Riesz angles of Orlicz sequence spaces

Yan Yaqiang

Abstract. We introduce some practical calculation of the Riesz angles in Orlicz sequence spaces equipped with Luxemburg norm and Orlicz norm. For an N-function $\Phi(u)$ whose index function is monotonous, the exact value $a(l^{(\Phi)})$ of the Orlicz sequence space with Luxemburg norm is $a(l^{(\Phi)}) = 2^{\frac{1}{C^0_\Phi}}$ or $a(l^{(\Phi)}) = \frac{\Phi^{-1}(1)}{\Phi^{-1}(\frac{1}{2})}$. The Riesz angles of Orlicz space l^Φ with Orlicz norm has the estimation $\max(2\beta^0_\Psi, 2\beta'_\Psi) \le a(l^\Phi) \le \frac{2}{\theta^0_\Phi}$.

Keywords: Orlicz space, N-function, index function, Riesz angle

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1. Introduction

In 1984, Borwein and Sims [2, p. 345] introduced the following

Definition 1.1. For a Banach lattice X, the Riesz angle $a(X)$ of X is defined as

$$a(X) = \sup \{\|(|x| \vee |y|)\| : \|x\| \le 1, \|y\| \le 1\}, \tag{1}$$

where $|x| \vee |y| = \max(|x|, |y|)$.

Clearly, $1 \le a(X) \le 2$. The notion of Riesz angle has attracted many researchers (see Borwein and Sims [2], Chen [3], Cui, Hudzik and Li [4]). It is an important geometric coefficient in Banach lattices. This paper is devoted to the computation (estimation) of $a(X)$ when X is an Orlicz sequence space with Luxemburg norm or Orlicz norm, i.e., $a(l^{(\Phi)})$ and $a(l^\Phi)$.

Let

$$\Phi(u) = \int_0^{|u|} \phi(t)\,dt \quad \text{and} \quad \Psi(v) = \int_0^{|v|} \psi(s)\,ds$$

be a pair of complementary N-functions. The Orlicz sequence spaces $l^{(\Phi)}$ and l^Φ are defined to be the sets $\left\{x : \rho_\Phi(\lambda x) = \sum_{n=1}^{\infty} \Phi(\lambda|x(i)|) < \infty \text{ for some } \lambda > 0\right\}$ equipped with Luxemburg norm $\|\cdot\|_{(\Phi)}$ and Orlicz norm $\|\cdot\|_\Phi$, respectively, where

$$\|x\|_{(\Phi)} = \inf\left\{c > 0 : \rho_\Phi\left(\frac{x}{c}\right) \le 1\right\}, \quad \text{and} \quad \|x\|_\Phi = \inf_{k>0} \frac{1}{k}[1 + \rho_\Phi(kx)].$$

$\Phi(u)$ is said to satisfy the Δ_2-condition for small u, in symbol $\Phi \in \Delta_2(0)$, if there exist $u_0 > 0$ and $k > 2$ such that $\Phi(2u) \le k\Phi(u)$ for $0 \le u \le u_0$. An N-function $\Phi(v)$ is said to satisfy the ∇_2-condition if its complementary N-function satisfies the Δ_2-condition.

In Orlicz spaces, Borwein and Sims [2, p. 347] proved: If $\Phi \in \nabla_2(0)$, then $a(l^{(\Phi)}) < 2$. Chen [3, p. 118] showed that if $\Phi \notin \nabla_2(0)$, then $a(l^{(\Phi)}) = 2$. Thus, we can state the results as follows:

Proposition 1.2 ([2], [3]). *Let Φ be an N-function; then $\Phi \in \nabla_2(0)$ if and only if $a(l^{(\Phi)}) < 2$.*

The result analogous to the above proposition for spaces with Orlicz norm will be given in this paper (see Corollary 3.4). We first introduce some propositions on indices and index functions. Simonenko [8] and Semenov [7] introduced the following indices:

$$A_\Phi^0 = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)}\,, \qquad B_\Phi^0 = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)}\,; \tag{2}$$

$$\alpha_\Phi^0 = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\,, \qquad \beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\,. \tag{3}$$

The same indices can be applied to $\Psi(v)$. Later on, these indices were extensively studied (see Maligranda [6]) and were used by Rao and Ren (see [9]) to calculate the packing sphere and the other important constants. We have the relationship:

Proposition 1.3 (Lindenstrauss and Tzafriri [5], Yan [10], Rao and Ren [9]). *Let Φ, Ψ be a pair of complementary N-functions. Then, we have*

$$\frac{1}{A_\Phi^0} + \frac{1}{B_\Psi^0} = 1 = \frac{1}{A_\Psi^0} + \frac{1}{B_\Phi^0}\,, \tag{4}$$

$$2\alpha_\Phi^0 \beta_\Psi^0 = 1 = 2\alpha_\Psi^0 \beta_\Phi^0, \tag{5}$$

$$2^{-\frac{1}{A_\Phi^0}} \le \alpha_\Phi^0 \le \beta_\Phi^0 \le 2^{-\frac{1}{B_\Phi^0}}. \tag{6}$$

For the index functions of N-functions, the author obtained the following results.

Proposition 1.4 (Yan [10]). *Let Φ be an N-function, $\phi(t)$ its right derivative. Denote*

$$F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}\,, \qquad G_\Phi(c,u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(cu)}\,, \qquad (c > 1). \tag{7}$$

Then $F_\Phi(t)$ is increasing (decreasing) on $(0, \Phi^{-1}(u_0)]$ if and only if $G_\Phi(c,u)$ is increasing (decreasing) on $(0, \frac{u_0}{c}]$ for every $c > 1$.

Proposition 1.5 (Yan [10]). *Let Φ, Ψ be a pair of N-functions, and ϕ, ψ be their right derivative, respectively. $C > 0$, $F_\Phi(t)$ and $F_\Psi(s)$ are defined as in* (7), *i.e.,*

$$F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}, \qquad F_\Psi(s) = \frac{s\psi(s)}{\Psi(s)}, \qquad t, s > 0.$$

Then

(i) *$F_\Phi(t)$ is increasing (decreasing) on $(0, \psi(C)]$ if and only if $F_\Psi(s)$ is decreasing (increasing) on $(0, C]$.*

(ii) *Denote*

$$a^*_\Phi = \inf\{F_\Phi(t) : t \in (0, \psi(C)]\}, \tag{8}$$

$$b^*_\Psi = \sup\{F_\Psi(s) : s \in (0, C]\}. \tag{9}$$

Then

$$\frac{1}{a^*_\Phi} + \frac{1}{b^*_\Psi} = 1. \tag{10}$$

2. Riesz angles of Orlicz sequence spaces with Luxemburg norm

Theorem 2.1. *Let Φ be an N-function; then*

$$\max\left(\frac{1}{\alpha'_\Phi}, \frac{1}{\alpha^0_\Phi}\right) \le a(l^{(\Phi)}) \le \frac{1}{\tilde{\alpha}_\Phi}, \tag{11}$$

where α^0_Φ is defined as (3), *and*

$$\alpha'_\Phi = \inf\left\{\frac{\Phi^{-1}(\frac{1}{2k})}{\Phi^{-1}(\frac{1}{k})} : k = 1, 2, \cdots\right\}, \tag{12}$$

$$\tilde{\alpha}_\Phi = \inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \le \frac{1}{2}\right\}. \tag{13}$$

Proof: We first prove

$$\frac{1}{\alpha^0_\Phi} \le a(l^{(\Phi)}). \tag{14}$$

The method is similar to that of Rao and Ren [9, Lemma 2.2(i)]. By the definition of α^0_Φ, there exists a sequence $1 > u_n \searrow 0$ such that

$$\lim_{n\to\infty} \frac{\Phi^{-1}(u_n)}{\Phi^{-1}(2u_n)} = \alpha^0_\Phi.$$

For any $0 < \varepsilon < 1$, select an integer $n_0 \geq 1$ such that $u_n < \frac{\varepsilon}{2}$ for $n \geq n_0$ and $[\Phi^{-1}(u_n)/\Phi^{-1}(2u_n)] < \alpha_\Phi^0 + \varepsilon$. For convenience, put $u_0 = u_{n_0}$. Then

$$2u_0 < \varepsilon, \qquad \Phi^{-1}(2u_0) > \frac{\Phi^{-1}(u_0)}{\alpha_\Phi^0 + \varepsilon}. \tag{15}$$

Let $k_0 = [\frac{1}{2u_0}]$ be the integral part of $\frac{1}{2u_0}$. Then $k_0 \leq \frac{1}{2u_0} < k_0 + 1$, and so $\frac{1}{2k_0} < \frac{u_0}{1-2u_0}$. Define

$$x_0 = \Phi^{-1}(2u_0) \sum_{i=1}^{k_0} e_i, \qquad y_0 = \Phi^{-1}(2u_0) \sum_{i=k_0+1}^{2k_0} e_i,$$

where $e_i = (\overbrace{0, \cdots, 0, 1}^{i}, 0, 0, \cdots)$. It is easy to see that

$$\|x_0\|_{(\Phi)} = \|y_0\|_{(\Phi)} = \frac{\Phi^{-1}(2u_0)}{\Phi^{-1}(\frac{1}{k_0})} \leq 1.$$

It follows from (15) that

$$\begin{aligned} \|(|x_0| \vee |y_0|)\|_{(\Phi)} = \|x_0 + y_0\|_{(\Phi)} &= \frac{\Phi^{-1}(2u_0)}{\Phi^{-1}(\frac{1}{2k_0})} \\ &> \frac{\Phi^{-1}(u_0)}{(\alpha_\Phi^0 + \varepsilon)\Phi^{-1}(\frac{1}{2k_0})} > \frac{(1-2u_0)\Phi^{-1}(\frac{u_0}{1-2u_0})}{(\alpha_\Phi^0 + \varepsilon)\Phi^{-1}(\frac{1}{2k_0})} \\ &> \frac{1-\varepsilon}{\alpha_\Phi^0 + \varepsilon}, \end{aligned}$$

so (14) holds since ε is arbitrary.

Next we show

$$\frac{1}{\alpha'_\Phi} \leq a(l^{(\Phi)}). \tag{16}$$

For every $k \geq 1$, define

$$Z_k = (0, 0, \cdots, 0), \qquad X_k = \Phi^{-1}(\frac{1}{k})(1, 1, \cdots, 1),$$

with $\dim Z_k = \dim X_k = k$. Denote

$$x_0 = (X_k, Z_k, Z_k, \cdots), \qquad y_0 = (Z_k, X_k, Z_k, \cdots).$$

Then $\|x_0\|_{(\Phi)} = \|y_0\|_{(\Phi)} = 1$ and

$$\|(|x_0| \vee |y_0|)\|_{(\Phi)} = \frac{\Phi^{-1}(\frac{1}{k})}{\Phi^{-1}(\frac{1}{2k})}\,.$$

Therefore

$$a(l^{(\Phi)}) \geq \sup\left\{\frac{\Phi^{-1}(\frac{1}{k})}{\Phi^{-1}(\frac{1}{2k})} : k = 1, 2, \cdots\right\} = \frac{1}{\alpha'_\Phi}\,.$$

Observe that (16) was proved by Borwein and Sims (see [2, p. 347]). However their proof is too complex and attached with an extra assumption $\Phi(1) = 1$.

Finally, it remains to show

$$a(l^{(\Phi)}) \leq \frac{1}{\tilde{\alpha}_\Phi}\,. \tag{17}$$

By the definition of $\tilde{\alpha}_\Phi$, $\tilde{\alpha}_\Phi \leq \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ for $0 < u \leq \frac{1}{2}$, so that

$$\Phi(\tilde{\alpha}_\Phi \Phi^{-1}(2u)) \leq u, \qquad 0 < u \leq \frac{1}{2}\,. \tag{18}$$

For each $x = (x(i))$, $y = (y(i)) \in l^{(\Phi)}$ satisfying $\|x\|_{(\Phi)} \leq 1$, $\|y\|_{(\Phi)} \leq 1$, we have $\rho_\Phi(x) \leq 1$, $\rho_\Phi(y) \leq 1$. Thus

$$\frac{1}{2}\Phi(|x(i)|) \leq \frac{1}{2}\,, \qquad \frac{1}{2}\Phi(|y(i)|) \leq \frac{1}{2}\,.$$

Note that (18) also holds for $u = 0$. Substitute $u_i = \frac{1}{2}\Phi(|x(i)|)$ and $u_i = \frac{1}{2}\Phi(|y(i)|)$ into (18); we have

$$\Phi(\tilde{\alpha}_\Phi |x(i)|) \leq \frac{1}{2}\Phi(|x(i)|), \qquad \Phi(\tilde{\alpha}_\Phi |y(i)|) \leq \frac{1}{2}\Phi(|y(i)|), \qquad i \geq 1. \tag{19}$$

It follows from (19) that

$$\begin{aligned}
\rho_\Phi(\tilde{\alpha}_\Phi(|x| \vee |y|)) &\leq \rho_\Phi(\tilde{\alpha}_\Phi|x|) + \rho_\Phi(\tilde{\alpha}_\Phi|y|) \\
&= \sum_{i=1}^{\infty} \Phi(\tilde{\alpha}_\Phi|x(i)|) + \sum_{i=1}^{\infty} \Phi(\tilde{\alpha}_\Phi|y(i)|) \\
&\leq \frac{1}{2}\sum_{i=1}^{\infty} \Phi(|x(i)|) + \frac{1}{2}\sum_{i=1}^{\infty} \Phi(|y(i)|) \\
&= \frac{1}{2}\rho_\Phi(x) + \frac{1}{2}\rho_\Phi(y) \leq 1.
\end{aligned}$$

Therefore,

$$\|(|x_0| \vee |y_0|)\|_{(\Phi)} \leq \frac{1}{\tilde{\alpha}_\Phi}\,.$$

Thus (16) holds since x and y are arbitrary. Consequently, (11) follows from (14), (16) and (17). (The idea of the proof for (17) is due to Benavides and Rodriguez [1] for estimation of the weakly convergent sequence coefficient $WCS(l^{(\Phi)})$.) $\square$

Remark 2.2. Proposition 1.2 may be deduced from (11). Indeed, if $\Phi \notin \nabla_2(0)$, then $\alpha_\Phi^0 = \frac{1}{2}$, and by (14), one has $2 \leq a(l^{(\Phi)})$, which implies $a(l^{(\Phi)}) = 2$ since $a(l^{(\Phi)}) \leq 2$ always holds. On the other hand, if $\Phi \in \nabla_2(0)$ then $\frac{1}{2} < \alpha_\Phi^0$. Observe that $[\Phi^{-1}(u)/\Phi^{-1}(2u)] > \frac{1}{2}$ for $0 < u \leq \frac{1}{2}$, therefore $\tilde{\alpha}_\Phi > \frac{1}{2}$, and hence, $a(l^{(\Phi)}) < 2$ by (11).

Example 2.3 (Borwein and Sims [2, p. 346]).

$$a(l^p) = 2^{\frac{1}{p}} (1 < p < \infty). \tag{20}$$

In [2], (20) was proved by means of Bohnenblust's inequality. Here we simply deduce it from Theorem 2.1. In fact, let $\Phi(u) = |u|^p$, then $l^p = l^{(\Phi)}$ and $\|\cdot\|_p = \|\cdot\|_{(\Phi)}$. Since

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-\frac{1}{p}}, \qquad 0 < u < \infty,$$

we have $\alpha_\Phi^0 = \tilde{\alpha}_\Phi = 2^{-\frac{1}{p}}$, and (20) holds.

We now present formulas for the exact value of $a(l^{(\Phi)})$.

Theorem 2.4. *For an N-function $\Phi \in \nabla_2(0)$, we have:*

(i) *if $F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}$ is increasing on $(0, \Phi^{-1}(1)]$, then*

$$a(l^{(\Phi)}) = 2^{\frac{1}{C_\Phi^0}}, \tag{21}$$

where $C_\Phi^0 = \lim_{t\to 0} \frac{t\phi(t)}{\Phi(t)}$;

(ii) *if $F_\Phi(t)$ is decreasing on $(0, \Phi^{-1}(1)]$, then*

$$a(l^{(\Phi)}) = \frac{\Phi^{-1}(1)}{\Phi^{-1}(\frac{1}{2})}. \tag{22}$$

Proof: (i) If $F_\Phi(t)$ is increasing on $(0, \Phi^{-1}(1)]$, then $C_\Phi^0 = \lim_{t\to 0} \frac{t\phi(t)}{\Phi(t)}$ exists; and $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is increasing on $(0, \frac{1}{2}]$ (see Proposition 1.4), so by Proposition 1.3(6),

$$\alpha'_\Phi = \alpha_\Phi^0 = \lim_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-\frac{1}{C_\Phi^0}},$$

and hence (21) follows from (11).

(ii) If $F_\Phi(t)$ is decreasing on $(0, \Phi^{-1}(1)]$, then $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is also decreasing on $(0, \frac{1}{2}]$. Therefore,

$$\alpha_\Phi^0 \geq \alpha'_\Phi = \tilde{\alpha}_\Phi = G_\Phi(\frac{1}{2}) = \frac{\Phi^{-1}(\frac{1}{2})}{\Phi^{-1}(1)}.$$

Thus (22) is derived from the above and (11). The proof is completed. □

Example 2.5. Suppose

$$M(u) = e^{|u|} - |u| - 1, \qquad N(v) = (1 + |v|)\ln(1 + |v|) - |v|. \tag{23}$$

Then it is not difficult to check that $F_M(t) = \frac{t(e^t-1)}{e^t-t-1}$ is increasing on $(0, \infty)$ and $C_M^0 = 2$ (see [9, Example 2.6]). Moreover, $F_N(s) = \frac{sN'(s)}{N(s)}$ is decreasing on $(0, \infty)$ and $C_N^0 = 2$. Thus, $M, N \in \nabla_2(0)$, which satisfy the condition of Theorem 2.4, so that

$$a(l^{(M)}) = \sqrt{2}, \tag{24}$$

$$a(l^{(N)}) = \frac{N^{-1}(1)}{N^{-1}(\frac{1}{2})} \approx 1.487. \tag{25}$$

3. Riesz angles of Orlicz sequence spaces with Orlicz norm

We first present some auxiliary lemmas. Observe that for a Banach lattice X, the Riesz angle $a(X)$ can be expressed as

$$a(X) = \sup\{\|(|x| \vee |y|)\| : x, y \in S(X)\}, \tag{26}$$

where $S(X)$ is the unit sphere of X.

Since

$$1 < \frac{C\Phi^{-1}(u_0)}{\Phi^{-1}(Cu_0)} < C, \qquad 0 < u_0 < \infty,$$

for any N-function Φ and $C > 1$, we have:

Proposition 3.1 (Chen [3])**.** *For an N-function Φ and a constant $C > 1$, one has:*

(i) $\Phi \in \nabla_2(0)$ *if and only if* $1 < \liminf_{u \to 0}[C\Phi^{-1}(u)/\Phi^{-1}(Cu)]$;

(ii) $\Phi \in \triangle_2(0)$ *if and only if* $\limsup_{u \to 0}[C\Phi^{-1}(u)/\Phi^{-1}(Cu)] < C$.

Lemma 3.2. *Let Φ and Ψ be a pair of N-functions. Denote*

$$Q_\Phi = \sup_{\|x\|_\Phi=1} \left\{ k_x > 1 : \|x\|_\Phi = \frac{1}{k_x}\,[1+\rho_\Phi(k_x x)] \right\} \tag{27}$$

and

$$q_\Phi = \inf_{\|x\|_\Phi=1} \left\{ k_x > 1 : \|x\|_\Phi = \frac{1}{k_x}\,[1+\rho_\Phi(k_x x)] \right\}. \tag{28}$$

Then we have

$$a^*_\Psi \le q_\Phi \le Q_\Phi \le b^*_\Psi, \tag{29}$$

where

$$a^*_\Psi = \inf\left\{\frac{s\psi(s)}{\Psi(s)} : 0 < s \le \Psi^{-1}(1)\right\},\ b^*_\Psi = \sup\left\{\frac{s\psi(s)}{\Psi(s)} : 0 < s \le \Psi^{-1}(1)\right\}. \tag{30}$$

Proof: The author [10] proved $Q_\Phi \le b^*_\Psi$. Considering (10), it remains to show

$$a^*_\Psi = \frac{b^*_\Phi}{b^*_\Phi - 1} \le q_\Phi. \tag{31}$$

In fact, if $\Phi \notin \triangle_2(0)$, then $\Psi \notin \nabla_2(0)$, and so $b^*_\Phi = \infty$, $\frac{b^*_\Phi}{b^*_\Phi-1} = 1 = a^*_\Psi$, which means (31) holds. We show that (31) holds for $\Phi \in \triangle_2(0)$.

For every $x = \{x(i)\} \in l^\Phi$, $\|x\|_\Phi = 1$, we have that $\rho_\Phi(4k_x x) < \infty$. Since

$$\Phi(4u) \ge 2u\phi(2u) \ge \Psi(\phi(2u))$$

we have $\rho_\Psi(\phi(2k_x x)) < \infty$. Now we shall prove

$$\lim_{\eta\to 0} \rho_\Psi[\phi((1+\eta)k_x x)] = \rho_\Psi[\phi(k_x x)]. \tag{32}$$

Indeed, for an arbitrary $\varepsilon > 0$, choose an i_0 satisfying

$$\sum_{i>i_0} \Psi[\phi(2k_x|x(i)|)] < \frac{\varepsilon}{2}\,.$$

We have for a sufficiently small η

$$\Psi[\phi(1+\eta)k_x|x(i)|)] - \Psi[\phi(k_x|x(i)|)] < \frac{\varepsilon}{2i_0} \qquad (i = 1, 2, \cdots, i_0)$$

by the right continuity of $\Psi(\phi(\cdot))$. It follows that

$$\begin{aligned}
&\rho_\Psi[\phi((1+\eta)k_x x)] - \rho_\Psi[\phi(k_x x)] \\
&\le \sum_{i=1}^{i_0} \{\Psi[\phi(1+\eta)k_x|x(i)|)] - \Psi[\phi(k_x|x(i)|)]\} + \sum_{i>i_0} \Psi[\phi(1+\eta)k_x|x(i)|)] \\
&\le \frac{\varepsilon}{2i_0} i_0 + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which implies (32). Since $\rho_\Psi[\phi((1+\eta)k_x x)] \ge 1$ for all $\eta > 0$, we deduce from (32) that

$$\rho_\Psi[\phi(k_x x)] \ge 1. \tag{33}$$

Analogously, we obtain

$$\rho_\Psi[\phi_{-}(k_x x)] \le 1, \tag{34}$$

where $\phi_{-}(t)$ is the left derivative of $\Phi(u)$. Therefore, $\Psi[\phi_{-}(k_x|x(i)|)] \le 1$ for every i, and hence,

$$k_x|x(i)| \le \psi_{-}[\phi_{-}(k_x|x(i)|)] \le \psi_{-}[\Psi^{-1}(1)] \le \psi[\Psi^{-1}(1)] \qquad i = 1, 2, \cdots. \tag{35}$$

Finally, by(34), (35) and the definition of b_Φ^*,

$$\begin{aligned}
1 \le \rho_\Psi[\phi(k_x x)] &= \sum_{i=1}^{\infty} \Psi[\phi(k_x|x(i)|)] \\
&= \sum_{i=1}^{\infty} [k_x|x(i)|\phi(k_x|x(i)|) - \Phi(k_x|x(i)|)] \\
&\le (b_\Phi^* - 1) \sum_{i=1}^{\infty} \Phi(k_x|x(i)|) \\
&= (b_\Phi^* - 1)\,[\|(k_x)x\|_\Phi - 1] \\
&= (b_\Phi^* - 1)(k_x - 1).
\end{aligned}$$

It follows that $\frac{b_\Phi^*}{b_\Phi^* - 1} \le k_x$. So (31) holds since x is arbitrary. $\square$

Theorem 3.3. (i) *Let Φ be an N-function and Ψ its complementary N-function. Then*

$$\max(2\beta_\Psi^0, 2\beta_\Psi') \le a(l^\Phi), \tag{36}$$

where

$$\beta_\Psi^0 = \limsup_{v\to 0} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}, \qquad \beta'_\Psi = \sup\left\{ \frac{\Psi^{-1}(\frac{1}{2k})}{\Psi^{-1}(\frac{1}{k})} : k = 1, 2, \cdots \right\};$$

(ii) *if* $\Phi \in \nabla_2(0)$ *then*

$$a(l^\Phi) \le \frac{2}{\theta_\Phi^0}, \tag{37}$$

where

$$\theta_\Phi^0 = \inf\left\{ \frac{(1+\frac{q_\Phi}{Q_\Phi})\Phi^{-1}(u)}{\Phi^{-1}[(1+\frac{q_\Phi}{Q_\Phi})u]} : 0 < u \le \frac{Q_\Phi}{q_\Phi + Q_\Phi}\Phi(\psi[\Psi^{-1}(1)]) \right\}. \tag{38}$$

Proof: (i) By the definition of β_Ψ^0, there exists a sequence $1 > v_n \searrow 0$ such that

$$\lim_{n\to\infty}[\Psi^{-1}(v_n)/\Psi^{-1}(2v_n)] = \beta_\Psi^0.$$

For any $0 < \varepsilon < \frac{1}{2}$, select a $v \in \{v_n : n \ge 1\}$ satisfying $2v < \varepsilon$ and

$$\frac{1}{\Psi^{-1}(2v)} > \frac{\beta_\Psi^0 - \varepsilon}{\Psi^{-1}(v)}. \tag{39}$$

Let $k = [\frac{1}{2v}]$ be the integral part of $(2v)^{-1}$. Then $k \le (2v)^{-1} < k+1$. Define

$$x = \frac{2v}{\Psi^{-1}(2v)}\sum_{i=1}^{k} e_i, \qquad y = \frac{2v}{\Psi^{-1}(2v)}\sum_{i=k+1}^{2k} e_i.$$

By the property of N-functions, for $u_2 \ge u_1 > 0$ we have

$$\frac{u_2}{\Psi^{-1}(u_2)} \ge \frac{u_1}{\Psi^{-1}(u_1)},$$

so that

$$\|x\|_\Phi = \|y\|_\Phi = \frac{2v}{\Psi^{-1}(2v)} k \Psi^{-1}(\frac{1}{k}) \le 1.$$

On the other hand, one deduces from (39) that

$$\begin{aligned}
\|(|x| \vee |y|)\|_\Phi &= \|x+y\|_\Phi \\
&= \frac{2v}{\Psi^{-1}(2v)} 2k\Psi^{-1}(\frac{1}{2k}) \\
&> 2(\beta_\Psi^0 - \varepsilon)\frac{v}{\Psi^{-1}(v)} 2k\Psi^{-1}(\frac{1}{2k}) \\
&\geq 2(\beta_\Psi^0 - \varepsilon)\frac{1}{2(k+1)\Psi^{-1}(\frac{1}{2(k+1)})} 2k\Psi^{-1}(\frac{1}{2k}) \\
&> 2(\beta_\Psi^0 - \varepsilon)\frac{k}{k+1} = 2(\beta_\Psi^0 - \varepsilon)(1 - \frac{1}{k+1}) \\
&> 2(\beta_\Psi^0 - \varepsilon)(1-\varepsilon).
\end{aligned}$$

So that $a(l^\Phi) \geq 2\beta_\Psi^0$ since ε is arbitrary.

Next we show $a(l^\Phi) \geq 2\beta'_\Psi$. In fact, for any fixed integer $k \geq 1$, let $Z_k = (0, 0, \cdots, 0)$, and $X_k = [k\Psi^{-1}(\frac{1}{k})]^{-1}(1, 1, \cdots, 1)$, with $\dim Z_k = \dim X_k = k$. Define

$$x_0 = (X_k, Z_k, Z_k, \cdots), \qquad y_0 = (Z_k, X_k, Z_k, \cdots).$$

Then $\|x_0\|_\Phi = \|y_0\|_\Phi = 1$, and

$$\|(|x_0| \vee |y_0|)\|_\Phi = \frac{2\Psi^{-1}(\frac{1}{2k})}{\Psi^{-1}(\frac{1}{k})}.$$

Therefore,

$$a(l^{(\Phi)}) \geq \sup\left\{\frac{2\Psi^{-1}(\frac{1}{2k})}{\Psi^{-1}(\frac{1}{k})} : k = 1, 2, \cdots\right\} = 2\beta'_\Psi.$$

(ii) If $\Phi \in \nabla_2(0)$, then $Q_\Phi < \infty$. By Proposition 3.1, we have

$$1 < \theta_\Phi^0 \leq 1 + \frac{q_\Phi}{Q_\Phi}. \tag{40}$$

By (38) one finds that if $0 < u \leq \frac{Q_\Phi}{q_\Phi + Q_\Phi}\Phi(\psi[\Psi^{-1}(1)])$, then

$$\theta_\Phi^0 \Phi^{-1}\Big[\Big(1 + \frac{q_\Phi}{Q_\Phi}\Big)u\Big] \leq \Big(1 + \frac{q_\Phi}{Q_\Phi}\Big)\Phi^{-1}(u).$$

Put $v = \Phi^{-1}\left[(1 + \frac{q_\Phi}{Q_\Phi})u\right]$ in the above inequality, then

$$\Phi\Big(\frac{\theta_\Phi^0 Q_\Phi v}{q_\Phi + Q_\Phi}\Big) \leq \frac{Q_\Phi}{q_\Phi + Q_\Phi}\Phi(v), \qquad 0 < v \leq \psi[\Psi^{-1}(1)]. \tag{41}$$

Let $x = (x(i)) \in S(l^\Phi)$, $y = (y(i)) \in S(l^\Phi)$. We have $q_\Phi < k \le Q_\Phi$ and $q_\Phi < h \le Q_\Phi$ satisfying

$$\frac{1}{k}[1+\rho_\Phi(kx)] = \|x\|_\Phi = 1 = \|y\|_\Phi = \frac{1}{h}[1+\rho_\Phi(hy)]. \tag{42}$$

Clearly,

$$\max\left(\frac{k}{k+h}, \frac{h}{k+h}\right) \le \frac{Q_\Phi}{q_\Phi + Q_\Phi}. \tag{43}$$

Since

$$\frac{\Phi(\theta_\Phi^0 b_1 v)}{b_1 \Phi(v)} \le \frac{\Phi(\theta_\Phi^0 b_2 v)}{b_2 \Phi(v)} \tag{44}$$

for $0 < b_1 \le b_2$ and $v > 0$ by the property of N-functions, it follows from (41) and (43) that

$$\Phi\left(\frac{\theta_\Phi^0 kv}{k+h}\right) \le \frac{k}{k+h}\Phi(v), \qquad \Phi\left(\frac{\theta_\Phi^0 hv}{k+h}\right) \le \frac{h}{k+h}\Phi(v) \tag{45}$$

for $0 < v \le \psi[\Psi^{-1}(1)]$. Note that $\rho_\Psi[\phi(k|x|)] \le 1$ and $\rho_\Psi[\phi(h|y|)] \le 1$. Thus we have

$$\max(k|x(i)|, h|y(i)|) \le \psi[\Psi^{-1}(1)] \tag{46}$$

for any $i \ge 1$. Consequently, we deduce from (46), (45) and (42) that

$$\begin{aligned}
\|\theta_\Phi^0(|x| \vee |y|)\|_\Phi &\le \frac{k+h}{kh}\left\{1 + \rho_\Phi\left[\frac{\theta_\Phi^0 kh}{k+h}(|x| \vee |y|)\right]\right\} \\
&\le \frac{k+h}{kh}\left\{1 + \sum_{i=1}^\infty \Phi\left(\frac{\theta_\Phi^0 h}{k+h}k|x(i)|\right)\right. \\
&\quad \left. + \sum_{i=1}^\infty \Phi\left(\frac{\theta_\Phi^0 k}{k+h}h|y(i)|\right)\right\} \\
&\le \frac{k+h}{kh}\left\{1 + \frac{h}{k+h}\rho_\Phi(kx) + \frac{k}{k+h}\rho_\Phi(hy)\right\} \\
&= 2.
\end{aligned}$$

So that (37) holds. The proof is completed. □

Corollary 3.4. $\Phi \in \nabla_2(0)$ *if and only if* $a(l^\Phi) < 2$.

PROOF: If $\Phi \notin \nabla_2(0)$ then $\Psi \notin \triangle_2(0)$, therefore $\beta^0_\Psi = 1$. Hence $a(l^\Phi) = 2$ by Theorem 3.3(i). If $\Phi \in \nabla_2(0)$, then $a(l^\Phi) < 2$ by (37) and (40). $\square$

The inverse function $\Phi^{-1}(u)$ of an N-function is usually hard to be expressed, so it is still difficult to estimate Riesz angles directly by means of (36) and (37). In order to make the calculation practical, we introduce the following proposition. The proof is the same as that of Proposition 1.5 in [9], replacing 2 by c.

Proposition 3.5. *For an N-function $\Phi(u) = \int_0^{|u|} \phi(t)\,dt$ and $c > 1$, define*

$$\alpha^0_\Phi(c) = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(cu)}, \qquad \beta^0_\Phi(c) = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(cu)}.$$

Then

$$c^{-\frac{1}{A^0_\Phi}} \le \alpha^0_\Phi(c) \le \beta^0_\Phi(c) \le c^{-\frac{1}{B^0_\Phi}} \tag{47}$$

where A^0_Φ and B^0_Φ are defined as in (2).

Example 3.6. Let a pair of N-functions be defined as in Example 2.5(23), i.e.,

$$M(u) = e^{|u|} - |u| - 1 \quad\text{and}\quad N(v) = (1+|v|)\ln(1+|v|) - |v|.$$

Now we estimate $a(l^M)$ and $a(l^N)$.

Since $F_M(t) = \frac{tM'(t)}{M(t)}$ is increasing on $(0,\infty)$ ([9, Example 2.6]), we have

$$a^*_M = \lim_{u\to 0} F_M(u) = 2,$$
$$b^*_M = F_M(u)|_{u=N'[N^{-1}(1)]} = F_M(u)|_{u=1} = \frac{e-1}{e-2},$$

and

$$a^*_N = F_N(v)|_{v=N^{-1}(1)} = F_N(v)|_{v=e-1} = e-1,$$
$$b^*_N = \lim_{v\to 0} F_N(v) = 2.$$

It follows from Lemma 3.2 that

$$1.718 \approx e-1 \le q_M \le Q_M \le 2; \tag{48}$$

$$2 = a^*_M \le q_N \le Q_N \le b^*_M = \frac{e-1}{e-2} \approx 2.392. \tag{49}$$

Since $G_M(c,u)=\frac{M^{-1}(u)}{M^{-1}(cu)}$ is also increasing on $(0,\infty)$ by Proposition 1.4, where $c=1+\frac{q_M}{Q_M}$, we get

$$\begin{aligned}\theta_M^0 &= \lim_{u\to 0}\frac{(1+\frac{q_M}{Q_M})M^{-1}(u)}{M^{-1}[(1+\frac{q_M}{Q_M})u]} = \Big(1+\frac{q_M}{Q_M}\Big)\Big(1+\frac{q_M}{Q_M}\Big)^{-\frac{1}{2}} = \sqrt{\Big(1+\frac{q_M}{Q_M}\Big)}\\ &\geq 1.3635\end{aligned}$$

by Proposition 3.5 ($A_M^0=B_M^0=C_M^0=2$).

On the other hand, $F_N(v)$ is decreasing on $(0,\infty)$, it follows also from Proposition 1.4 that $G_N(v)=\frac{N^{-1}(v)}{N^{-1}(2v)}$ is decreasing, and so

$$2\beta_N'=2\beta_N^0=\frac{1}{\alpha_M^0}=2^{\frac{1}{C_M^0}}=\sqrt{2}.$$

By (36) and (37) we have

$$1.414\approx\sqrt{2}\leq a(l^M)\leq\frac{2\sqrt{2}}{\sqrt{3}}\approx 1.469. \tag{50}$$

Finally we estimate $a(l^N)$. Note that the index function $G_N\left((1+\frac{q_N}{Q_N}),v\right)$ (take $c=1+\frac{q_N}{Q_N}$ in Proposition 1.4) is monotonically decreasing, hence we have (see Lemma 3.5)

$$\begin{aligned}\theta_N^0 &= \frac{(1+\frac{q_N}{Q_N})N^{-1}(v)}{N^{-1}[(1+\frac{q_N}{Q_N})v]}\Bigg|_{v=\frac{Q_N}{q_N+Q_N}N(M'[M^{-1}(1)])}\\ &\geq \frac{(1+\frac{q_N}{Q_N})N^{-1}(v)}{N^{-1}[(1+\frac{q_N}{Q_N})v]}\Bigg|_{v=\frac{b_M^*}{a_M^*+b_M^*}N(M'[M^{-1}(1)])}\\ &\approx \frac{(1+\frac{q_N}{Q_N})N^{-1}(v)}{N^{-1}[(1+\frac{q_N}{Q_N})v]}\Bigg|_{v=0.77118}\\ &\geq \frac{(1+\frac{a_M^*}{b_M^*})N^{-1}(v)}{N^{-1}[(1+\frac{a_M^*}{b_M^*})v]}\Bigg|_{v=0.77118}\\ &\approx 1.26502.\end{aligned}$$

Because

$$2\beta_M^0\leq 2\beta_M'=2\frac{M^{-1}(\frac{1}{2})}{M^{-1}(1)}\approx 1.4966,$$

we obtain the final estimation from (36) and (37) of Theorem 3.3

$$1.4966 \approx 2\frac{M^{-1}(\frac{1}{2})}{M^{-1}(1)} \leq a(l^N) \leq \frac{2}{\theta_N^0} \approx 1.5810. \tag{51}$$

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Department of Mathematics, Suzhou University, Jiangsu 215006, P.R. China
E-mail: yanyq@pub.sz.jsinfo.net

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