Stability of the geodesic flow for the energy

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Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. We study the stability of the geodesic flow ξ as a critical point for the energy functional when the base space is a compact orientable quotient of a two-point homogeneous space.

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Classification: 53C20, 53C22, 53C30, 58E20

1. Introduction

Consider a Riemannian manifold (M,g) and its unit tangent sphere bundle T_1M equipped with the Sasaki metric g_S. A unit vector field ξ on (M,g), if it exists, defines a map $\xi\colon (M,g) \to (T_1M, g_S)\colon x \mapsto \xi_x$ between Riemannian manifolds, embedding M into T_1M. In this way, it makes sense to say that ξ is *harmonic* (defining a harmonic map) or *minimal* (defining a minimal immersion). Of course, when M is compact and orientable, this corresponds to ξ being a critical point for the energy functional or the volume functional, respectively. (We refer to [2] for more information and further references.)

In previous work, we have looked at a distinguished vector field on the unit tangent bundle T_1M of a Riemannian manifold (M,g), namely the geodesic flow vector field ξ which is unit for g_S. We showed in [2] that ξ is both harmonic and minimal if the base manifold (M,g) is locally isometric to a two-point homogeneous space. When (M,g) is in addition compact and orientable, this raises the question about the stability of ξ as a critical point for energy and volume.

In this paper, we restrict our attention to the energy functional E and leave all considerations about the more complicated case of the volume aside. Moreover, we only look at the energy functional E *restricted to those maps* $\varphi\colon T_1M \to T_1(T_1M)$ *which arise from unit vector fields on* (T_1M, g_S) in the way described above. Already in this restricted setting, we find that the geodesic flow ξ on the unit tangent sphere bundle of a compact orientable quotient of a two-point

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homogeneous space is unstable in quite a number of cases. For that purpose, we calculate explicit expressions for the Hessian $(\mathrm{Hess}\, E)_\xi$ of the energy in specific directions orthogonal to ξ.

We start by lifting vector fields X on M to T_1M in such a way that the lifts are everywhere orthogonal to ξ. In this way we obtain *tangential* and *modified horizontal lifts* and we evaluate the Hessian $(\mathrm{Hess}\, E)_\xi$ for these lifts (Lemma 1). By taking special vector fields X on M, we can obtain negative values in some cases, thereby proving instability of ξ. The positively and negatively curved spaces require different types of vector fields. For the negatively curved ones, we lift vector fields X for which the dual one-form $X^\flat$ is harmonic. In this way, we show: *if (M^n, g), $n \geq 3$, is a compact and orientable quotient of a two-point homogeneous space of non-positive curvature with non-vanishing first Betti number $b_1(M)$, then the geodesic flow ξ on T_1M is unstable for the energy functional* (Theorem 3). We note that, according to [3], compact quotients always exist. For the case of positive curvature, we lift Killing vector fields X. The results now are less clear-cut. Informally speaking, we obtain: *if (M^n, g), $n \geq 3$, is a compact and orientable quotient of a two-point homogeneous space of positive curvature, then the existence of non-zero Killing vector fields implies instability of ξ for the energy functional for well-defined ranges of the dimension n and the curvature* (see Theorem 5 and the comments thereafter for a more precise statement).

For Kähler manifolds (M, g, J), we can use the complex structure J to define natural unit vector fields on (T_1M, g_S) orthogonal to ξ and different from the lifts mentioned above. In this way, we derive additional results concerning the instability of ξ. These also give information for the case of a two-dimensional surface of constant curvature, for which the method with lifts provided no answers. In the final section, we look at two-dimensional space forms in some more detail, based on earlier results by the second and third author about left-invariant unit vector fields on Lie groups ([6]).

With these results, certain questions concerning stability of ξ as a critical point of E remain wide open. The most intriguing one concerns the unit spheres $S^n(1)$ of dimension $n > 2$. Our method does not give any decisive answers in this case.

2. The Hessian on specific vector fields

We first recall a few of the basic facts and formulas about the unit tangent sphere bundle of a Riemannian manifold. A more elaborate exposition and further references can be found in [1].

The tangent bundle TM of a Riemannian manifold (M, g) consists of pairs (x, u) where x is a point in M and u a tangent vector to M at x. The mapping $\pi\colon TM \to M\colon (x, u) \mapsto x$ is the natural projection from TM onto M. It is well-known that the tangent space to TM at a point (x, u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker \pi_{*|(x,u)}$ and the horizontal subspace

$HTM_{(x,u)}$ with respect to the Levi Civita connection ∇ of (M,g): $T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}$.

For $w \in T_xM$, there exists a unique horizontal vector $w^h \in HTM_{(x,u)}$ for which $\pi_*(w^h) = w$. It is called the *horizontal lift* of w to (x,u). There is also a unique vertical vector $w^v \in VTM_{(x,u)}$ for which $w^v(df) = w(f)$ for all functions f on M. It is called the *vertical lift* of w to (x,u). These lifts define isomorphisms between T_xM and $HTM_{(x,u)}$ and $VTM_{(x,u)}$ respectively. Hence, every tangent vector to TM at (x,u) can be written as the sum of a horizontal and a vertical lift of uniquely defined tangent vectors to M at x. The *horizontal* (respectively *vertical*) *lift of a vector field* X on M to TM is defined in the same way by lifting X pointwise. Further, if T is a tensor field of type $(1,s)$ on M and $X_1,\dots,X_{s-1}$ are vector fields on M, then we denote by $T(X_1,\dots,u,\dots,X_{s-1})^v$ the vertical vector field on TM which at (x,w) takes the value $T(X_{1x},\dots,w,\dots,X_{s-1\,x})^v$, and similarly for the horizontal lift. In general, these are *not* the vertical or horizontal lifts of a vector field on M.

The *Sasaki metric* g_S on TM is completely determined by

$$g_S(X^h,Y^h) = g_S(X^v,Y^v) = g(X,Y)\circ\pi, \quad g_S(X^h,Y^v) = 0$$

for vector fields X and Y on M.

Our interest lies in the unit tangent sphere bundle T_1M which is a hypersurface of TM consisting of all unit tangent vectors to (M,g). It is given implicitly by the equation $g_x(u,u) = 1$. A unit normal vector field N to T_1M is given by the vertical vector field u^v. We see that horizontal lifts to $(x,u) \in T_1M$ are tangent to T_1M, but vertical lifts in general are not. For that reason, we define the *tangential lift* w^t of $w \in T_xM$ to $(x,u) \in T_1M$ by $w^t = w^v - g(w,u)N$. Clearly, the tangent space to T_1M at (x,u) is spanned by horizontal and tangential lifts of tangent vectors to M at x. One defines the *tangential lift of a vector field* X on M in the obvious way.

If we consider T_1M with the metric induced from the Sasaki metric g_S of TM, also denoted by g_S, we turn T_1M into a Riemannian manifold. Its Levi Civita connection $\bar\nabla$ is described completely by

$$\begin{aligned}
&\bar\nabla_{X^t}Y^t = -g(Y,u)X^t,\\
&\bar\nabla_{X^t}Y^h = \frac{1}{2}\,(R(u,X)Y)^h,\\
&\bar\nabla_{X^h}Y^t = (\nabla_X Y)^t + \frac{1}{2}\,(R(u,Y)X)^h,\\
&\bar\nabla_{X^h}Y^h = (\nabla_X Y)^h - \frac{1}{2}\,(R(X,Y)u)^t
\end{aligned} \tag{1}$$

for vector fields X and Y on M. We now have the necessary formulas for the computations which follow.

The geodesic flow vector field $\xi = u^h$ on T_1M is a unit vector field for the Sasaki metric g_S. For an arbitrary vector field X on M, the tangential lift X^t is orthogonal to ξ, but the horizontal lift in general is not. For that reason, we define the *modified horizontal lift* $\bar{X}^h$ of X by

$$\bar{X}^h = X^h - g(X,u)\,\xi.$$

This vector field on T_1M is orthogonal to ξ and tangent to T_1M. The aim of this section is to compute explicit expressions for $(\mathrm{Hess}\,E)_\xi(X^t, X^t)$ and for $(\mathrm{Hess}\,E)_\xi(\bar{X}^h, \bar{X}^h)$ for arbitrary vector fields X on a compact and orientable quotient (M,g) of a two-point homogeneous space, which we always suppose to be connected. We do not give the computations in full detail, but only give intermediate results, as the calculations are fairly routine. They use the technical apparatus developed in [1] and also used in the paper [2]. In particular, when summing over an orthonormal basis of $T_{(x,u)}(T_1M)$, we always take a basis of the form $\{e_1^t, \dots, e_{n-1}^t, e_1^h, \dots, e_n^h\}$ where $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_xM with $e_n = u$.

From the general expression (see [12], e.g.)

$$(\mathrm{Hess}\,E)_\xi(T,T) = \int_{T_1M} (|\bar{\nabla} T|^2 - |T|^2 |\bar{\nabla}\xi|^2)\, \mu_{T_1M}$$

for any vector field T on T_1M such that $g_S(T,\xi) = 0$, we see that we have to determine the quantities $|\bar{\nabla} X^t|$, $|\bar{\nabla}\bar{X}^h|$, $|X^t|$, $|\bar{X}^h|$ and $|\bar{\nabla}\xi|$ first.

The components of $\bar{\nabla}\xi$ have been found already in [2, Section 4]. We get at the point (x,u)

$$|\bar{\nabla}\xi|^2 = (n-1) - \rho(u,u) + \frac{1}{2}\,|R_u|^2$$

where ρ denotes the Ricci tensor of (M,g) and R_u the Jacobi operator associated to u. As (M,g) is locally isometric to a two-point homogeneous space, it is both Einstein, i.e., $\rho = \frac{\tau}{n}\,g$, τ being the scalar curvature, and two-stein, i.e., $\rho = \frac{\tau}{n}\,g$ and $\sum_{i,j=1}^n R_{uiuj}^2 = \alpha g(u,u)^2$. In particular, $\alpha = |R_u|^2 = \frac{1}{n(n+2)}\big(|\rho|^2 + \frac{3}{2}\,|R|^2\big)$. So we find

$$|\bar{\nabla}\xi|^2 = (n-1) - \frac{\tau}{n} + \frac{\tau^2}{2n^2(n+2)} + \frac{3|R|^2}{4n(n+2)}\,, \tag{2}$$

which is a constant function on T_1M, i.e., independent of both the point $x \in M$ and the unit vector $u \in T_xM$.

Next, we easily find

$$|X^t|^2 = |\bar{X}^h|^2 = |X|^2 - g(X,u)^2. \tag{3}$$

For computing $|\bar{\nabla} X^t|$, we need the formulas (1) for the Levi Civita connection $\bar{\nabla}$ on (T_1M, g_S) to obtain

$$(4)\quad |\bar{\nabla} X^t|^2 = (n-1)\, g(X,u)^2 + |\nabla X|^2 - \sum_{i=1}^{n} g(u, \nabla_{e_i} X)^2 + \frac{1}{4}\sum_{i=1}^{n} |R(u,X)e_i|^2$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for the tangent space at x. The calculation for $|\bar{\nabla}\bar{X}^h|$ is somewhat more involved. First, we derive the components of $\bar{\nabla}\bar{X}^h$ using once more (1):

$$\begin{aligned}
\bar{\nabla}_{Y^t}\bar{X}^h =\ & \frac{1}{2}\big(R(u,Y)X + g(X,u)R_uY\big)^h \\
& - \big(g(X,Y) - 2g(X,u)g(Y,u)\big)\, u^h - g(X,u)\, Y^h, \\
\bar{\nabla}_{Y^h}\bar{X}^h =\ & (\nabla_Y X)^h - g(\nabla_Y X, u)\, u^h + \frac{1}{2}\big(R(X,Y)u + g(X,u)R_uY\big)^t.
\end{aligned}$$

Then a tedious but straightforward computation yields

$$(5)\quad \begin{aligned}
|\bar{\nabla}\bar{X}^h|^2 =\ & |X|^2 + (n-2)g(X,u)^2 + |\nabla X|^2 - \sum_{i=1}^{n} g(u, \nabla_{e_i}X)^2 \\
& - g(R_uX, X) + \frac{1}{2} g(X,u)^2 |R_u|^2 \\
& + \frac{1}{2}\sum_{i=1}^{n} |R(u,e_i)X|^2 + g(X,u)\sum_{i=1}^{n} g(R(u,e_i)X, R_ue_i).
\end{aligned}$$

The next step is to integrate these expressions over T_1M. We do this in two steps: first we integrate over the fiber $S^{n-1}(1)$ above a point $x \in M$, using the formulas given in [5, Lemma 6.3]. We obtain

$$\begin{aligned}
\int_{u\in S^{n-1}} |X^t|^2(x,u)\, \mu_{S^{n-1}} =\ & \int_{u\in S^{n-1}} |\bar{X}^h|^2(x,u)\, \mu_{S^{n-1}} = c_{n-1}\,\frac{n-1}{n}|X|^2(x), \\
\int_{u\in S^{n-1}} |\bar{\nabla} X^t|^2(x,u)\, \mu_{S^{n-1}} =\ & \frac{c_{n-1}}{n}\Big\{(n-1)|\nabla X|^2 + \Big(\frac{|R|^2}{4n} + n - 1\Big)|X|^2\Big\}(x), \\
\int_{u\in S^{n-1}} |\bar{\nabla}\bar{X}^h|^2(x,u)\, \mu_{S^{n-1}} =\ & \frac{c_{n-1}}{n}\Big\{(n-1)|\nabla X|^2 \\
& + \Big(\frac{(2n+1)|R|^2}{4n(n+2)} - \frac{\tau^2}{2n^2(n+2)} \\
& - \frac{\tau}{n} + 2(n-1)\Big)|X|^2\Big\}(x)
\end{aligned}$$

where c_{n-1} is the volume of $S^{n-1}(1)$. Note that the coefficients of $|X|^2$ and $|\nabla X|^2$ in these expressions do not depend on the point $x \in M$ because we deal with two-point homogeneous spaces.

Combining the above, we obtain

Lemma 1. *Let X be a vector field on a compact orientable quotient (M^n, g) of a two-point homogeneous space. Then we have*

$$(\text{Hess}\, E)_\xi(X^t, X^t) = \frac{(n-1)c_{n-1}}{n}\,(\|\nabla X\|^2 + A_t\|X\|^2), \tag{6}$$

$$(\text{Hess}\, E)_\xi(\bar{X}^h, \bar{X}^h) = \frac{(n-1)c_{n-1}}{n}\,(\|\nabla X\|^2 + A_h\|X\|^2) \tag{7}$$

where $\|X\|^2 = \int_M |X|^2(x)\mu_M$, $\|\nabla X\|^2 = \int_M |\nabla X|^2(x)\mu_M$ and the numbers A_t and A_h are given explicitly as

$$A_t = \frac{(5-2n)|R|^2}{4n(n-1)(n+2)} - \frac{\tau^2}{2n^2(n+2)} + \frac{\tau}{n} - (n-2), \tag{8}$$

$$A_h = \frac{(4-n)|R|^2}{4n(n-1)(n+2)} - \frac{\tau^2}{2n(n-1)(n+2)} + \frac{(n-2)\tau}{n(n-1)} - (n-3). \tag{9}$$

3. The case of non-positive curvature

In order to derive explicit results about the instability of the geodesic flow ξ from the formulas (6) and (7), we must estimate the factor $\|\nabla X\|^2$. A first useful estimate is given in [11].

Proposition 2 ([11])**.** *Let (M^n, g), $n \geq 2$, be a compact and orientable manifold and X any vector field on M. Then we have*

$$\int_M |\nabla X|^2 \mu_M \geq -\int_M \rho(X, X)\mu_M \tag{10}$$

and equality holds if and only if $X^\flat$, the dual one-form of X, is a harmonic one-form.

Of course, for spaces of positive Ricci curvature, the right-hand side of the inequality is negative, and the inequality is trivial. But in that case, because of the Weitzenböck formula for one-forms, harmonic one-forms do not exist. Therefore, we consider the case of non-positive curvature and suppose $b_1(M) \neq 0$.

We look first at a compact and orientable space (M^n, g) of constant curvature $\lambda \leq 0$, and we take X to be a non-zero vector field on M such that $X^\flat$ is harmonic. Then (see, e.g., [7, Table II]):

$$\tau = n(n-1)\lambda, \quad |R|^2 = 2n(n-1)\lambda^2, \quad \|\nabla X\|^2 = -(n-1)\lambda\|X\|^2,$$

and the formulas (6), (7) reduce to

$$(\text{Hess}\, E)_\xi(X^t, X^t) = -\frac{(n-1)(n-2)}{n}\, c_{n-1}\, \frac{\lambda^2+2}{2}\, \|X\|^2,$$

$$(\text{Hess}\, E)_\xi(\bar{X}^h, \bar{X}^h) = \frac{n-1}{n}\, c_{n-1}\, \Big(\frac{2-n}{2}\,\lambda^2 - \lambda - (n-3)\Big)\|X\|^2.$$

When $n \geq 4$, both expressions are strictly negative. For $n = 3$, the first is still negative, but the second is negative only when $\lambda < -2$.

Next, consider a compact and orientable Kähler space $(M^{n=2m}, g)$ of constant negative holomorphic sectional curvature μ and let X be again a non-zero vector field on M with harmonic dual one-form $X^\flat$. Then

$$\tau = m(m+1)\mu, \quad |R|^2 = 2m(m+1)\mu^2, \quad \|\nabla X\|^2 = -\frac{m+1}{2}\,\mu\|X\|^2. \tag{11}$$

The formulas (6) and (7) now simplify to

$$\begin{aligned}
(\operatorname{Hess} E)_\xi(X^t, X^t) &= -\frac{m-1}{2m}\, c_{2m-1}\big(\frac{2m+11}{16}\,\mu^2 + 2(2m-1)\big)\,\|X\|^2,\\
(\operatorname{Hess} E)_\xi(\bar{X}^h, \bar{X}^h) &= \frac{1}{2m}\, c_{2m-1}\,\big(\frac{(1-m)(4+m)}{8}\,\mu^2 - \frac{m+1}{2}\,\mu\\
&\qquad - (2m-3)(2m-1)\big)\,\|X\|^2.
\end{aligned}$$

One easily checks that, when $m > 1$, both expressions are strictly negative.

Now take a compact orientable quaternionic Kähler space $(M^{n=4m}, g)$ of constant negative Q-sectional curvature ν and X as before. Then

$$\tau = 4m(m+2)\nu, \quad |R|^2 = 4m(5m+1)\nu^2, \quad \|\nabla X\|^2 = -(m+2)\nu\|X\|^2.$$

The formulas (6), (7) then read

$$\begin{aligned}
(\operatorname{Hess} E)_\xi(X^t, X^t) &= -\frac{1}{4m}\, c_{4m-1}\big(\frac{4m^2+33m-13}{8}\,\nu^2\\
&\qquad + 2(2m-1)(4m-1)\big)\,\|X\|^2,\\
(\operatorname{Hess} E)_\xi(\bar{X}^h, \bar{X}^h) &= \frac{1}{4m}\, c_{4m-1}\big(\frac{-m^2-6m+1}{2}\,\nu^2 - (m+2)\nu\\
&\qquad - (4m-1)(4m-3)\big)\,\|X\|^2.
\end{aligned}$$

Again, both expressions are strictly negative.

Finally, take a compact and orientable quotient of the Cayley plane $\mathrm{Cay}\,H^2(\zeta)$ with minimal sectional curvature $\zeta < 0$ and take X as before. Then

$$\tau = 144\zeta, \quad |R|^2 = 576\zeta^2, \quad \|\nabla X\|^2 = -9\zeta\|X\|^2$$

and we have

$$\begin{aligned}
(\operatorname{Hess} E)_\xi(X^t, X^t) &= -\frac{21}{64}\, c_{15}\,(9\zeta^2 + 40)\|X\|^2,\\
(\operatorname{Hess} E)_\xi(\bar{X}^h, \bar{X}^h) &= -\frac{3}{16}\, c_{15}\,(14\zeta^2 + 3\zeta + 65)\|X\|^2.
\end{aligned}$$

Again, these expressions are both negative.

Summarizing the above, we have

Theorem 3. *Let (M^n, g), $n \geq 3$, be a compact and orientable quotient of a two-point homogeneous space of non-positive curvature with $b_1(M) \neq 0$. Then the geodesic flow ξ on T_1M is an unstable critical point for the energy functional. The index is at least $2b_1(M)$, except possibly in the case when (M^n, g) is a three-dimensional space of constant curvature λ, $-2 \leq \lambda \leq 0$. Then the index is at least $b_1(M)$.*

Note: For a two-dimensional surface of constant curvature λ, the formulas (6) and (7) are given by the simple expressions

$$(\operatorname{Hess} E)_\xi(X^t, X^t) = \pi(\|\nabla X\|^2 + \lambda\|X\|^2),$$
$$(\operatorname{Hess} E)_\xi(\bar{X}^h, \bar{X}^h) = \pi(\|\nabla X\|^2 + \|X\|^2).$$

From Proposition 2 we see that both are non-negative. Hence, we obtain no information about the stability of the geodesic flow for two-dimensional two-point homogeneous spaces starting from Lemma 1. In Sections 5 and 6, we will find some answers about the instability of ξ for these spaces using vector fields on T_1M different from the tangential and modified horizontal lifts.

4. The case of positive curvature

In [11], also a second estimate for $\|\nabla X\|^2$ in terms of the curvature is given.

Proposition 4 ([11])**.** *Let (M^n, g), $n \geq 2$, be a compact and orientable manifold and X any vector field on M. Then we have*

$$\int_M (|\nabla X|^2 + (\operatorname{div} X)^2)\mu_M \geq \int_M \rho(X, X)\mu_M$$

and equality holds if and only if X is a Killing vector field (so that, in particular, $\operatorname{div} X = 0$).

The above inequality is trivial for spaces of negative Ricci curvature, for then the right-hand side is negative. However, by a theorem of Bochner, Killing vector fields are non-existent on such spaces. On the other hand, there is a multitude of Killing vector fields on the spheres, the complex and the quaternionic projective spaces and on the positively curved Cayley plane.

We consider first a compact and orientable manifold (M^n, g) of constant positive curvature λ, and we take a non-zero Killing vector field X on M (supposing it exists). Then we have

$$\tau = n(n-1)\lambda, \quad |R|^2 = 2n(n-1)\lambda^2, \quad \|\nabla X\|^2 = (n-1)\lambda\|X\|^2.$$

The formulas (6), (7) reduce to

$$(\operatorname{Hess} E)_\xi(X^t, X^t) = \frac{n-1}{n} c_{n-1}\Big(\frac{2-n}{2}\lambda^2 + 2(n-1)\lambda - (n-2)\Big)\|X\|^2,$$
$$(\operatorname{Hess} E)_\xi(\bar{X}^h, \bar{X}^h) = \frac{n-1}{n} c_{n-1}\Big(\frac{2-n}{2}\lambda^2 + (2n-3)\lambda - (n-3)\Big)\|X\|^2.$$

From these it follows, for $n \geq 3$,

$$
\begin{aligned}
&(\operatorname{Hess} E)_\xi(X^t, X^t) \geq 0 \\
&\qquad \Leftrightarrow \frac{2(n-1)-\sqrt{2n^2-4}}{n-2} \leq \lambda \leq \frac{2(n-1)+\sqrt{2n^2-4}}{n-2}, \\
&(\operatorname{Hess} E)_\xi(\bar{X}^h, \bar{X}^h) \geq 0 \\
&\qquad \Leftrightarrow \frac{2n-3-\sqrt{2n^2-2n-3}}{n-2} \leq \lambda \leq \frac{2n-3+\sqrt{2n^2-2n-3}}{n-2}.
\end{aligned}
$$

If we put

$$
\begin{aligned}
(n-2)\alpha_n &= 2(n-1)-\sqrt{2n^2-4}, \\
(n-2)\beta_n &= 2(n-1)+\sqrt{2n^2-4}, \\
(n-2)\gamma_n &= 2n-3-\sqrt{2n^2-2n-3}, \\
(n-2)\delta_n &= 2n-3+\sqrt{2n^2-2n-3}, \\
N &= \text{the number of linearly independent Killing vector fields on } M,
\end{aligned}
$$

then we have

Theorem 5. *Let (M^n, g), $n \geq 3$, be a compact and orientable space of constant curvature $\lambda > 0$. Then*

(1) *the index of the geodesic flow ξ for the energy functional is at least N for $\lambda \in [\gamma_n, \alpha_n) \cup (\delta_n, \beta_n]$;*
(2) *the index is at least $2N$ for $\lambda \in (0, \gamma_n) \cup (\beta_n, +\infty)$.*

The phenomenon we observe here also holds for the other positively curved two-point homogeneous spaces, though with different (and more complicated) expressions for α_n, β_n, γ_n and δ_n and with λ replaced by either the constant holomorphic sectional curvature μ of a Kähler space, or by the constant Q-sectional curvature ν of a quaternion Kähler space, or by the maximum sectional curvature ζ of the Cayley plane $\operatorname{Cay} P^2(\zeta)$. In particular, there are combinations of (λ, n), (μ, n), (ν, n) and $(\zeta, 16)$ for which we can derive no information about the stability or the instability of ξ from Lemma 1. These include the unit spheres $S^n(1)$, $n \geq 3$.

5. Complex space forms

The tangential lift and the modified horizontal lift of a vector field X on (M, g) to the unit tangent sphere bundle are not the only possible choices for vector fields on T_1M orthogonal to the geodesic flow vector field ξ. In this section, we consider a very natural unit vector field on T_1M of a different type when the base manifold (M, g, J) is a complex space form. In contrast to the previous results, this will allow us to comment on the instability of ξ also in the case when the base manifold is two-dimensional.

Suppose (M, g, J) is a compact Kähler space of dimension $n = 2m$ with constant holomorphic sectional curvature μ. Consider the unit vector field $\xi_1 = (Ju)^t$ on (T_1M, g_S). Clearly, this is everywhere orthogonal to ξ. We compute

$$(\mathrm{Hess}\, E)_\xi(\xi_1, \xi_1) = \int_{T_1M} (|\bar{\nabla}\xi_1|^2 - |\xi_1|^2 |\bar{\nabla}\xi|^2) \mu_{T_1M}. \tag{12}$$

For $|\bar{\nabla}\xi|^2$, we combine (2) and (11) to obtain

$$|\bar{\nabla}\xi|^2 = (2m-1) - \frac{m+1}{2}\,\mu + \frac{m+7}{16}\,\mu^2.$$

The components of $\bar{\nabla}\xi_1$ have been calculated in [2, Section 5]. We found there

$$\bar{\nabla}_{X^t}\xi_1 = (JX)^t - g(X,u)\xi_1, \quad \bar{\nabla}_{X^h}\xi_1 = \frac{1}{2}(R(u,Ju)X)^h.$$

So, using also the explicit form for the curvature tensor of a complex space form

$$\begin{aligned} R(X,Y)Z = \frac{\mu}{4}\big(&g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - 2g(JX,Y)JZ - g(JX,Z)JY\big), \end{aligned}$$

we get

$$|\bar{\nabla}\xi_1|^2 = 2(m-1) + \frac{m+3}{8}\,\mu^2.$$

Consequently, we obtain

$$(\mathrm{Hess}\, E)_\xi(\xi_1, \xi_1) = \big(\frac{m-1}{16}\,\mu^2 + \frac{m+1}{2}\,\mu - 1\big)\,\mathrm{vol}(T_1M). \tag{13}$$

Theorem 6. *Let (M^{2m}, g, J), $m > 1$, be a compact Kähler space of constant holomorphic sectional curvature μ. If*

$$\mu \in \Big(\frac{-4(m+1) - 4\sqrt{m^2+3m}}{m-1}, \frac{-4(m+1) + 4\sqrt{m^2+3m}}{m-1}\Big),$$

then the geodesic flow vector field ξ on T_1M is an unstable critical point for the energy functional.

Next, we return to a two-dimensional Riemannian manifold (M^2, g). If it is orientable, one can equip it with a parallel complex structure J, which takes a non-zero tangent vector u at x to the unique vector v at x, orthogonal to u, of the same length and such that $\{u, v\}$ is positively oriented. The Kähler space (M^2, g, J) has constant holomorphic sectional curvature μ if and only if (M^2, g) has constant sectional curvature μ. From (13) with $m = 1$, we then obtain

Theorem 7. *Let (M^2, g) be a compact and orientable surface of constant curvature $\mu < 1$. Then the geodesic flow vector field ξ on T_1M is an unstable critical point for the energy functional.*

Remark 1. The first and third author proved in [2] that $\xi_1 = (Ju)^t$ is itself harmonic. From (12), we see that

$$(\operatorname{Hess} E)_{\xi_1}(\xi, \xi) = -(\operatorname{Hess} E)_{\xi}(\xi_1, \xi_1).$$

Hence, we also get

Proposition 8. *Let (M^{2m}, g, J), $m > 1$, be a compact Kähler space of constant holomorphic sectional curvature μ. If*

$$\mu \in (-\infty, \frac{-4(m+1) - 4\sqrt{m^2+3m}}{m-1}) \cup (\frac{-4(m+1) + 4\sqrt{m^2+3m}}{m-1}, +\infty),$$

then the unit vector field ξ_1 on T_1M is an unstable critical point for the energy functional.

Proposition 9. *Let (M^2, g) be a compact and orientable surface of constant curvature $\mu > 1$. Then the unit vector field ξ_1 on T_1M is an unstable critical point for the energy functional.*

Remark 2. If (M, g, J) is a complex space form, then there is a third distinguished harmonic unit vector field ξ_2 on (T_1M, g_S), namely $\xi_2 = (Ju)^h$. For this vector field, using again the formulas in [2, Section 5], one easily finds that $|\bar{\nabla}\xi_2| = |\bar{\nabla}\xi|$, and hence

$$\begin{aligned} (\operatorname{Hess} E)_{\xi}(\xi_2, \xi_2) &= (\operatorname{Hess} E)_{\xi_2}(\xi, \xi) = 0, \\ -(\operatorname{Hess} E)_{\xi_1}(\xi_2, \xi_2) &= (\operatorname{Hess} E)_{\xi_2}(\xi_1, \xi_1) = (\operatorname{Hess} E)_{\xi}(\xi_1, \xi_1). \end{aligned}$$

So, the Theorems 6 and 7 also hold with ξ replaced by ξ_2.

6. Two-dimensional spaces of constant curvature

Let (M^2, g) be a compact and orientable surface of constant curvature λ. Let J be the complex structure associated to a choice of orientation on M as in the previous section. Then $\{\xi_1 = (Ju)^t, \xi_2 = (Ju)^h, \xi = u^h\}$ is a global orthonormal frame field on (T_1M^2, g_S). Moreover, from $[T, S] = \bar{\nabla}_T S - \bar{\nabla}_S T$ and the formulas in [2], we find at once

$$[\xi_1, \xi_2] = -\xi, \quad [\xi_2, \xi] = -\lambda\xi_1, \quad [\xi, \xi_1] = -\xi_2. \tag{14}$$

As λ is constant, Proposition 1.9 of [10] says that the universal covering $\widetilde{T_1M}$ of T_1M can be equipped with a Lie group structure for which the lifts of ξ_1, ξ_2

and ξ are left-invariant. So, we can consider (T_1M, g_S) as the compact quotient $\Gamma\backslash G$ of a three-dimensional Lie group G with a left-invariant metric. Because of (14), G is necessarily unimodular. From Milnor's classification of three-dimensional metric Lie groups in [9] (see also [6]), it follows that $G = SU(2)$ for $\lambda > 0$, $G = E(2)$ for $\lambda = 0$ and $G = \widetilde{SL}(2, I\!R)$ for $\lambda < 0$.

The second and the third author have studied the stability of left-invariant harmonic unit vector fields on compact quotients of three-dimensional Lie groups in [6]. We rephrase the relevant results for the present case of the unit tangent sphere bundle of a two-dimensional space form. We note that in [6] the energy functional is the *restricted one,* i.e., the energy restricted to maps arising from unit vector fields. In particular, stability of a unit vector field for this functional does not necessarily imply stability for the energy functional in the larger sense, i.e., for the energy on all maps. However, instability for the restricted energy clearly implies instability in the larger sense with index at least as big.

Theorem 10. *Let (M^2, g) be a compact and orientable surface of constant curvature λ.*

(1) *If $\lambda > 1$, then ξ_1 is an unstable critical point for the energy with index at least 2.*
(2) *If $\lambda = 1$, then any vector field $V = a\xi_1 + b\xi_2 + c\xi$ $(a, b, c \in I\!R,\ a^2 + b^2 + c^2 = 1)$ minimizes the energy and $E(V) = \frac{7}{4}\ \mathrm{vol}(T_1M)$.*
(3) *If $0 \le \lambda < 1$, then ξ_1 is an absolute minimizer of the energy with $E(\xi_1) = \frac{\lambda^2+6}{4}\ \mathrm{vol}(T_1M)$. Any vector field $V = a\xi_2 + b\xi$ $(a, b \in I\!R,\ a^2 + b^2 = 1)$ is an unstable critical point with index at least 2.*
(4) *If $\lambda < 0$, then any vector field $V = a\xi_2 + b\xi$ $(a, b \in I\!R,\ a^2 + b^2 = 1)$ is an unstable critical point for the energy with index at least 2.*

Remark. We see that the only case where we actually find that the geodesic flow vector field ξ is stable (for the restricted energy), is when (M^2, g) has constant curvature 1. In that case, (T_1M^2, g_S) is locally isometric to the three-sphere S^3 of radius 2 and ξ corresponds to a Hopf vector field on it. (We note that critical point considerations for the energy are invariant under homothetic changes of the metric.) It was proved already by G. Wiegmink in [13] that Hopf vector fields on S^3 are stable critical points for the restricted energy. Recently, it has been shown in [4] and [8] that Hopf vector fields are the unique minimizers for the energy functional on unit vector fields on S^3. Note that this is no longer true for Hopf vector fields on higher-dimensional spheres S^{2m+1}, $m > 1$ ([14]).

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