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A proof for the Blair-Hager-Johnson theorem on absolute z-embedding

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Abstract. In this paper, a simple proof is given for the following theorem due to Blair [7], Blair-Hager [8] and Hager-Johnson [12]: A Tychonoff space X is z-embedded in every larger Tychonoff space if and only if X is almost compact or Lindelöf. We also give a simple proof of a recent theorem of Bella-Yaschenko [6] on absolute embeddings.

Keywords: absolute z-embedding, absolute C-embedding, absolute C^*-embedding, absolute embeddings, almost compact, Lindelöf, compact, pseudocompact

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All spaces are assumed to be Tychonoff spaces. For a subspace A of a space X, A is said to be C (resp. C^*)-*embedded* in X if every real-valued (resp. every bounded real-valued) continuous function on A can be continuously extended over X. A subspace A of a space X is said to be z-*embedded* in X if for every zero-set Z of A there exists a zero-set Z' of$X$ with $Z' \cap A = Z$. Clearly, C-embedding implies C^*-embedding, and the latter implies z-embedding. As it is known, there have been several results on z-embeddings, which are closely related to those on C or C^*-embeddings and other extension properties (see [1]). Let us recall the following theorem which describes the so-called absolute C-embedding or absolute C^*-embedding; a Tychonoff space X is said to be *almost compact* if $|\beta X - X| \leq 1$, where βX denotes the Stone-Čech compactification of X.

Theorem 1 (Doss [9], Hewitt [14], Smirnov [16]; see also [1], [11]). *A Tychonoff space X is C (or equivalently, C^*)-embedded in every larger Tychonoff space if and only if X is almost compact.*

As concerns z-embeddings, recall the following result due to Jerison (see [13, Lemma 5.3] or [1, Theorem 7.8]).

Theorem 2 (Jerison). *If X is a Lindelöf subspace of a Tychonoff space Y, then X is z-embedded in Y.*

Corresponding to Theorem 1, the result characterizing the so-called absolute z-embedding has been established by Blair [7], Blair-Hager [8] and Hager-Johnson [12] as the following; the "if" part directly follows from Theorems 1 and 2.

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Theorem 3 (Blair [7], Blair-Hager [8], Hager-Johnson [12]). *A Tychonoff space X is z-embedded in every larger Tychonoff space if and only if X is almost compact or Lindelöf.*

The proofs of the "only if" part of Theorem 3 given in [7], [8] and [12] are obtained through several consequences under their own interests on realcompactness or rings of continuous functions. In the present paper, we give an alternative and simple proof to this theorem, with only a use of the following well-known fact: a Tychonoff space X is Lindelöf if and only if for every compact subspace F of βX with $F \subset \beta X - X$ there exists a zero-set Z of βX such that $F \subset Z \subset \beta X - X$ (see [10, 3.12.25(b)]). Other terminology may be found in [1], [10] and [11].

Proof of the "only if" part of Theorem 3: Assume that X is z-embedded in every larger Tychonoff space. Suppose that X is not almost compact. We show that X is Lindelöf. Let F be a compact subspace of βX with $F \subset \beta X - X$.

Claim. *For every $x \in F$, there exist an open neighborhood U_x of x in the subspace F and a zero-set Z_x of βX such that $U_x \subset Z_x \subset \beta X - X$.*

Proof of Claim: Let $x \in F$. Since $|\beta X - X| \geq 2$, pick up a point $y \in \beta X - X$ with $y \neq x$. Let $f : \beta X \to [0,1]$ be a continuous function satisfying that $f(x) = 0$ and $f(y) = 1$. Let $Y = \beta X/(F \cup \{y\})$ be the quotient space obtained from βX by identifying $F \cup \{y\}$ with a single point and $q : \beta X \to Y$ be the natural quotient map. Since $f^{-1}([0,1/2)) \cap X$ is a cozero-set of X and X is z-embedded in Y, there exists a cozero-set V of Y such that $V \cap X = f^{-1}([0,1/2)) \cap X$. Then, $q(x) \notin V$. Indeed, if $q(x) \in V$, then $y \in q^{-1}(V) \cap f^{-1}((1/2,1]) \subset \beta X - X$, a contradiction. Put $U_x = f^{-1}([0,1/3)) \cap F$ and $Z_x = f^{-1}([0,1/3]) - q^{-1}(V)$. These are the required sets. This completes the proof of Claim. □

Finally, for some finite points $x_1, \ldots, x_n \in F$ with $F = \bigcup_{i=1}^n U_{x_i}$, put $Z = \bigcup_{i=1}^n Z_{x_i}$. Then, Z is a zero-set of βX and $F \subset Z \subset \beta X - X$. Hence X is Lindelöf. This completes the proof. □

We apply our technique in the above proof to give a simple proof to the following theorem which was recently obtained by Bella and Yaschenko [6], their proof is long and complicated.

Theorem 4 (Bella-Yaschenko [6]). *For a Tychonoff space X, the following statements are equivalent:*

(1) *if a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be separated in Y by open sets;*
(2) *X is Lindelöf.*

See [3] for the motivation of Theorem 4. For the proof, let us consider another condition $(*)$ below; recall that *separated subsets* mean those subsets A and B of X with $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

($*$) *If a Tychonoff space Y contains two copies X_1 and X_2 of X which are separated subsets, then these copies can be separated in Y by cozero-sets.*

Proof of Theorem 4: (1) $\Rightarrow$ (2): Assume (1). We shall first prove that ($*$) holds. Let X_1 and X_2 be copies of X and assume that they are separated subsets of a Tychonoff space Y. Embed Y into the Tychonoff cube T. Moreover, embed T into the product space $T \times [0,1]$ as a subspace $T \times \{0\}$ and denote the subspace $(T \times (0,1]) \cup ((X_1 \cup X_2) \times \{0\})$ by Z. Then, X_1 and X_2 are disjoint closed subsets in Z. From the assumption, $X_i \times \{0\}$, $i = 1, 2$, can be separated by open sets U_1 and U_2 in Z. Since Z is a dense subset of $T \times [0,1]$, these open sets can be extended to disjoint open sets U_1^* and U_2^* in $T \times I$. Since $T \times I$ is a Tychonoff cube, it is known that U_1^* and U_2^* can be separated by cozero-sets G_1 and G_2 in $T \times I$, use [15, Theorem 2] and [17, Theorem 2]. Then, it follows that $X_i \times \{0\} \subset G_i \cap (Y \times \{0\})$, $i = 1, 2$. Hence, X_1 and X_2 can be separated by cozero-sets in Y. So, ($*$) holds.

We now prove X is Lindelöf. Let F be a compact subspace of βX with $F \subset \beta X - X$ and let $X_1 = X_2 = X$. Denote F in βX_i by F_i, $i = 1, 2$. Let $Y = (\beta X_1 \oplus \beta X_2)/(F_1 \cup F_2)$ be the quotient space obtained from $\beta X_1 \oplus \beta X_2$ identifying $F_1 \cup F_2$ to a single point, and $q : \beta X_1 \oplus \beta X_2 \to Y$ the natural quotient map. Since $q(X_1)$ and $q(X_2)$ are separated subsets in Y, by ($*$), there exists disjoint cozero-sets U_1 and U_2 of Y such that $q(X_i) \subset U_i$, $i = 1, 2$. We may assume $q(F_1 \cup F_2) \notin U_1$. Then, $X_1 \subset q^{-1}(U_1) \cap \beta X_1 \subset \beta X_1 - F_1$ and $q^{-1}(U_1) \cap \beta X_1$ is a cozero-set of βX_1. Hence X_1 is (that is, X is) Lindelöf.

(2) $\Rightarrow$ (1): Easy. $\square$

It should be noted in Theorem 1 (resp. Theorem 3) that X may be equivalently assumed to be C or C^* (resp. z)-embedded in every Tychonoff space in which X is embedded as a closed subset ([4], [14], [16]). This can be proved similarly to the above proof showing ($*$).

Yajima has given some generalizations of Theorem 4 and characterizations of paracompactness [18].

Next we give some results related to Theorem 4. The technique of making the adjunction space determined by a space and the Tychonoff plank, which is popular in the theory of relative topological properties (cf. [2]), will be used.

Theorem 5. *For a Tychonoff space X, the following statements are equivalent:*

(1) *if a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be completely separated in Y;*
(2) *X is compact.*

Proof: (1) $\Rightarrow$ (2): Assume (1). By Theorem 4, X is Lindelöf. It suffices to show that every closed discrete set of X is finite. To prove this, assume the contrary and let $\{x_n : n < \omega\}$ be a closed discrete set in X consisting of distinct points.

Let $X_1 = X_2 = X$ and denote $\{x_n : n < \omega\}$ in X_i by $\{x_n^i : n < \omega\}$, $i = 1, 2$. Let $Z = (\omega_1 + 1) \times (\omega + 1) - \{(\omega_1, \omega)\}$ be the Tychonoff plank. Let $Z_1 = Z_2 = Z$, and denote the right edge $\{(\omega_1, n) : n < \omega\}$ of Z_i by $\{(\omega_1, n)^i : n < \omega\}$, and the top edge $\{(\alpha, \omega) : \alpha < \omega_1\}$ of Z_i by $\{(\alpha, \omega)^i : \alpha < \omega_1\}$, $i = 1, 2$. For $i = 1, 2$, define a map $f_i : \{x_n^i : n < \omega\} \to Z_i$ by $f_i(x_n^i) = (\omega_1, n)^i$, $n < \omega$. Consider the adjunction spaces $X_i \cup_{f_i} Z_i$, $i = 1, 2$ (see [10, p. 93]). Define a map $g : \{(\alpha, \omega)^1 : \alpha < \omega_1\} \to X_2 \cup_{f_2} Z_2$ by $g((\alpha, \omega)^1) = (\alpha, \omega)^2$, $\alpha < \omega_1$. Let Y be the adjunction space $(X_1 \cup_{f_1} Z_1) \cup_g (X_2 \cup_{f_2} Z_2)$. Since Y is a Tychonoff space and X_1 and X_2 are closed subsets of Y, by the assumption, X_1 and X_2 must be completely separated in Y. But this is a contradiction.

(2) $\Rightarrow$ (1): Easy. $\square$

The following result should be compared with Theorems 4 and 5; this is probably known and is proved similarly to Theorem 5.

Theorem 6. *For a Tychonoff space X, the following statements are equivalent:*

(1) *if a Tychonoff space Y contains a copy X_1 of X and a closed subset F disjoint from X_1, then X_1 and F can be completely separated in Y;*
(2) *if a Tychonoff space Y contains a closed copy X_1 of X and a closed subset F disjoint from X_1, then X_1 and F can be separated in Y by open sets;*
(3) *X is compact.*

Analogously, the following holds: (1) $\Leftrightarrow$ (3) is due to Blair-Hager [8, Proposition 4.3], (2) $\Leftrightarrow$ (3) is due to Aull [5, Theorem 1(b)]. Here, we give a direct proof of (2) $\Rightarrow$ (3).

Theorem 7 (Blair-Hager [8], Aull [5]). *For a Tychonoff space X, the following statements are equivalent:*

(1) *if a Tychonoff space Y contains a copy X_1 of X and a zero-set F disjoint from X_1, then X_1 and F can be completely separated in Y;*
(2) *if a Tychonoff space Y contains a closed copy X_1 of X and a zero-set F disjoint from X_1, then X_1 and F can be separated in Y by open sets;*
(3) *X is pseudocompact.*

Proof: (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Assume (2) and assume X is not pseudocompact. Let $f : X \to (0, +\infty)$ be a positive unbounded continuous function. Then, there exists a closed discrete subset $\{x_n : n < \omega\}$ of X such that $n \leq f(x_n)$ for every $n < \omega$. Consider the adjunction space $X \cup_g Z$, where Z is the Tychonoff plank, and $g : \{x_n : n < \omega\} \to Z$ is a map defined by $g(x_n) = (\omega_1, n)$, $n < \omega$. Then, notice that the top edge of Z is a zero-set of $X \cup_g Z$. Indeed, define a continuous function $h : X \cup_g Z \to [0, +\infty)$ by $h(x) = 1/f(x)$ if $x \in X$; $h(x) = 1/f(x_n)$ if $x = (\alpha, n) \in Z$, $\alpha \leq \omega_1$, $n < \omega$; $h(x) = 0$ if $x = (\alpha, \omega) \in Z$, $\alpha < \omega_1$. Since the top edge of Z is $h^{-1}(\{0\})$, the condition (2) raises a contradiction.

(3) $\Rightarrow$ (1): See [8]. □

By [8, Corollary 3.6] (see also [1]), a subspace A of a space X is C-embedded in X if and only if A is z-embedded and well-embedded in X, where A is said to be *well-embedded* in X if any zero-set of X disjoint from A can be completely separated in X. Hence, it should be noted that Theorems 3 and 7 induce Theorem 1.

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