Metrics with homogeneous geodesics on flag manifolds

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Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. A geodesic of a homogeneous Riemannian manifold $(M = G/K, g)$ is called homogeneous if it is an orbit of an one-parameter subgroup of G. In the case when $M = G/H$ is a naturally reductive space, that is the G-invariant metric g is defined by some non degenerate biinvariant symmetric bilinear form B, all geodesics of M are homogeneous. We consider the case when $M = G/K$ is a flag manifold, i.e. an adjoint orbit of a compact semisimple Lie group G, and we give a simple necessary condition that M admits a non-naturally reductive invariant metric with homogeneous geodesics. Using this, we enumerate flag manifolds of a classical Lie group G which may admit a non-naturally reductive G-invariant metric with homogeneous geodesics.

Keywords: homogeneous Riemannian spaces, homogeneous geodesics, flag manifolds

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1. Introduction

A classical problem of differential geometry is to study geodesics of Riemannian manifolds (M, g). Of particular interest are geodesics with some special properties, for example homogeneous geodesics. A geodesic of a Riemannian manifold (M, g) is called homogeneous if it is an orbit of a one-parameter group of isometries of M.

Homogeneous geodesics have important applications to mechanics. For example, the equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M. Homogeneous geodesics of M are called by V.I. Arnold "relative equilibriums". The description of such relative equilibria is important for qualitative description of the behaviour of the corresponding mechanical system with symmetries. There is a big literature in mechanics devoted to the investigation of relative equilibria.

In differential geometry homogeneous geodesics have been studied by many authors. In 1965 R. Hermann showed that homogeneous geodesics which are orbits of a given 1-parameter group of isometries $a(t)$ correspond to the critical points of the square norm $g(X, X)$ of the Killing vector field X which generates $a(t)$. B. Kostant [Kost] and E.B. Vinberg [Vin] found a simple condition that the orbit $\gamma(t) = a(t)o$ through the point $o = eK$ of an 1-parameter subgroup $a(t) = \exp tX \subset G$ of the isometry group G of a homogeneous Riemannian manifold $M = G/K$, is a geodesic.

If all geodesics in a Riemannian manifold (M, g) are homogeneous, then M is called a *g.o. space*, and the metric g is called a *g.o. metric*. The terminology was introduced by O. Kowalski and L. Vanhecke, who initiated a systematic study of such spaces. In [Ko-Va] many interesting results had been proved. The class of g.o. spaces includes the subclass of naturally reductive spaces, i.e. homogeneous Riemannian manifolds (M, g) whose metric g is induced by a non-degenerate biinvariant bilinear form B on the Lie algebra $\mathfrak{g}$ of some transitive group G of isometries. If B is proportional to the Killing form of $\mathfrak{g}$ then the metric g is called standard. In particular, O. Kowalski and L. Vanhecke gave the first example of a compact g.o. space which is not naturally reductive, and classified all such g.o. spaces in dimension ≤ 6. The structure of the g.o. spaces was clarified by C. Gordon [Go]. In fact, she reduced the classification of g.o. spaces M to three special cases in which (a) M is a nilmanifold (i.e. a nilpotent Lie group with left-invariant Riemannian metric), (b) M is compact, or (c) M admits a transitive non-compact semisimple Lie group of isometries. She described g.o. spaces in case (a). Another approach for description of g.o. spaces was proposed by O. Kowalski, S.Ž. Nikčević and Z. Vlášek in the works [Ko-Ni] and [Ko-Ni-Vl], as well as by Z. Dušek in [Du1] and [Du2].

The problem of classification of compact non-naturally reductive g.o. spaces M remains open. In this paper we study it for the case when M is a flag manifold, that is a homogeneous manifold G/K which is an adjoint orbit of a compact semisimple Lie group G. This means that the stabilizer K is the centralizer of a torus S in G. We associate with a flag manifold $M = G/K$ the so called T-root system R_T ([A-P]), which consists of the restriction of the roots of the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathrm{Lie}\, G^{\mathbb{C}}$ to the center of the stability subalgebra $\mathfrak{k} = \mathrm{Lie}\, K$. We define the notion of the connected components of R_T and we prove that if R_T is connected (i.e. it has only one connected component) then the standard metric on M, defined by a multiple of the Killing form of $\mathfrak{g}$, is the only metric with homogeneous geodesics. For the case of the classical Lie groups, we describe all flag manifolds $M = G/K$ with non-connected T-root system R_T. As a corollary, we get the following theorem.

Theorem. *Let $M = G/K$ be a Riemannian flag manifold of a classical Lie group G. Assume that M is a g.o. space with respect to a non-standard G-invariant metric. Then M must be of the form $SO(2\ell+1)/U(\ell-m) \times SO(2m+1)$ for some $\ell \geq 2$, $m \geq 0$, (the manifold of all CR structures in $\mathbb{R}^{2\ell+1}$).*

For $\ell = 2, m = 0$ one obtains the example $SO(5)/U(2)$ of O. Kowalski–L. Vanhecke [Ko-Va] of a g.o. space which is in no way naturally reductive.

2. Homogeneous geodesics on a Riemannian homogeneous space

A Riemannian manifold (M, g) is called homogeneous if it admits a transitive connected Lie group G of isometries. We will identify such a manifold with the coset space G/K, where K is the stabilizer of a point $o \in M$. We will assume that the Lie algebra $\mathfrak{g}$ of G has an Ad^G-invariant non-degenerate symmetric bilinear form B such that $\mathfrak{k}$ (the Lie algebra of K) is non-degenerate, and we denote by $\mathfrak{m} = \mathfrak{k}^{\perp}$ the orthogonal complement to $\mathfrak{k}$ with respect to B. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive decomposition of $\mathfrak{g}$, that is $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. We may identify $\mathfrak{m}$ with the tangent space $T_oM = T_oG/K$. Then the isotropy representation of K is identified with the restriction $\mathrm{Ad}^K_{|\mathfrak{m}}$ of the adjoint representation of K on $\mathfrak{g}$ to $\mathfrak{m}$.

The metric g on M induces an Ad^K-invariant inner product g_o on $\mathfrak{m} \cong T_oM$ $(o = eK)$ which can be written as $g_o(x, y) = B(Ax, y)$ $(x, y \in \mathfrak{m})$, where A is an Ad^K-invariant B-symmetric operator on $\mathfrak{m}$. If $B_{|\mathfrak{m}}$ is positively defined, then the operator A is positively defined. Conversely, any such operator A determines an Ad^K-invariant scalar product $< \cdot, \cdot >= B(A\cdot, \cdot)$ on $\mathfrak{m}$, which defines an invariant Riemannian metric g on M. We will say that A is the operator associated with the metric g, and that g is generated by the operator A.

Proposition 1. *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold with the metric g generated by an operator A, and let $a \in \mathfrak{k}$, $x \in \mathfrak{m}$. Then the orbit $\gamma(t) = \exp t(a + x) \cdot o$ of the one-parameter subgroup $\exp t(a + x)$ through the point $o = eK$ is a geodesic of M if and only if one of the following conditions is fulfilled:*

(1) $[a + x, Ax] \in \mathfrak{k}$;
(2) $\langle [a, x], y\rangle = \langle x, [x, y]_{\mathfrak{m}}\rangle$ *for all* $y \in \mathfrak{m}$;
(3) $\langle [a + x, y]_{\mathfrak{m}}, x\rangle = 0$ *for all* $y \in \mathfrak{m}$.

Here $Z_{\mathfrak{m}}$ is the $\mathfrak{m}$-component of a vector $Z \in \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

Condition (3) was established by B. Kostant [Kost], E.B. Vinberg [Vin], and O. Kowalski–L. Vanhecke [Ko-Va]. Condition (1) is its reformulation in terms of the operator A, and obviously is equivalent to condition (2).

An element $a+x \in \mathfrak{g}$ which satisfies one of the equivalent conditions (1), (2), (3) is called a *geodesic vector*.

A homogeneous Riemannian manifold (M, g) is called a *g.o. space*, if all its geodesics are homogeneous geodesics.

Corollary 2. *A homogeneous Riemannian manifold $(G/K, g)$ is a g.o. space if and only if for every $x \in \mathfrak{m}$ there exists an $a(x) \in \mathfrak{k}$ such that*

$$[a(x) + x, Ax] \in \mathfrak{k}. \tag{1}$$

Examples of g.o. spaces are the naturally reductive spaces. A Riemannian manifold (M, g) and its metric g is called *naturally reductive* (or more precisely G-naturally reductive) if it admits a transitive Lie group G of isometries such that the Lie algebra $\mathfrak{g}$ has a non-degenerate Ad^G-invariant symmetric bilinear form B which is positively defined on $\mathfrak{m} = \mathfrak{k}^{\perp}$, and such that the metric g on $M = G/K$

is induced by the scalar product $B_{|\mathfrak{m}}$. Here $\mathfrak{k}$ is the stability subalgebra of the point $o = eK \in M = G/K$. If B is proportional to the Killing form of the Lie algebra $\mathfrak{g}$, then the associated metric is called *standard*. Note that if G is a simple compact Lie group, then any G-naturally reductive metric on a homogeneous space $M = G/K$ is standard.

Since the metric g is generated by the identity endomorphism $A = \mathrm{Id}$, a naturally reductive manifold is a g.o. space and any vector from $\mathfrak{m}$ is a geodesic vector. The converse statement is not true even if $M = G/K$ is a homogeneous manifold of a compact semisimple Lie group G. The first example of a non-standard compact homogeneous Riemannian manifold $M = G/K$ with homogeneous geodesics was discovered by O. Kowalski and L. Vanhecke [Ko-Va]. They proved that the manifold $SO(5)/U(2)$ is a g.o. space which is in no way naturally reductive.

3. Riemannian flag manifolds

A homogeneous manifold $M = G/K$ of a compact semisimple Lie group G is called a *flag manifold* if it is isomorphic to an adjoint orbit of the group G. This means that the stabilizer K is the centralizer of a torus in G.

A flag manifold $M = G/K$ equipped with a G-invariant Riemannian metric g is called a *Riemannian flag manifold*. Let $M = G/K$ be a flag manifold. We denote by $\mathfrak{g}, \mathfrak{k}$ the Lie algebras of the groups G, K and by $\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}$ their complexifications. Let $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$, hence also of $\mathfrak{g}^{\mathbb{C}}$. Then we have the following Cartan decompositions

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_{\alpha}, \qquad \mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_{\alpha}$$

where R (respectively R_K) is the root system of $\mathfrak{g}^{\mathbb{C}}$ (respectively of $\mathfrak{k}^{\mathbb{C}}$) with respect to $\mathfrak{h}^{\mathbb{C}}$. We denote by $R_M = R \setminus R_K$ the set of *complementary roots*. Then

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathfrak{g}_{\alpha}$$

and root vectors $\{E_{\beta} \in \mathfrak{g}_{\alpha} \colon \beta \in R_M\}$ form a basis of $\mathfrak{m}^{\mathbb{C}}$.

We denote by $\mathfrak{h} = \mathfrak{h}^{\mathbb{C}} \cap i\mathfrak{k}$ the real ad-diagonal subalgebra, and by

$$\mathfrak{t} = Z(\mathfrak{k}^{\mathbb{C}}) \cap \mathfrak{h}$$

the intersection of the center $Z(\mathfrak{k}^{\mathbb{C}})$ with $\mathfrak{h}$. Then $\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{k}'^{\mathbb{C}}$ where $\mathfrak{k}'^{\mathbb{C}}$ is the semisimple part of $\mathfrak{k}^{\mathbb{C}}$.

We consider the restriction map

$$\kappa \colon \mathfrak{h}^* \to \mathfrak{t}^*, \quad \alpha \mapsto \alpha\big|_{\mathfrak{t}}$$

and set $R_T = \kappa(R) = \kappa(R_M)$. The elements of R_T are called *T-roots*.

There exists a 1-1 correspondence between T-roots ξ and irreducible submodules $\mathfrak{m}_\xi$ of the $\mathrm{ad}^{\mathfrak{k}^{\mathbb{C}}}$-module $\mathfrak{m}^{\mathbb{C}}$ which is given by

$$R_T \ni \xi \leftrightarrow \mathfrak{m}_\xi = \sum_{\kappa(\alpha)=\xi} \mathfrak{g}_\alpha.$$

We get the following decomposition

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_T} \mathfrak{m}_\xi$$

of $\mathfrak{m}^{\mathbb{C}}$ into a sum of non equivalent irreducible $\mathrm{ad}^{\mathfrak{k}^{\mathbb{C}}}$-submodules.

From now on we will denote by B the negative of the Killing form of the Lie algebra $\mathfrak{g}$ which is positively defined. We remark that the complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$ interchanges $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, hence also $\mathfrak{m}_\xi$ and $\mathfrak{m}_{-\xi}$. This implies that any G-invariant Riemannian metric g on $M = G/K$ is defined by the scalar product $B(A\cdot, \cdot)$ on $\mathfrak{m}$, where the operator A is given by

$$A = \sum_{\xi \in R_T^+} \lambda_\xi \mathrm{Id}_{(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi})}.$$

Here $R_T^+ = \kappa(R^+)$ is the set of all positive T-roots (i.e. the restriction to $\mathfrak{t}$ of the system R^+ of positive roots of R), and λ_ξ are positive constants. We remark that λ_ξ are the eigenvalues of the operator A.

The scalar operator $A = \lambda \mathrm{Id}$ corresponds to the standard metric of the flag manifold $M = G/K$.

4. A necessary condition that a flag manifold admits a non-standard invariant metric with homogeneous geodesics

We give a necessary condition that a Riemannian flag manifold $M = G/K$ admits a non-standard invariant metric with homogeneous geodesics in terms of the connectedness of the associated T-root system $R_T = R|\mathfrak{t}$.

Definition. Two non-proportional T-roots ξ, η are called adjacent if $\xi + \eta \in R_T$ or $\xi - \eta \in R_T$.

We start from the following statement, which is a corollary of Proposition 1.

Proposition 3. *Let $(M = G/K, g)$ be a Riemannian flag manifold which is a g.o. space, where the invariant metric g is generated by the operator A with eigenvalues $\lambda_\xi, \xi_\eta \in R_T^+$. If ξ, η are two adjacent T-roots then $\lambda_\xi = \lambda_\eta$.*

Proof: By Corollary 2, $[a + x,\ Ax] \in \mathfrak{k}$ for all $x \in \mathfrak{m}$ and some $a = a(x) \in \mathfrak{k}$. We will assume that $\xi + \eta \in R_T$ and choose

$$x = x_\xi + x_{-\xi} + x_\eta + x_{-\eta} \in \mathfrak{m} \cap (\mathfrak{m}_\xi + \mathfrak{m}_{-\xi} + \mathfrak{m}_\eta + \mathfrak{m}_{-\eta})$$

such that $0 \neq [x_\xi, x_\eta] \in \mathfrak{m}_{\xi+\eta}$. Then condition (1) can be written as

$$\begin{gathered}[a + x_\xi + x_{-\xi} + x_\eta + x_{-\eta}, \lambda(x_\xi + x_{-\xi}) + \mu(x_\eta + x_{-\eta})] \equiv \\ (\mu - \lambda)([x_\xi, x_\eta] + [x_{-\xi}, x_{-\eta}] + [x_\xi, x_{-\eta}] + [x_{-\xi}, x_\eta]) \\ \bmod (\mathfrak{m}_\xi + \mathfrak{m}_\eta + \mathfrak{m}_{-\xi} + \mathfrak{m}_{-\eta} + \mathfrak{k}),\end{gathered}$$

where $\lambda = \lambda_\xi$, $\mu = \lambda_\eta$. Since the first term belongs to $\mathfrak{m}_{\xi+\eta}$ and the other terms belong to other $\mathfrak{k}$-modulus, it follows that $\lambda = \mu$. □

Definition. Two T-roots $\xi, \eta \in R_T$ are called connected if there exists a chain of T-roots

$$\xi = \xi_1, \xi_2, \ldots, \xi_s = \pm\eta$$

such that ξ_i, ξ_{i+1} are adjacent for $i = 1, \ldots, s-1$.

We define ξ and $-\xi$ to be connected. If ξ and 2ξ are the only T-roots, these are not connected.

The connectedness is an equivalence relation. Hence the set R_T of T-roots is decomposed into a disjoint union

$$R_T = R^1 \cup \cdots \cup R^r$$

of subsets R^i consisting from mutually connected T-roots. We denote by R^i $(i = 1, \ldots, r)$ the connected components of R_T, and we say that R_T is connected if $r = 1$.

Proposition 4. *Let $(M = G/K, g)$ be a Riemannian flag manifold. If M is a g.o. space, then*

$$\lambda_\xi = \lambda_\eta \ \text{ for } \ \xi, \eta \in R^i, \ (i = 1, \ldots, r).$$

Hence we obtain the following:

Theorem 5. *If the T-root system R_T of a flag manifold $M = G/K$ is connected, then the standard metric is the only G-invariant metric of M that makes M a g.o. space.*

Recall that any flag manifold $M = G/K$ is simply connected and has the canonically defined decomposition

$$M = G/K = G_1/K_1 \times G_2/K_2 \times \cdots \times G_n/K_n$$

where $G_1, \ldots, G_n$ are simple factors of the (connected) Lie group G. This decomposition is the de Rham decomposition of M equipped with any invariant metric g. In particular, (M, g) has homogeneous geodesics if and only if all factors $(M_i = G_i/K_i, g_i = g_{|M_i})$ have homogeneous geodesics. This reduces the problem of the description of invariant metrics with homogeneous geodesics on a flag manifold $M = G/K$ to the case when the group G is simple. By using Theorem 6 we solve this problem for the flag manifolds $M = G/K$ of the classical simple Lie groups $G = SU(n), SO(n)$ and $Sp(n)$.

5. Flag manifolds of classical groups that are g.o. spaces with respect to a non-standard invariant metric

By Theorem 5, if a flag manifold $M = G/K$ admits a non standard invariant metric with homogeneous geodesics then the associated system R_T of T-roots is not connected. We consider the cases when G is one of the classical groups A_ℓ, B_ℓ, C_ℓ and D_ℓ, and describe the flag manifolds G/K with non connected T-root system R_T.

Case of A_ℓ.

A flag manifold of the group $A_\ell = SU(n)$, $n = \ell+1$ is determined by an integer vector $\bar{n} = (n_1, \dots, n_s)$ such that $n_1 \geq n_2 \geq \dots \geq n_s \geq 1$ and $n = n_1 + \dots + n_s$, and it has the form

$$A(\bar{n}) = SU(n)/S(U(n_1) \times \dots \times U(n_s)).$$

We describe the associated T-root system R_T as follows (see [A-P], [A]):

Let $\epsilon = \{\epsilon_1, \dots, \epsilon_n\}$ be the standard basis of $\mathbb{R}^n$. It is more convenient to pass to dual indexes of the vectors of the basis ϵ, such that
$\epsilon = \{\epsilon^1_1, \dots, \epsilon^1_{n_1}, \epsilon^2_1, \dots, \epsilon^2_{n_2}, \dots, \epsilon^s_1, \dots, \epsilon^s_{n_s}\}$.

Then we may assume that $R_K = \{\epsilon^a_i - \epsilon^a_j\}$ and $R_M = \{\epsilon^a_i - \epsilon^b_j\colon a \neq b\}$. By deleting the lower indexes, we get the T-root system

$$R_T = \{\epsilon^a - \epsilon^b\colon a, b = 1, \dots, s\}$$

which is the root system of type A_{s-1}. Hence, it is connected. We obtain

Proposition 6. *The T-root system of the flag manifold $A(\bar{n}) = SU(n)/S(U(n_1) \times \dots \times U(n_s))$ is connected, hence $A(\bar{n})$ is a g.o. space with respect to the standard metric only.*

Case of $G = B_\ell$, C_ℓ or D_ℓ.

Now following [A-P] we describe the root systems $R, R_K, R_M = R \setminus R_K$ for all flag manifolds of the classical groups $B_l = SO(2\ell+1)$, $C\ell = Sp(\ell)$, or $D_\ell = SO(2\ell)$. Any such flag manifold is defined by an integer vector $\bar{\ell} = (\ell_1, \dots, \ell_k, m)$, such that

$$\ell_1 \geq \dots \geq \ell_k \geq 1,\ m \geq 0,\ k \geq 0,\ \ell = \ell_1 + \dots + \ell_k + m,$$

and it has the form

$$B(\bar{\ell}) = SO(2\ell+1)/U(\ell_1)\times\cdots\times U(\ell_k)\times SO(2m+1),$$
$$C(\bar{\ell}) = Sp(\ell)/U(\ell_1)\times\cdots\times U(\ell_k)\times Sp(m),$$
$$D(\bar{\ell}) = SO(2\ell)/U(\ell_1)\times\cdots\times U(\ell_k)\times SO(2m).$$

Let $\epsilon = \{\epsilon_i^a, \pi_j\}$ be an orthonormal basis of $\mathbb{R}^\ell$, where $a = 1,\dots,k$, $j = 1,\dots,m$, and for a given a the index i takes the values $1,\dots,\ell_a$. Then we can describe the root systems R, R_K, associated with the flag manifolds as follows :

$$R = \{\pm\epsilon_i^a \pm \epsilon_j^b,\ \pm\epsilon_i^a \pm \pi_j,\ \pm\pi_i \pm \pi_j,\ \pm\mu\epsilon_i^a,\ \pm\mu\pi_j\},$$
$$R_K = \{\pm(\epsilon_i^a - \epsilon_j^a),\ \pm\pi_j \pm \pi_k,\ \pm\mu\pi_j\},$$

where $\mu = 1$ in the case B_ℓ, $\mu = 2$ for C_ℓ and $\mu = \emptyset$ for D_ℓ.

In the case of B_ℓ

$$R_M^+ = R^+ \setminus R_K^+ = \{\epsilon_i^a + \epsilon_{i'}^a,\ \epsilon_i^a \pm \epsilon_j^b,\ \epsilon_i^a \pm \pi_j,\ \epsilon_i^a \colon i < i',\, a < b\}.$$

The system of positive T-roots is given by

$$R_T^+ = \{(2\epsilon^a),\, \epsilon^a \pm \epsilon^b,\, \epsilon^a\}$$

where the vector $2\epsilon^a$ is absent if $\ell_a = 1$. If $k = 1$ it takes the form $R_T^+ = \{2\epsilon,\, \epsilon\}$ and it is not connected. In all other cases it is connected. Hence we obtain:

Proposition 7. *A flag manifold of the group $G = B_\ell$ with a non-connected R_T has the form $M = SO(2\ell+1)/U(\ell-m)\times SO(2m+1)$. Only these manifolds may be g.o. spaces with respect to a non-standard $SO(2\ell+1)$-invariant metric.*

Similarly in the cases C_ℓ and D_ℓ the T-root system is given as follows:

Case C_ℓ:

$$R_M^+ = \{2\epsilon_i^a,\ \epsilon_i^a + \epsilon_{i'}^a,\, \epsilon_i^a \pm \epsilon_j^b,\ \epsilon_i^a \pm \pi_j\},$$
$$R_T^+ = \{2\epsilon^a,\ \epsilon^a \pm \pi,\ \epsilon^a \pm \epsilon^b\}.$$

Case D_ℓ:

$$R_M^+ = \{\epsilon_i^a + \epsilon_{i'}^a,\ \epsilon_i^a \pm \epsilon_j^b,\ \epsilon_i^a \pm \pi_j\},$$
$$R_T^+ = \{2\epsilon^a,\ \epsilon^a \pm \epsilon^b,\ \epsilon^a \pm \pi\}.$$

One can check that R_T is always connected. Hence we get the following final result.

Theorem 8. *Let $M = G/K$ be a flag manifold of a classical Lie group $G = A_\ell, B_\ell, C_\ell$, or D_ℓ. Assume that M is a g.o. space with respect to a non-standard G-invariant metric. Then $G = B_\ell$, and M has the form $M = SO(2\ell+1)/U(\ell-m)\times SO(2m+1)$ for some $\ell \geq 2$, $m \geq 0$.*

6. Homogeneous geodesics in flag manifolds

In order to further analyze whether the flag manifold $SO(2\ell+1)/U(\ell-m)\times SO(2m+1)$ is a g.o. space, we will firstly give an equivalent formulation of Corollary 2 for the case of a general flag manifold G/K.

Recall the reductive decomposition of $\mathfrak{g}^{\mathbb{C}}$ as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha\in R_K} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in R_M} \mathfrak{g}_\alpha.$$

Then the real Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} = \sum_{\gamma=1}^{n} i\mathbb{R}H_\gamma \oplus \sum_{\alpha\in R^+} \mathbb{R}(E_\alpha - E_{-\alpha}) \oplus \sum_{\alpha\in R^+} i\mathbb{R}(E_\alpha + E_{-\alpha}),$$

where $\{H_1,\dots,H_n; E_\alpha\ (\alpha\in R)\}$ is a Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$. Then a vector x in $\mathfrak{m}$ has the form

$$x = \sum_{\alpha\in R_M^+} z_\alpha E_\alpha - \sum_{\alpha\in R_M^+} \bar{z}_\alpha E_{-\alpha} \quad (z_\alpha\in\mathbb{C})$$

and a vector a in $\mathfrak{k}$ has the form

$$a = \sum_{\gamma=1}^{n} y_\gamma H_\gamma + \sum_{\phi\in R_K^+} w_\phi E_\phi - \sum_{\phi\in R_K^+} \bar{w}_\phi E_{-\phi} \quad (w_\phi\in\mathbb{C}, y_\gamma\in i\mathbb{R}).$$

Then M is a g.o. space if for all x in $\mathfrak{m}$, there exists an $a=a(x)\in\mathfrak{k}$ such that

$$[a(x), Ax] + [x, Ax] \in \mathfrak{k}. \tag{2}$$

We obtain the following:

Proposition 9. *The flag manifold $(M=G/K,g)$ is a g.o. space if and only if for each $z_\alpha, \bar{z}_\alpha$ $(\alpha\in R_M^+)$ the following linear system of $|R_M^+|$ equations has a solution in y_γ $(\gamma=1,\dots,n)$, w_ϕ, $\bar{w}_\phi$ $(\phi\in R_K^+)$:*

$$\begin{aligned}
&z_\delta\lambda_\delta \sum_{\gamma=1}^{n} \frac{2(\delta,\gamma)}{(\gamma,\gamma)} y_\gamma \\
&+ \sum_{\phi\in R_K^-(\delta)} w_\phi z_{\delta-\phi}\lambda_{\delta-\phi}N_{\phi,\delta-\phi} - \sum_{\phi\in R_K^+(\delta)} \bar{w}_\phi z_{\delta+\phi}\lambda_{\delta+\phi}N_{-\phi,\delta+\phi} \\
&+ \sum_{\alpha\in R_M^-(\delta)} z_\alpha z_{\delta-\alpha}\lambda_{\delta-\alpha}N_{\alpha,\delta-\alpha} - \sum_{\alpha\in R_M^+(\delta)} \bar{z}_\alpha z_{\delta+\alpha}\lambda_{\delta+\alpha}N_{-\alpha,\delta+\alpha} = 0
\end{aligned}$$

for all $\delta \in R_M^+$. Here $R_K^{\pm}(\delta) = \{\phi \in R_K^+ : \delta \pm \phi \in R_M^+\}$, $R_M^{\pm}(\delta) = \{\alpha \in R_M^+ : \delta \pm \alpha \in R_M^+\}$, and λ_δ are the eigenvalues of the operator A that generates the G-invariant metric g.

Example.

Let $M = SO(5)/U(2)$. A Cartan subalgebra has the form $\{\mathrm{diag}(\epsilon_1, \epsilon_2) : \epsilon_i \in \mathbb{C}\}$. Then $R_K = \{\pm(\epsilon_1 - \epsilon_2)\}$, $R_M = \{\pm(\epsilon_1 + \epsilon_2), \pm\epsilon_1, \pm\epsilon_2\}$, hence $R_T^+ = \{2\epsilon, \epsilon\}$. An $SO(5)$-invariant metric, hence the operator A, depends on two parameters $\lambda_1 = \lambda_{\epsilon_1} = \lambda_{\epsilon_2} = \lambda_\epsilon$ and $\lambda_2 = \lambda_{\epsilon_1+\epsilon_2} = \lambda_{2\epsilon}$.

A vector $x \in \mathfrak{m}$ has the form

$$x = z_{\epsilon_1+\epsilon_2} E_{\epsilon_1+\epsilon_2} + z_{\epsilon_1} E_{\epsilon_1} + z_{\epsilon_2} E_{\epsilon_2} - \bar{z}_{\epsilon_1+\epsilon_2} E_{-(\epsilon_1+\epsilon_2)} - \bar{z}_{\epsilon_1} E_{-\epsilon_1} - \bar{z}_{\epsilon_2} E_{-\epsilon_2},$$

and an $a = a(x) \in \mathfrak{k}$ has the form

$$a = y_1 H_{\epsilon_1-\epsilon_2} + y_2 H_{\epsilon_2} + w_{\epsilon_1-\epsilon_2} E_{\epsilon_1-\epsilon_2} - \bar{w}_{\epsilon_1-\epsilon_2} E_{-(\epsilon_1-\epsilon_2)}.$$

Then the system of Proposition 9 reduces to the following:

$$\begin{aligned}
&2 z_{\epsilon_1+\epsilon_2} \lambda_1 y_2 = 0 \\
&z_{\epsilon_1} \lambda_2 y_2 + z_{\epsilon_2} \lambda_2 N_{\epsilon_1-\epsilon_2,\epsilon_2} w_{\epsilon_1-\epsilon_2} = \bar{z}_{\epsilon_2} z_{\epsilon_1+\epsilon_2} \lambda_1 N_{-\epsilon_2,\epsilon_1+\epsilon_2} \\
&- z_{\epsilon_2} \lambda_2 y_1 + 2 z_{\epsilon_2} \lambda_2 y_2 - z_{\epsilon_1} \lambda_2 N_{-(\epsilon_1-\epsilon_2),\epsilon_1} \bar{w}_{\epsilon_1-\epsilon_2} = \bar{z}_{\epsilon_1} z_{\epsilon_1+\epsilon_2} \lambda_1 N_{-\epsilon_1,\epsilon_1+\epsilon_2}
\end{aligned}$$

which has a solution for every $z_{\epsilon_1+\epsilon_2}, z_{\epsilon_1}, z_{\epsilon_2}, \bar{z}_{\epsilon_1}, \bar{z}_{\epsilon_2}$. Hence $SO(5)/U(2)$ is a g.o. space with respect to a non-standard $SO(5)$-invariant metric, which agrees with the result of O. Kowalski and L. Vanhecke.

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