On p-injectivity, YJ-injectivity and quasi-Frobeniusean rings

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Dedicated to Professor Carl Faith on his 75-th birthday

Abstract. A new characteristic property of von Neumann regular rings is proposed in terms of annihilators of elements. An ELT fully idempotent ring is a regular ring whose simple left (or right) modules are either injective or projective. Artinian rings are characterized in terms of Noetherian rings. Strongly regular rings and rings whose two-sided ideals are generated by central idempotents are characterized in terms of special annihilators. Quasi-Frobeniusean rings are characterized in terms of p-injectivity. Also, a commutative YJ-injective ring with maximum condition on annihilators and finitely generated socle is quasi-Frobeniusean.

Keywords: von Neumann regular, V-ring, Artinian ring, p-injectivity, YJ-injectivity, quasi-Frobeniusean

Classification: 16D30, 16D36, 16D50

Introduction

Von Neumann regular rings, V-rings, self-injective rings and generalizations are extensively studied since several years (cf. for example, [1], [3]–[16], [28]–[30]). This sequel of [21] and [24] contains the following results for a ring A: (1) A is von Neumann regular iff A is a semi-prime ring such that every finitely generated one-sided ideal is the annihilator of an element of A (Theorem 1); (2) A is strongly regular iff for every $b \in A$, $Ab + I(AbA)$ is a complement right ideal of A (Proposition 3); (3) Left Noetherian rings whose essential right ideals are idempotent two-sided ideals are left Artinian (Proposition 4); (4) Special two-sided ideals are used to characterize rings whose two-sided ideals are generated by central idempotents (Proposition 8), (5) A is quasi-Frobeniusean iff A has a p-injective left generator and projective p-injective left A-modules are injective (Theorem 9); (6) If every simple right A-module is flat and every maximal left ideal of A is either injective or a two-sided ideal of A, then A is either a left self-injective regular left V-ring or strongly regular; (7) If A is commutative, then A is quasi-Frobeniusean iff A is a YJ-injective ring with maximum condition on annihilators and a finitely generated socle (Theorem 11).

Throughout, A denotes an associative ring with identity and A-modules are unital. Recall that a left A-module M is p-injective (resp f-injective) if, for any principal (resp. finitely generated) left ideal I of A, every left A-homomorphism of I into M extends to A (cf. [4, p. 122], [14, p. 340], [17]). A is called left p-injective (resp. f-injective) if ${}_AA$ is p-injective (resp. f-injective). Similarly, p-injectivity and f-injectivity are definited on the right side. (P-injectivity is also called principal injectivity in the literature). Following C. Faith, A is called a left V-ring if every simple left A-module is injective. A well-known theorem of I. Kaplansky asserts that a commutative ring is von Neumann regular iff it is a V-ring. In general, von Neumann regular rings need not be V-rings and the converse is not true either. A theorem of M. Ikeda-T. Nakayama asserts that A is a left p-injective ring iff every principal right ideal of A is a right annihilator. It is well-known that A is von Neumann regular iff every left (right) A-module is flat ([4, p. 91]). This remains true if "flat" is replaced by "p-injective" ([17]).

Our first result is motivated by [19, Question 1] and [27, Question 1].

Theorem 1. *The following conditions are equivalent:*

(1) *A is von Neumann regular;*
(2) *A is a semi-prime ring whose finitely generated one-sided ideals are annihilators of an element of A;*
(3) *A is a semi-prime ring such that every finitely generated left ideal is the left annihilator of an element of A and every principal right ideal of A is the right annihilator of an element of A.*

Proof: It is clear that (1) implies (2) while (2) implies (3). Assume (3). Let F be a finitely generated left ideal of A. Then $F = I(u)$, $u \in A$. Since $uA = r(w)$, $w \in A$, $F = I(uA) = I(r(w)) = I(r(Aw))$. But Aw is a left annihilator which implies that $Aw = I(r(Aw))$. Therefore $F = Aw$ which shows that every finitely generated left ideal of A is principal. Since A is semi-prime, A is left semi-hereditary by [4, Theorem 7.5B] and it follows that every principal left ideal of A is projective. Since every principal right ideal is a right annihilator, by Ikeda-Nakayama's theorem, A is left p-injective. Now A being a left *p.p.* ring is equivalent to "every quotient of a p-injective left A-module is p-injective" ([18, Remark 2]). Since A is left p-injective, every cyclic left A-module is p-injective which yields that A is von Neumann regular ([17, Lemma 2]). Thus (3) implies (1). □

In Theorem 1, the term "semi-prime" cannot be omitted (otherwise, any principal ideal quasi-Frobeniusean ring would be von Neumann regular !).

Remark 1. If every finitely generated one-sided ideal of A is the annihilator of an element, then A is a left and right IF-ring whose finitely generated one-sided ideals are principal (cf. [4, Theorem 6.9]).

Remark 2. The fact that a strongly regular ring is unit-regular follows from a cancellation theorem of G. Ehrlich (cf. [4, Corollary 6.3C] and [5, Corollary 4.2]).

The proof of [17, Proposition 1] shows in an elementary way that this result holds. Note that a left and right V-ring whose essential left ideals are two-sided is a unit-regular ring (whose prime factor rings are Artinian).

We may note the following characterization of principal ideal rings in terms of p-injectivity.

Remark 3. A is a principal left ideal ring iff every finitely generated left ideal of A is principal and every p-injective left A-module is injective.

Another result on annihilators.

Proposition 2. *If every finitely generated left ideal of A is the left annihilator of a finite subset of A and every finitely generated right ideal of A is a right annihilator, then A is left f-injective and right p-injective.*

Proof: Since every principal one-sided ideal of A is an annihilator, A is a left and right p-injective ring by Ikeda-Nakayama's theorem. Let F, K be finitely generated left ideals of A. By hypothesis, we have $F = I(U)$, $K = I(V)$, where U, V are finitely generated right ideals of A. Then $U = r(I(U))$, $V = r(I(V))$ which imply $r(F) + r(K) = r(I(U)) + r(I(V)) = U + V = r(I(U + V)) = r(I(U) \cap I(V)) = r(F \cap K)$. By Ikeda-Nakayama's theorem, A is left f-injective. $\square$

Question 1. If A is left p-injective such that every finitely generated left ideal of A is the left annihilator of an element of A, is A left f-injective?

Remark 4. In Proposition 2, $\operatorname{Soc}({}_A A) = \operatorname{Soc}(A_A)$.

Proposition 3. *The following conditions are equivalent:*

(1) *A is strongly regular;*
(2) *for any $b \in A$, $Ab + I(AbA)$ is a complement right ideal of A.*

Proof: Assume (1). For any $b \in A$, $Ab = Ae$, where e is a central idempotent. Then $I(AbA) = I(b) = A(1-e)$ and $Ab + I(AbA) = Ae \oplus A(1-e) = A$. Therefore (1) implies (2) evidently.

Assume (2). Suppose that $c \in A$ such that $(Ac)^2 = 0$. Then $I(AcA)$ is an essential right ideal of A. By hypothesis, $I(AcA) = Ac + I(AcA)$ is a complement right ideal of A which proves that $A = I(AcA)$, whence $c = 0$. This proves that A must be semi-prime. It follows that for any $b \in A$, $I = Ab \oplus I(AbA)$ (because $Ab \cap I(Ab) = 0$) is a complement right ideal of A. Now there exists a complement right ideal C of A such that $I \oplus C$ is an essential right ideal of A. Then $CAb \subseteq C \cap Ab \subseteq C \cap I = 0$ implies that $C \subseteq I(AbA) \subseteq I$ and hence $C \subseteq C \cap I = 0$. Therefore I is an essential right ideal of A which yields $I = A$. Therefore $A = Ab \oplus I(AbA)$ which proves that A is von Neumann regular. Since $AbA \cap I(AbA) = 0$ (in as much as A is semi-prime), then $A = Ab \oplus I(AbA) \subseteq AbA \oplus I(AbA)$ which yields $A = AbA \oplus I(AbA)$. For any $u \in AbA$, $u = v + w$,

$v \in Ab$, $w \in I(AbA)$. Then $u - v \in AbA \cap I(AbA) = 0$ which implies $u = v \in Ab$, proving that $Ab = AbA$ is generated by a central idempotent (because A is semi-prime). Thus (2) implies (1).

□

Question 2. If $Ab + r(AbA)$ is a complement left ideal of A for every $b \in A$, is A regular, biregular?

We now give a sufficient condition for left Noetherian rings to be left Artinian.

Proposition 4. *If A is a left Noetherian ring such that every essential right ideal of A is an idempotent two-sided ideal, then A is left Artinian.*

Proof: Let B be a prime factor ring of A. Then every essential right ideal of B is an idempotent two-sided ideal of B. For any $0 \neq b \in B$, set $T = BbB$. Let K be a complement right subideal of T such that $E = bB \oplus K$ is an essential right subideal of T. Since T is an essential right ideal of B (B being prime), E is an essential right ideal of B which implies that E is an idempotent two-sided ideal of B. Now $b \in E = E^2$ implies that $b = \sum_{i=1}^{n}(bb_i + k_i)(bc_i + s_i)$, $b_i, c_i \in B$ and $k_i, s_i \in K$, whence $b - \sum_{i=1}^{n} bb_i(bc_i + s_i) = \sum_{i=1}^{n} k_i(bc_i + s_i) \in bB \cap K = 0$. Then $b \in bBbB$ which proves that every right ideal of B is idempotent. Since every essential right ideal of B is two-sided, then B is von Neumann regular by [1, Theorem 3.1]. Since B is left Noetherian, it is well-known that B must be simple Artinian. If A is prime, then A is Artinian as just seen. If A is not prime, then by [3, Lemma 18.34B], A is left Artinian. This establishes the proposition. □

Note that the ring in Proposition 4 needs not be right duo. The proof of Proposition 4 yields a characterization of Artinian rings.

Theorem 5. *The following conditions are equivalent:*

(1) *A is left Artinian;*
(2) *A is a left Noetherian ring such that for any prime factor ring B of A, every essential right ideal of B is an idempotent two-sided ideal of B.*

An ideal of A will always mean a two-sided ideal of A. Recall that A is ELT (resp. ERT) if every essential left (resp. right) ideal of A is an ideal of A. As usual, A is called fully (resp. (1) fully left; (2) fully right) idempotent if every ideal (resp. (1) left ideal; (2) right ideal) of A is idempotent.

Note that if A is fully idempotent and every maximal left (with even every maximal right) ideal of A is an ideal, then A needs not be von Neumann regular ([28, Theorem 1]).

Theorem 6. *If A is an* ELT *fully idempotent ring, then A is a von Neumann regular ring whose simple right* (*or left*) *modules are either projective or injective.*

Proof: Let B be a prime factor ring of A. Then B is an ELT fully idempotent ring. The proof of Proposition 4 shows that B is fully left idempotent. By

[1, Theorem 3.1], B is von Neumann regular. Looking carefully at the proof of [1, Theorem 3.1], we see that A is also ERT. Let M be a maximal right ideal of A. If A/M_A is not projective, then M_A is essential in A_A which implies that M is an ideal of A and is therefore a maximal left ideal of A. For any $y \in M$, $y \in yAyA \subseteq yM$ which implies that the simple left A-module A/M is flat ([2, p. 458]). By [21, Lemma 1], A/M_A is injective. This proves that every simple right A-module is either injective or projective. Similarly, every simple left A-module is either injective or projective. □

Corollary 7. *An* ELT *fully idempotent ring is either regular with non-zero socle or strongly regular.*

The next remark is connected with [6, Corollary 6], [18, Lemma 1] and [26, Question 1].

Remark 5. If A is an ERT (or ELT) ring whose simple left modules are p-injective, then A is regular and every simple one-sided A-module is either injective or projective.

Remark 6. If A is a semi-prime ELT ring containing an injective maximal left ideal, then A is a left and right self-injective, left and right V-ring of bounded index. Consequently, A is left and right FPF by a theorem of S. Page [4, Theorem 5.49].

We now consider a particular class of biregular rings which generalizes simple non-Artinian rings and semi-simple Artinian rings.

We introduce two definitions.

Definitions. Let E, T be ideals of A, $E \subseteq T$. We say that

(1) T is an essential extension of E (or E is essential in T) if $E \cap N \neq 0$ for any non-zero ideal N of A contained in T;

(2) E is a complement ideal of A if E has no proper essential extension in A.

Proposition 8. *The following conditions are equivalent for a ring A:*

(1) *every ideal of A is generated by a central idempotent;*

(2) *for every ideal T of A, $T + (I(T) \cap r(T))$ is a complement ideal of A.*

Proof: Assume (1). Let T be an ideal of A. If $I = T + (I(T) \cap r(T))$, since $T = Ae$, where e is a central idempotent, then $I(T) = I(eA) = A(1-e) = (1-e)A = r(T)$ and $A = Ae \oplus A(1-e) = T \oplus (I(T) \cap r(T))$. Therefore (1) implies (2).

Assume (2). Let T be an ideal of A such that $T^2 = 0$. Then $r(T)$ is an essential left ideal of A which implies $r(T)$ is an essential ideal of A. Similarly, $I(T)$ is an essential ideal of A. Therefore $r(T) \cap I(T)$ is an essential ideal of A which implies that $T + I(T) \cap r(T)$ is an essential ideal of A. By hypothesis, $T + (I(T) \cap r(T))$ is a complement ideal of A which yields $T + (I(T) \cap r(T)) = A$.

But $T^2 = 0$ implies that $T \subseteq I(T) \cap r(T)$, whence $A = I(T) \cap r(T)$, yielding $A = I(T) = r(T)$. This implies $T = 0$ and proves that A must be semi-prime. Now for any ideal U of A, $U \cap I(U) = 0$ and $I(U) = r(U)$. If $I = U + (I(U) \cap r(U))$, then $I = U + I(U) = U \oplus I(U)$. Suppose that I is not an essential ideal of A: there exists a non-zero ideal N of A such that $I \cap N = 0$. Now $NU \subseteq N \cap U \subseteq N \cap I = 0$ which implies that $N \subseteq I(U) = r(U)$, whence $N = N \cap I(U) \subseteq N \cap I = 0$. This contradiction proves that I must be essential in A. By hypothesis, $I = A$ which proves that U is generated by a central idempotent (in as much as A is semi-prime). Thus (2) implies (1).

□

The following property of p-injectivity seems interesting.

Remark 7. If T is an ideal of A such that ${}_AA/T$ is p-injective, then the factor ring A/T is left p-injective.

Quasi-Frobeniusean rings are now characterized in terms of p-injectivity.

Theorem 9. *The following conditions are equivalent:*

(1) *A is quasi-Frobeniusean;*
(2) *A is left pseudo-Frobeniusean and projective p-injective right A-modules are injective;*
(3) *there exists a p-injective left generator of A-*Mod *and projective p-injective left A-modules are injective.*

Proof: (1) implies (2) and (3) by [3, Theorem 24.20].

Assume (2). Since A is left pseudo-Frobeniusean, then every left ideal of A is a left annihilator which implies that A is right p-injective. For any projective right A-module P, there exist B, a direct sum of copies of A_A, and an epimorphism p of B onto P_A. Then $B/\ker p \approx P_A$ implies that $B \approx \ker p \oplus B/\ker p$. Since B is a direct sum of p-injective right A-modules, then B_A is p-injective. Therefore $B/\ker p$ is a p-injective right A-module and hence P_A is p-injective. By hypothesis, P_A is injective. Then (2) implies (1) by [3, Theorem 24.20].

Assume (3). Let G be a p-injective left generator of A-Mod. For any projective left A-module F, there exists C, a direct sum of copies of ${}_AG$, and an epimorphism $q : {}_AC \to {}_AF$. As before, we obtain a p-injective left A-module F which is injective by hypothesis. Thus (3) implies (1) by [3, Theorem 24.20]. □

Recall that a left A-module M is YJ-injective if, for any $o \neq a \in A$, there exists a positive integer n with $a^n \neq 0$ such that every left A-homomorphism of Aa^n into M extends to A (cf. [15], [24], [30]). A is called left YJ-injective if ${}_AA$ is YJ-injective. Similarly, YJ-injectivity is defined on the right side. In [15], it is shown that YJ-injectivity generalizes p-injectivity even for rings (quasi-injectivity generalizes injectivity but the two concepts coincide for rings). Also left YJ-injective rings generalize right IF-rings. If A is left YJ-injective, then every non-zero-divisor is invertible in A. Consequently, A coincides with $Q^l_{cl}(A)$ and $Q^r_{cl}(A)$, the classical left and right quotient rings of A. It is well-known that A is left non-singular iff A has a maximal left quotient ring, noted $Q^l_{\max}(A)$, which is a left self-injective regular ring. If A is a reduced ring, $Q^l_{\max}(A)$, is not necessarily strongly regular. However, if A is reduced and admits a classical left quotient ring $Q^l_{cl}(A)$, then by [22, Proposition 1.5], $Q^l_{cl}(A)$ is a reduced ring (this is the case when A is left non-singular, left duo). In that case, if $Q^l_{cl}(A)$ is left or right YJ-injective, then it is strongly regular. Consequently, a reduced left YJ-injective ring is strongly regular.

A well-known theorem of Y. Utumi asserts that if A is left and right non-singular, then $Q^l_{\max}(A)$ and $Q^r_{\max}(A)$ coincide iff every complement one-sided ideal of A is an annihilator. (The terms "annihilator" and "complement" should be permuted in [4, p. 181]).

Left (or right) Johns rings are studied in [4].

Question 3. Is a left Johns, left YJ-injective ring quasi-Frobeniusean?

(We know that a left p-injective left Johns ring is quasi-Frobeniusean).

Now let J, Y, Z denote respectively the Jacobson radical, the right singular ideal and the left singular ideal of the ring A.

Proposition 10. *Let A be a ring whose simple right modules are flat. If every maximal left ideal of A is either injective or an ideal of A, then either A is a left self-injective regular left V-ring or A is strongly regular. Consequently, A must be a regular left V-ring.*

Proof: First suppose that every maximal left ideal of A is an ideal of A. For any maximal left ideal N of A, N is a maximal right ideal of A and by hypothesis, A/N_A is flat. Then ${}_AA/N$ is injective by [21, Lemma 1] which implies that A is a left V-ring, whence A is strongly regular (cf [17, Proposition 3]). Now suppose that there exists a maximal left ideal M of A which is not an ideal of A. Then ${}_AM$ is injective which implies $A = M \oplus U$, where U is a minimal projective left ideal of A. Let V be an arbitrary minimal projective left ideal of A. Write $V = Av$, $0 \neq v \in A$. If $MV = 0$, then $MAv = 0$ which implies that $MA = M$ (because $MA \neq A$). This contradicts the hypothesis that M is not an ideal of A. Therefore $MV \neq 0$ and $Mw \neq 0$ for some $0 \neq w \in V$. Now $Mw = V$ and we have an epimorphism $p : M \to V$ defined by $p(m) = mw$ for all $m \in M$. Then $M/\ker p \approx V$ which yields $M \approx \ker p \oplus M/\ker p$ (in as much as ${}_AV$ is projective). Since ${}_AM$ is injective, then so is $M/\ker p$, proving that ${}_AV$ is injective. In particular, ${}_AU$ is injective which implies that A is left self-injective. Now for any maximal left ideal L of A, if ${}_AL$ is injective, then ${}_AA/L$ is injective as just seen. If L is an ideal of A, then A/L_A is flat which implies that ${}_AA/L$ is injective ([21, Lemma 1]). In any case, ${}_AA/L$ is injective, proving that A is a left V-ring, whence $J = 0$ (cf. [18, Lemma 1]). Since A is left self-injective, $Z = J = 0$

and hence A is von Neumann regular. We conclude that A must be a regular, left V-ring. □

We now turn to a characterization of commutative quasi-Frobeniusean rings.

Theorem 11. *The following conditions are equivalent for a commutative ring A:*

(1) *A is quasi-Frobeniusean;*
(2) *A is a* YJ*-injective ring with maximum condition on annihilators and* $\mathrm{Soc}(A)$*, the socle of A, is finitely generated.*

Proof: It is clear that (1) implies (2).

Assume (2). Since A is a commutative YJ-injective, then A coincides with its classical quotient ring. Since A satisfies the maximum condition on annihilators, then A/J is Artinian ([4, Theorem 16.31]) and also Y is nilpotent. Now A being YJ-injective implies that $J = Y$ ([24, p. 103]), whence A is semi-primary. Therefore A has an essential socle. Given $\mathrm{Soc}(A)$ finitely generated, we then have a finitely embedded ring A satisfying the maximum condition on annihilators which yields A Artinian ([4, p. 164]). Since A is YJ-injective, every minimal ideal of A must be an annihilator. In that case, A is quasi-Frobeniusean by a theorem of H.H. Storrer. Thus (2) implies (1). □

Remark 8. A right YJ-injective ring whose simple right modules are either p-injective or projective is fully right idempotent (this is because $Y = J = Y \cap J = 0$).

Theorem 11 motivates the next question on YJ-injectivity.

Question 4. Is a commutative YJ-injective ring with maximum condition on annihilators quasi-Frobeniusean?

We add a remark on flatness and p-injectivity.

Remark 9. We know that A is von Neumann regular if every cyclic singular left A-module is flat (Math. J. Okayama Univ. 20 (1978), 123–129 (Theorem 5)). If every singular left A-module is injective, A needs not be von Neumann regular ([4, p. 92]). Consequently, this is also the case when every cyclic singular left A-module is p-injective. However, A is von Neumann regular if A is also left p-injective. We may also recall the following: If I is a p-injective left ideal of A, then ${}_AA/I$ is flat.

In 1974, we introduced the concept of p-injective modules to study von Neumann regular rings, V-rings and associated rings ([17]). In 1985, this is generalized to YJ-injective modules ([24]). In 1998, Xue Weimin showed that even for rings, YJ-injectivity effectively generalizes p-injectivity [15]. Finally, Zhang-Wu [30] proved that if every left A-module is YJ-injective, then A is von Neumann regular (which answers a question raised in [24]).

K.R. Goodearl's volume on von Neumann regular rings [5] has motivated numerous papers in that area during the last twenty years. It is now a classic for people interested in VNR rings (cf. [4]).

In view of [17, Proposition 1] and our Theorem 1 here, we raise the last question.

Question 5. Is A von Neumann regular if A is a semi-prime ring such that every principal one-sided ideal is the annihilator of an element of A?

Note that semi-prime rings whose principal one-sided ideals are annihilators need not be von Neumann regular (cf. K. Beidar–R. Wisbauer, Comm. Algebra 23 (1995), 841–861 (Example 4.8), which answered in the negative a question raised in 1981).

In recent years, Xue Weimin and Zhang Jule solved several problems raised in my papers. Among the still unanswered questions, we recall the following:

U.Q.1. Is A von Neumann regular if A satisfies any one of the following conditions: (a) A is left semi-hereditary and every maximal left ideal of A is p-injective; (b) A is a left p-injective left V-ring; (c) every principal left ideal of A is a projective left annihilator; (d) A is left semi-hereditary and every simple left A-module is flat; (e) A is a semi-prime left self-injective ring whose maximal essential left ideals are two-sided.

U.Q.2. Is A strongly regular if A is a reduced ring whose principal left ideals are complement left ideals?

U.Q.3. Is A fully left idempotent if every simple left A-module is YJ-injective? (The answer is "yes" if "YJ-injective" is replaced by "p-injective").

U.Q.4. Is A Artinian if A is a prime left self-injective ring whose maximal essential left ideals are two-sided?

U.Q.5. Is A left pseudo-Frobeniusean if A is a left Kasch ring containing an injective maximal left ideal?

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(*Received* July 19, 2001)