Let S be a sequentially compact set. Then S is a complete bounded set. So, for any $\varepsilon > 0$ there exists a finite $\frac{\varepsilon}{2}$-net of S $\{g_1, g_2, \cdots, g_N\}$ such that $S \subset \bigcup_{i=1}^{N} B(g_i, \frac{\varepsilon}{2})$, where $B(g_i, \frac{\varepsilon}{2}) = \{f \in L^{(M^{-1})}(G) : \|f - g_i\|_{(M^{-1})} < \frac{\varepsilon}{2}\}$. Clearly, for $\frac{\varepsilon}{2} > 0$, there is $\delta > 0$ such that for $\mu(e) < \delta$ we have $\int_e M^{-1}(|g_i(t)|)\,dt < \frac{\varepsilon}{2}$, $i = 1, 2, \cdots, N$. For any $f \in S$ there exists g_{i_0} such that $f \in B(g_{i_0}, \frac{\varepsilon}{2})$. For $e \subset G$, $\mu(e) < \delta$, we have

$$\begin{aligned}\|f \cdot \chi_e\|_{(M^{-1})} &= \int_e M^{-1}(|f(t)|)\,dt \\ &= \int_e M^{-1}(|f(t) - g_{i_0}(t)|)\,dt + \int_e M^{-1}(|g_{i_0}(t)|)\,dt \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus for $e \subset G$ with $\mu(e) < \delta$ we have $\sup_{f \in S} \|f \cdot \chi_e\|_{(M^{-1})} < \varepsilon$. □

Corollary 3.1. *For $L^p[a, b]$ $(0 < p < 1)$ with F-norm $\|x\|_p = \int_a^b |x(t)|^p\,dt$, $A \subset L^p[a, b]$ is sequentially compact if and only if*

(i) *A is sequentially compact in measure,*
(ii) *A is equi-absolutely continuous in $\|\cdot\|_p$.*

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References

[1] Rao M.M., Ren Z.D., *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.
[2] Maligranda L., *Orlicz Spaces and Interpolation*, Seminars in Mathematics 5, Univ. Estadual de Campinas, Capinas SP Brasil, 1989.
[3] Rao M.M., Ren Z.D., *Applications of Orlicz Spaces*, Marcel Dekker, New York, 2001, to appear.
[4] Riesz F., *Untersuchungen über Systeme integrierbarer Funktionen*, Math. Ann. **69** (1910), 449–497.
[5] Kantorovich L.V., Akilov G.P., *Functional Analysis*, Pergamon Press, New York, 1982.
[6] Krasnosel'skii M.A., Rutickii Ya.B., *Convex Functions and Orlicz Space*, Noordhoff Ltd. Groningen, 1961.
[7] Takahashi T., *On the compactness of the function-set by the convergence in mean of general type*, Studia Math. **5** (1935), 141–150.
[8] Gribanov Yu., *Nonlinear operators in Orlicz spaces* (in Russian), Uč. zap. kazansk. Unta **115** (7) (1955).
[9] Tsuji M., *On the compactness of space $L^p (p > 0)$ and its application to integral equation*, Mathematical Institute, Tokyo University, 1951.

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