An example of a $\mathcal{C}^{1,1}$ function, which is not a d.c. function

Miroslav Zelený

Abstract. Let $X = \ell_p$, $p \in (2, +\infty)$. We construct a function $f : X \to \mathbb{R}$ which has Lipschitz Fréchet derivative on X but is not a d.c. function.

Keywords: Lipschitz Fréchet derivative, d.c. functions

Classification: 46B20, 26B25

We start with the following two definitions.

Definition 1. Let X be a normed linear space and $f : X \to \mathbb{R}$ be a function. We say that f is a *d.c. function* if f is a difference of two continuous convex functions on X.

It is easy to see that $f : X \to \mathbb{R}$ is a d.c. function if and only if there exists a continuous convex function h on X such that $f + h$ and $-f + h$ are continuous convex functions. Every such h is called a *control function for f*.

Definition 2. Let X be a normed linear space and $f : X \to \mathbb{R}$ be a function. We say that f is a $\mathcal{C}^{1,1}$ *function* if its Fréchet derivative $f'(x)$ exists at each point $x \in X$ and the mapping f' is Lipschitz on X.

The reader may consult [VZ] and [DVZ] for basic properties and also for generalizations of these notions.

The main aim of this note is to answer the following question posed in [DVZ].

Question. Does there exists a Banach space X and a $\mathcal{C}^{1,1}$ function on X, which is not d.c.?

The question is answered in the positive by the following theorem.

Theorem. *Let $X = \ell_p$, $p \in (2, +\infty)$. Then there exists a $\mathcal{C}^{1,1}$ function $f : X \to \mathbb{R}$, which is not a d.c. function.*

Remark. Let us remark that the class of d.c. functions contains the class of $\mathcal{C}^{1,1}$ functions on ℓ_p, where $p \in (1, 2]$. This result is a consequence of a more general theorem due to Duda, Veselý and Zajíček ([DVZ, Theorem 11]).

Research supported by Research Grant GAUK 160/1999, GAČR 201/00/0767 and MSM 113200007.

We denote the set of all finite sequences from $\{0,1\}$ by $\mathrm{Seq}\{0,1\}$ and if $s \in \mathrm{Seq}\{0,1\}$, then $s\frown 0$ ($s\frown 1$, respectively) stands for the concatenation of the sequences s and (0) (s and (1), respectively). The length of $s \in \mathrm{Seq}\{0,1\}$ is denoted by $|s|$. Let X be a normed linear space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x,r)$.

The following auxiliary notion will be helpful in the sequel.

Definition 3. Let X be a Banach space. We say that points x_s, $s \in \mathrm{Seq}\{0,1\}$, form an *$S$-family in X*, if there exists a sequence $\{r_n\}_{n=0}^{\infty}$ of positive real numbers such that the following conditions are satisfied:

(a) $\frac{1}{2}(x_{s\hat{}0} + x_{s\hat{}1}) = x_s$ for every $s \in \mathrm{Seq}\{0,1\}$,
(b) the set $\{x_s;\ s \in \mathrm{Seq}\{0,1\}\}$ is bounded,
(c) $\|x_s - x_t\| \geq \max\{r_{|s|}, r_{|t|}\}$ for every $s,t \in \mathrm{Seq}\{0,1\}$, $s \neq t$,
(d) $\sum_{n=0}^{\infty} r_{2n}^2 = +\infty$,
(e) $\lim r_n = 0$.

Lemma 1. *Let X be a Banach space, let $T = (x_s)_{s\in\mathrm{Seq}\{0,1\}}$ be an indexed set with elements in X. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $h : X \to \mathbb{R}$ is a function satisfying*

$$(\star) \qquad \forall\, s \in \mathrm{Seq}\{0,1\} : \ \frac{1}{2}(h(x_{s\hat{}0}) + h(x_{s\hat{}1})) - h(x_s) \geq c_{|s|+1},$$

then for every $n \in \mathbb{N}\cup\{0\}$ there exists $s \in \{0,1\}^n$ with $h(x_s) \geq h(x_\emptyset) + \sum_{j=1}^{n} c_j$.

Proof: We will proceed by induction over n. The case $n = 0$ is obvious. (Note that we use the convention saying that $\sum_{j=1}^{0} c_j = 0$.) Suppose that the assertion holds for n and we will deal with the case "$n+1$". Using induction hypothesis we have $h(x_s) \geq h(x_\emptyset) + \sum_{j=1}^{n} c_j$ for some $s \in \{0,1\}^n$. According to $(\star)$ we have $h(x_{s\hat{}i}) \geq h(x_s) + c_{n+1}$ for some $i \in \{0,1\}$. Thus we conclude

$$h(x_{s\hat{}i}) \geq h(x_\emptyset) + \left(\sum_{j=1}^{n} c_j\right) + c_{n+1}$$

and we are done. □

The next lemma is very easy to prove, so the proof will be omitted.

Lemma 2. *Let X be a Banach space and f be a d.c. function on X with a control function h. Then for every $x \in X$ and $v \in X$ we have*

$$\frac{1}{2}(h(x+v) + h(x-v)) - h(x) \geq \left|\frac{1}{2}(f(x+v) + f(x-v)) - f(x)\right|.$$

The next lemma uses the notion of bump function, which means a function with nonempty bounded support.

Lemma 3. *Let X be a Banach space with a $\mathcal{C}^{1,1}$ bump function. Suppose that there exists an S-family in X. Then there exists a $\mathcal{C}^{1,1}$ function $f : X \to \mathbb{R}$ which is not a d.c. function.*

Proof: Let $T = (x_s)_{s\in\mathrm{Seq}\{0,1\}}$ be an S-family in X and let $\{r_n\}_{n=0}^{\infty}$ be the corresponding sequence of real numbers from Definition 3. Let φ be a $\mathcal{C}^{1,1}$ bump function on X. We may assume that the support of φ is contained in the unit ball of X and $\varphi(0) = 1$. We may also assume that $0 \le \varphi(x) \le 1$ for every $x \in X$. Indeed, we can use $h \circ \varphi$, where $h : \mathbb{R} \to [0,1]$ is a $\mathcal{C}^{\infty}$ function with $h(0) = 0$ and $h(1) = 1$, instead of φ, if necessary. Denote $E = \{s \in \mathrm{Seq}\{0,1\};\ |s| \text{ is even}\}$. For every $s \in E$ we define a function $\psi_s : X \to \mathbb{R}$ by

$$\psi_s(x) = r_{|s|}^2 \varphi\left(\frac{4}{r_{|s|}}(x - x_s)\right).$$

We denote $B_s = B(x_s, \frac{1}{4}r_{|s|})$ for $s \in E$. Now we define a function $\psi : X \to \mathbb{R}$ putting $\psi(x) = \sum_{s\in E} \psi_s(x)$. We will verify the following properties of ψ:

(i) ψ is well defined on X,
(ii) Fréchet derivative $\psi'(x)$ exists for each $x \in X$,
(iii) the mapping $x \mapsto \psi'(x)$ is Lipschitz.

(i) We have $\operatorname{supp}\psi_s \subset B_s$ for every $s \in E$. The system $\{B_s;\ s \in E\}$ of balls is disjoint by the property (c) of S-family T and thus ψ is well defined on X.

(ii) If $x \in \overline{B_s}$ for some $s \in E$, then $\psi'(x)$ exists since $\psi = \psi_s$ on some neighborhood of x. If $x \in X \setminus \overline{\bigcup_{s\in E} B_s}$, then $\psi'(x)$ exists since $\psi = 0$ on some neighborhood of x.

It remains to deal with $x \in \overline{\bigcup_{s\in E} B_s} \setminus \bigcup_{s\in E} \overline{B_s}$. Then we have $\psi(x) = 0$. We show that $\psi'(x) = 0$. Take $y \in X$, $y \ne x$. We distinguish two cases.

a) If $y \notin \bigcup_{s\in E} \overline{B_s}$, then $\psi(y) = 0$ and we have $|\psi(x) - \psi(y)|/\|x - y\| = 0$.

b) If $y \in \overline{B_s}$ for some $s \in E$, then $\|x - y\| \ge \frac{1}{4}r_{|s|}$ since $B(x_s, \frac{1}{2}r_{|s|})$ intersects no ball B_t, $t \in E$, $t \ne s$. We obtain

$$\frac{|\psi(x) - \psi(y)|}{\|x - y\|} \le \frac{r_{|s|}^2}{\frac{1}{4}r_{|s|}} = 4r_{|s|}.$$

Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \ge n_0$, we have $4r_n < \varepsilon$. Then we can find $\delta > 0$ such that $B(x, \delta)$ intersects only those B_s's with $|s| \ge n_0$. Now the above discussion gives

$$\frac{|\psi(x) - \psi(y)|}{\|x - y\|} < \varepsilon$$

for every $y \in B(x,\delta) \setminus \{x\}$. This proves $\psi'(x) = 0$.

(iii) Let $K_0 > 0$ be the Lipschitz constant of the mapping $x \mapsto \varphi'(x)$. According to the definition of ψ we have that the mapping $x \mapsto \psi'(x)$ is Lipschitz on $\overline{B_s}$, $s \in E$, with the Lipschitz constant $K_1 = 16K_0$. Now take $x, y \in X$ such that these points are not elements of the same B_t, $t \in E$. If $x \in B_s$ for some $s \in E$, then we find $\tilde{x} \in X$ such that $\tilde{x}$ is an element of the segment with endpoints x and y and lies on the boundary of B_s. If $x \in X \setminus \bigcup_{t\in E} B_t$ we put $\tilde{x} = x$. The element $\tilde{y}$ is defined in the analogical way. We have $\psi'(\tilde{x}) = 0$ and $\psi'(\tilde{y}) = 0$. We estimate

$$\begin{aligned} \|\psi'(x) - \psi'(y)\| &\le \|\psi'(x) - \psi'(\tilde{x})\| + \|\psi'(\tilde{x}) - \psi'(\tilde{y})\| + \|\psi'(\tilde{y}) - \psi'(y)\| \\ &\le K_1\|x - \tilde{x}\| + 0 + K_1\|\tilde{y} - y\| \le K_1\|x - y\|. \end{aligned}$$

Thus we have verified the property (iii).

Since T is a bounded set and $\lim r_n = 0$ we have that $\operatorname{supp}\psi$ is bounded. So take $R > 0$ with $\operatorname{supp}\psi \subset B(0,R)$. We find a sequence $\{B(z_n, d_n)\}_{n=1}^{\infty}$ of balls with disjoint closures such that $\lim z_n = 0$ and $0 < 2d_n < \|z_n\|$. The desired function f is defined as follows:

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \text{ where } f_n(x) = d_n^2 \psi\left(\frac{R}{d_n}(x - z_n)\right).$$

We have to verify the following properties:

(iv) f is well defined on X,
(v) $f'(x)$ exists for each $x \in X$,
(vi) the mapping $x \mapsto f'(x)$ is Lipschitz,
(vii) f is not a d.c. function.

(iv) The supports of f_n's are disjoint and thus f is well defined.

(v) The function ψ is obviously bounded. Let C be a constant such that $|\psi(x)| \le C$ for every $x \in X$. If $x \in X \setminus \{0\}$, then $f = f_n$ for some $n \in \mathbb{N}$ on some neighborhood of x. Thus the derivative $f'(x)$ clearly exists for every $x \ne 0$. We show that $f'(0) = 0$. We have $f(0) = 0$, therefore it is sufficient to show that $\lim_{y\to 0} |f(y)|/\|y\| = 0$.

Fix $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $Cd_n < \varepsilon$ for every $n \in \mathbb{N}$, $n \ge n_0$. Find $\delta > 0$ such that $B(0,\delta)$ intersects no ball $B(z_n, d_n)$ with $n < n_0$. Take $y \in B(0,\delta) \setminus \{0\}$. If $y \notin \bigcup_{n=1}^{\infty} B(z_n, d_n)$, then $f(y) = 0$ and therefore $|f(y)|/\|y\| = 0$. If $y \in B(z_n, d_n)$ for some $n \in \mathbb{N}$, then $n \ge n_0$ and we have $|f(y)|/\|y\| \le Cd_n^2/d_n = Cd_n < \varepsilon$. Thus we have $|f(y)|/\|y\| < \varepsilon$ for every $y \in B(0,\delta) \setminus \{0\}$. This gives $f'(0) = 0$.

(vi) The mapping $x \mapsto f_n'(x)$ is Lipschitz on $\overline{B(z_n, d_n)}$ with the Lipschitz constant K_1R^2. Using the same method as in the proof of the property (iii) we obtain that $x \mapsto f'(x)$ is Lipschitz with the constant K_1R^2.

(vii) Suppose to the contrary that f is a d.c. function. Let h be a control function for f. Since h is continuous there exists $\tau > 0$ such that $|h(x)| < 1$ for every $x \in B(0,\tau)$. Then there exists $m \in \mathbb{N}$ with $B(z_m, d_m) \subset B(0,\tau)$. Put

$$y_s := \frac{d_m}{R} x_s + z_m, \qquad s \in \operatorname{Seq}\{0,1\}\,.$$

Using Lemma 2 we have that

$$\frac{1}{2}(h(y_{s\hat{}0}) + h(y_{s\hat{}1})) - h(y_s) \geq \left|\frac{1}{2}(f(y_{s\hat{}0}) + f(y_{s\hat{}1})) - f(y_s)\right|$$

for every $s \in \operatorname{Seq}\{0,1\}$ and $i \in \{0,1\}$. The construction of f and ψ_s's gives

$$f(y_s) = f_m(y_s) = d_m^2 \psi(x_s) = \begin{cases} 0, & |s| \text{ is odd;} \\ d_m^2 r_{|s|}^2, & |s| \text{ is even.} \end{cases}$$

Thus we have

$$\frac{1}{2}(h(y_{s\hat{}0}) + h(y_{s\hat{}1})) - h(y_s) \geq \begin{cases} d_m^2 r_{|s|+1}^2, & |s| \text{ is odd;} \\ d_m^2 r_{|s|}^2, & |s| \text{ is even.} \end{cases}$$

Put $c_{2n-1} = d_m^2 r_{2n-2}^2$ and $c_{2n} = d_m^2 r_{2n}^2$ for $n \in \mathbb{N}$. For every $s \in \operatorname{Seq}\{0,1\}$ we have

$$\frac{1}{2}(h(y_{s\hat{}0}) + h(y_{s\hat{}1})) - h(y_s) \geq c_{|s|+1}.$$

Using Lemma 1 and the fact that $\sum_{n=0}^{\infty} r_{2n}^2 = +\infty$ we obtain that there exists $y_s \in B(z_m, d_m) \subset B(0,\tau)$ with $h(y_s) > 1$, a contradiction. $\square$

For the sake of completeness we prove the following well-known result.

Lemma 4. *Let $X = \ell_p$, $p \in (2,+\infty)$. Then there exists a $\mathcal{C}^{1,1}$ bump function on X.*

PROOF: Fix $p \in (2,+\infty)$. Using [DGZ, Theorem 1.1., p. 184] it is easy to see that the function $w : X \to \mathbb{R}$ defined by $w(x) = \|x\|^p$ has bounded second Fréchet derivative on the unit ball. (The symbol $\|.\|$ stands for the canonical norm on ℓ_p.) Let $\tau : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ bump with $\operatorname{supp}\tau \subset [-1,1]$. Putting $g = \tau \circ w$ we obtain the desired bump. $\square$

PROOF OF THEOREM: According to Lemma 4 there exists a $\mathcal{C}^{1,1}$ bump on X. Thus it is sufficient to show that X contains an S-family. Such a set can be defined as follows. We put

$$x_\emptyset = (0,0,0,\dots)$$

$$x_s = \Big((-1)^{s_1}, (-1)^{s_2}/\sqrt{2}, \dots, (-1)^{s_n}/\sqrt{n}, 0, 0, \dots\Big), \quad s = (s_1,\dots,s_n) \in \{0,1\}^n.$$

The corresponding r_n's are defined by $r_n = 1/\sqrt{n+1}$, $n \in \mathbb{N} \cup \{0\}$. A direct calculation shows that $T = (x_s)_{s \in \operatorname{Seq}\{0,1\}}$ satisfies the conditions (a)—(e) from Definition 3. Using Lemma 3 we are done. $\square$

References

[DGZ] Deville R., Godefroy G., Zizler V., *Smoothness and Renormings in Banach Spaces*, Longman, 1993.

[DVZ] Duda J., Veselý L., Zajíček L., *On d.c. functions and mappings*, submitted to Atti Sem. Mat. Fis. Univ. Modena.

[VZ] Veselý L., Zajíček L., *Delta-convex mappings between Banach spaces and applications*, Dissertationes Math. (Rozprawy mat.) **289** (1989), 52 pp.

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail: zeleny@karlin.mff.cuni.cz

(*Received* October 25, 2001)