The Jordan normal form of higher order Osserman algebraic curvature tensors

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Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. We construct new examples of algebraic curvature tensors so that the Jordan normal form of the higher order Jacobi operator is constant on the Grassmannian of subspaces of type (r, s) in a vector space of signature (p, q). We then use these examples to establish some results concerning higher order Osserman and higher order Jordan Osserman algebraic curvature tensors.

Keywords: higher order Jacobi operator, Osserman algebraic curvature tensors, Jordan Osserman algebraic curvature tensors

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§1 Introduction

A 4 tensor R is said to be an algebraic curvature tensor if it satisfies the well known symmetries of the Riemannian curvature tensor, i.e.

$$\begin{aligned}&R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y), \qquad \text{and}\\ &R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.\end{aligned}$$

It is clear that the Riemann curvature tensor defines an algebraic curvature tensor at each point of the manifold. Conversely, every algebraic curvature tensor is geometrically realizable ([10]). We remark that it is often convenient to study certain geometric problems in a purely algebraic setting.

Let R be an algebraic curvature tensor on a vector space V of signature (p, q). The *Jacobi operator* $\mathcal{J}_R$ is the self-adjoint linear map defined by:

$$\mathcal{J}_R(v)y := R(y, v)v.$$

Here the natural domains of definition are the *pseudo-spheres* of unit timelike $(-)$ and spacelike $(+)$ vectors in V:

$$S^{\pm}(v) := \{v \in V : (v, v) = \pm 1\}.$$

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Let (M,g) be a pseudo-Riemannian manifold of signature (p,q) and let ${}^gR$ be the curvature tensor of the Levi-Civita connection. If (M,g) is Riemannian (i.e.$p=0$), and if it is flat or it is a local rank 1 symmetric space, then the set of local isometries acts transitively on the unit sphere bundle $S(M,g)$. Consequently, the eigenvalues of $\mathcal{J}_{{}^gR}$ are constant on $S(M,g)$. Osserman [15] wondered if the converse holds; later authors called this problem the *Osserman conjecture*. The conjecture has been established by Chi [4] for Riemannian manifolds of dimension m, where $m=4$, where m is odd, or where $m\equiv 2 \bmod 4$. However, it is known ([8]) that there exist Riemannian Osserman algebraic curvature tensors which are not flat and which are not the curvature tensors of rank 1 symmetric spaces. We also refer to [5], [14] for related results.

In any signature, we say that an algebraic curvature tensor is *Osserman* if the eigenvalues of $\mathcal{J}_R$ are constant on $S^{\pm}(V)$. Similarly, we say that a pseudo-Riemannian manifold (M,g) is *Osserman* if the eigenvalues of $\mathcal{J}_{{}^gR}$ are constant on the pseudo-sphere bundles $S^{\pm}(M,g)$.

In the Lorentzian setting ($p=1$), it is known ([1], [6]) that an Osserman algebraic curvature tensor has constant sectional curvature. Thus we may draw the geometric consequence that a Lorentzian Osserman manifold has constant sectional curvature; the geometry of such manifolds is very special.

In higher signatures, although there are some partial results known, the classification is far from complete. In particular, it is known that there exist pseudo-Riemannian Osserman manifolds which are neither flat nor local rank 1 symmetric spaces ([2], [3], [7]).

In the Riemannian setting, any self-adjoint linear map is diagonalizable; thus the eigenvalues determine the Jordan normal form (i.e.the conjugacy class). However, this is not true in higher signature so we have to differentiate between the eigenvalue structure and the Jordan normal form. We say that an algebraic curvature tensor is *Jordan Osserman* if the Jordan normal form of $\mathcal{J}_R$ is constant on $S^{\pm}(V)$. There exist algebraic curvature tensors which are Osserman but which are not Jordan Osserman. Furthermore, there exist Jordan Osserman algebraic curvature tensors whose Jordan normal form is arbitrarily complicated ([11]).

The Jacobi operator was originally defined on $S^{\pm}(V)$. However, since we have $\mathcal{J}_R(tv)=t^2\mathcal{J}_R(v)$, we can also regard $\mathcal{J}_R$ as being defined on the projective spaces of non-degenerate lines in V by setting

$$\mathcal{J}_R(\mathrm{span}\{v\}):=(v,v)^{-1}\mathcal{J}_R(v)\quad\text{if}\quad(v,v)\neq 0. \tag{1.1.a}$$

Stanilov has extended $\mathcal{J}_R$ to non-degenerate subspaces of arbitrary dimension. Let σ be a non-degenerate subspace of V. If $\{v_i\}$ is a basis for σ, then let $h_{ij}:=(v_i,v_j)$ describe the restriction of the metric on V to the subspace σ. Since σ is non-degenerate, the matrix (h_{ij}) is invertible and we let (h^{ij}) be the inverse. The *higher order Jacobi operator* is defined ([16]) by generalizing equation (1.1.a):

$$\mathcal{J}_R(\sigma)y:=\textstyle\sum_{ij}h^{ij}R(y,v_i)v_j;$$

it is independent of the basis chosen. This extends the natural domains of $\mathcal{J}_R$ to the *Grassmannians* $\mathrm{Gr}_{r,s}(V)$ of subspaces of V which have signature (r, s). Let $\{e_1, \ldots, e_{r+s}\}$ be an orthonormal basis for $\sigma \in \mathrm{Gr}_{r,s}(V)$. Let $\varepsilon_i := (v_i, v_i)$. Then we can express $\mathcal{J}_R$ more simply as:

$$\mathcal{J}_R(\sigma) = \varepsilon_1 \mathcal{J}_R(e_1) + \cdots + \varepsilon_k \mathcal{J}_R(e_k). \tag{1.1.b}$$

Now we extend the notions 'Osserman' and 'Jordan Osserman' to the higher order context. We say that R is *Osserman of type* (r, s) if the eigenvalues of $\mathcal{J}_R(\cdot)$ are constant on $\mathrm{Gr}_{r,s}(V)$. Furthermore, R is said to be *Jordan Osserman of type* (r, s) if the Jordan normal form of $\mathcal{J}_R(\cdot)$ is constant on $\mathrm{Gr}_{r,s}(V)$. Since the Jordan normal form determines the eigenvalues, it is immediate that if R is Jordan Osserman of type (r, s), then R is Osserman of type (r, s); the reverse implication can fail — see Remark 2.7.

We say that a pair (r, s) is *admissible* if $\mathrm{Gr}_{r,s}(V)$ is non-empty and does not consist of a single point. Equivalently, this means that:

$$0 \le r \le p, \quad 0 \le s \le q, \quad \text{and} \quad 1 \le r + s \le \dim V - 1.$$

In Section 2, we state the main theorems of this paper concerning the higher order Jacobi operator, Osserman algebraic curvature tensors, and Jordan Osserman algebraic curvature tensors. In Theorem 2.1, we summarize previously known results for Osserman algebraic curvature tensors. Theorem 2.2 deals with Jordan Osserman duality. In Theorem 2.3, we present examples due to [13] and note that previously known results for these examples can be extended from the Osserman to the Jordan Osserman setting. In Theorem 2.4, we construct new examples of algebraic curvature tensors which are Jordan Osserman for certain but not all values of (r, s). We use these examples to draw certain conclusions about the relationship between the various concepts which we have introduced. In Section 3, we prove Theorem 2.1. In Section 4, we prove Theorem 2.4.

§2 Statement of results

In the following theorem, we summarize previously known facts concerning Osserman algebraic curvature tensors (see [13] for details). Assertion (2) is a duality result. Moreover, assertion (3) shows that when we consider the eigenvalue structure, only the value of $r + s$ is relevant. Consequently, we shall say that R is *k-Osserman* if R is Osserman of type (r, s) for any, and hence for all, admissible pairs (r, s) for which $k = r + s$. In particular, if $p > 0$ and if $q > 0$, then the eigenvalues of $\mathcal{J}_R$ are constant on $S^+(V)$ if and only if they are constant on $S^-(V)$.

2.1 Theorem. *Let R be an algebraic curvature tensor on a vector space V of signature (p, q). Let R be Osserman of type (r, s), where (r, s) is an admissible pair. Then we have:*

(1) *R is Einstein;*

(2) *R is Osserman of type $(p-r, q-s)$;*
(3) *if $(\tilde{r}, \tilde{s})$ is an admissible pair with $r+s=\tilde{r}+\tilde{s}$, then R is Osserman of type $(\tilde{r}, \tilde{s})$.*

Instead of the eigenvalue structure, we can consider the Jordan normal form and establish a similar duality result.

2.2 Theorem. *Let R be an algebraic curvature tensor on a vector space V of signature (p, q). Let (r, s) be an admissible pair. If R is Jordan Osserman of type (r, s), then R is Jordan Osserman of type $(p-r, q-s)$.*

The only examples given in the literature [13] may be described as follows. Let

$$R_{\mathrm{Id}}(x, y)z := (y, z)x - (x, z)y$$

denote the algebraic curvature tensor of constant sectional curvature. If ϕ is a skew-adjoint map of V, then we may define:

$$R_\phi(x, y)z := (\phi y, z)\phi x - (\phi x, z)\phi y - 2(\phi x, y)\phi z.$$

We showed [11] that R_ϕ is an algebraic curvature tensor. We then have

$$\begin{aligned} &\mathcal{J}_{R_{\mathrm{Id}}}(x)y = (x, x)y - (x, y)x \quad \text{and} \\ &\mathcal{J}_{R_\phi}(x)y = 3(\phi x, y)\phi x. \end{aligned} \tag{2.2.a}$$

Let ϕ be a skew-adjoint map with $\phi^2 = \pm\,\mathrm{Id}$. Let c_0 and c_1 be real constants. We set $R := c_0 R_{\mathrm{Id}} + c_1 R_\phi$. If σ is a non-degenerate subspace, then $\mathcal{J}_R(\sigma)$ is diagonalizable; thus the eigenvalue structure determines the Jordan normal form. The following result in the Jordan Osserman context then follows from the corresponding result in the Osserman context ([13]).

2.3 Theorem. *Let V be a vector space of signature (p, q). Let ϕ be a skew-adjoint map of V with $\phi^2 = \pm\,\mathrm{Id}$.*

(1) *The algebraic curvature tensors R_{Id} and R_ϕ are Jordan Osserman of type (r, s) for every admissible pair (r, s).*
(2) *Let c_0 and c_1 be non-zero constants. Let $R = c_0 R_{\mathrm{Id}} + c_1 R_\phi$. Then R is Jordan Osserman of types $(1, 0)$, $(0, 1)$, $(p-1, q)$, and $(p, q-1)$. Furthermore, R is not Osserman of type (r, s) for other values of (r, s).*

As noted above, the operators $\mathcal{J}_R(\sigma)$ associated to the algebraic curvature tensors discussed in Theorem 2.3 are all diagonalizable. We now construct algebraic curvature tensors so $\mathcal{J}_R(\sigma)$ has non-trivial Jordan normal form. Let $p \geq 2$ and let $q \geq 2$. Let $\{e_1^-, \dots, e_p^-, e_1^+, \dots, e_q^+\}$ be an orthonormal basis for V, where the vectors $\{e_1^-, \ldots, e_p^-\}$ are timelike and the vectors $\{e_1^+, \ldots, e_q^+\}$ are spacelike. Let a be a positive integer with $2a \le \min(p, q)$. We define a skew-adjoint linear map Φ_a of V by setting:

$$\phi_a e_k^{\pm} = \begin{cases} \pm(e_{2i}^- + e_{2i}^+) & \text{if } k = 2i-1 \le 2a, \\ \mp(e_{2i-1}^- + e_{2i-1}^+) & \text{if } k = 2i \le 2a, \\ 0 & \text{if } k > 2a. \end{cases} \tag{2.3.a}$$

The map Φ_a is the direct sum of a different 4×4 'blocks'; the subspaces spanned by $\{e_{2i-1}^-, e_{2i}^-, e_{2i-1}^+, e_{2i}^+\}$ are invariant under the action of Φ_a for $1 \le i \le a$.

We can interchange the roles of spacelike and timelike vectors by changing the sign of the inner product. Thus we may always assume that $p \le q$. The following result giving new families of examples is in many ways the main result of this paper.

2.4 Theorem. *Let Φ_a be the skew-adjoint linear map on the vector space V of signature (p, q) which is defined in equation (2.3.a). Let R_a be the associated curvature tensor. Assume $p \le q$. We have:*

(1) *R_a is k Osserman for $1 \le k \le \dim V - 1$.*
(2) *Suppose that $2a < p$. Then R_a is Jordan Osserman of type $(p, 0)$ and $(0, q)$; R_a is not Jordan Osserman of type (r, s) otherwise.*
(3) *Suppose that $2a = p < q$. Then R_a is Jordan Osserman of type $(r, 0)$ and of type (r, q) for any $1 \le r \le p-1$; R_a is not Jordan Osserman otherwise.*
(4) *Suppose that $2a = p = q$. Then R_a is Jordan Osserman of type $(r, 0)$, of type (r, q), of type $(0, s)$, and of type (p, s) for $1 \le r \le p-1$ and $1 \le s \le q-1$; R_a is not Jordan Osserman otherwise.*

2.5 Remark. We may use assertion (1) of Theorem 2.4 to see that assertion (2) of Theorem 2.1 does not generalize to the Jordan Osserman context; there exist algebraic curvature tensors which are k Osserman for all k, which are Jordan Osserman of type $(p, 0)$ and $(0, q)$, and which are not Jordan Osserman of type (r, s) for other values of (r, s). Thus we cannot determine whether or not R is Jordan Osserman of type (r, s) only from $k = r + s$.

2.6 Remark. We suppose $2 \le k \le \dim V - 2$ to ensure that we are truly in the higher order setting. The k Osserman algebraic curvature tensors have been classified in the Riemannian and in the Lorentzian settings ([9], [12]); all these curvature tensors have constant sectional curvature and hence are 1 Osserman. Again, we may use assertion (1) of Theorem 2.4 to see that a similar assertion fails for a Jordan Osserman algebraic curvature tensor in the higher signature setting.

2.7 Remark. The algebraic curvature tensors described in Theorem 2.4 show that Osserman of type (r,s) does not imply Jordan Osserman of type (r,s).

It is useful to give a graphical representation of Theorem 2.4. We may think of the values of (r,s) as the points with integer coordinates in the rectangle

$$\mathfrak{R} := \{(r,s) : 0 \le r \le p,\ 0 \le s \le q\}.$$

The two corners $(0,0)$ and (p,q) of $\mathfrak{R}$ are excluded as inadmissible; R_a is always Jordan Osserman at the other two corners $(p,0)$ and $(0,q)$. These two corner points are the only values for which R_a is Jordan Osserman if $2a < p \le q$. If $2a = p < q$, then R_a is Jordan Osserman on the two edges of $\mathfrak{R}$ parallel to the r axis. If $2a = p = q$, then R_a is Jordan Osserman on the boundary of $\mathfrak{R}$. The values for which R_a is Jordan Osserman are graphically represented by the three different pictures given below. Entries with '$\star$' are points where R is Jordan Osserman, entries with '$\circ$' are points where R is not Jordan Osserman, and entries with '$-$' are inadmissible points. The r-axis is horizontal and the s-axis is vertical.

$2a < p \le q$

$\star$	$\circ$	$\dots$	$\circ$	$-$
$\circ$	$\circ$	$\dots$	$\circ$	$\circ$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\circ$	$\circ$	$\dots$	$\circ$	$\circ$
$-$	$\circ$	$\dots$	$\circ$	$\star$

$2a = p < q$

$\star$	$\star$	$\dots$	$\star$	$-$
$\circ$	$\circ$	$\dots$	$\circ$	$\circ$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\circ$	$\circ$	$\dots$	$\circ$	$\circ$
$-$	$\star$	$\dots$	$\star$	$\star$

$2a = p = q$

$\star$	$\star$	$\dots$	$\star$	$-$
$\star$	$\circ$	$\dots$	$\circ$	$\star$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\star$	$\circ$	$\dots$	$\circ$	$\star$
$-$	$\star$	$\dots$	$\star$	$\star$

§3 Jordan Osserman duality

Theorem 2.1(2) is a duality result for Osserman algebraic curvature tensors which was proved in [13]. We generalize that proof to establish the corresponding duality result for Jordan Osserman algebraic curvature tensors given in Theorem 2.2.

Let R be Jordan Osserman of type (r,s) on a vector space V of signature (p,q). We must show that R is Jordan Osserman of type $(p-r, q-s)$, i.e. that the Jordan normal form of $\mathcal{J}_R(\cdot)$ is constant on $\mathrm{Gr}_{p-r,q-s}(V)$.

By Theorem 2.1(1), R is Einstein. Let $\{e_1, \dots, e_{p+q}\}$ be any orthonormal basis for V. If $\varepsilon_i := (e_i, e_i)$, then $\sum_i \varepsilon_i R(y, e_i, e_i, x) = c(y,x)$, where c is the Einstein constant. This implies that:

$$\text{(3.1.a)} \qquad \textstyle\sum_i \varepsilon_i \mathcal{J}_R(e_i) = c \cdot \mathrm{Id}\,.$$

Let $\tau \in \mathrm{Gr}_{p-r,p-s}(V)$. Let $\sigma := \tau^\perp \in \mathrm{Gr}_{r,s}(V)$ be the orthogonal complement of τ. We construct an adapted orthonormal basis for V as follows. Let

$\{e_1, \ldots, e_{r+s}\}$ be an orthonormal basis for σ and let $\{e_{r+s+1}, \ldots, e_{p+q}\}$ be an orthonormal basis for τ. We use equations (1.1.b) and (3.1.a) to see that

$$\mathcal{J}_R(\sigma) + \mathcal{J}_R(\tau) = \textstyle\sum_{1\le i\le r+s} \varepsilon_i \mathcal{J}_R(e_i) + \sum_{r+s+1\le i\le p+q} \varepsilon_i \mathcal{J}_R(e_i) = c \cdot \mathrm{Id}\,.$$

Thus the Jordan normal form of $\mathcal{J}_R(\tau)$ is determined by the Jordan normal form of $\mathcal{J}_R(\sigma)$. By hypothesis the Jordan normal form of $\mathcal{J}_R(\sigma)$ is constant on $\mathrm{Gr}_{r,s}(V)$. Thus we may conclude that the Jordan normal form of $\mathcal{J}_R(\tau)$ is also constant on $\mathrm{Gr}_{p-r,p-s}(V)$. □

§4 Examples of Jordan Osserman algebraic curvature tensors

Throughout this section, we shall let V be a vector space of signature (p,q), we shall let Φ_a be the skew-adjoint linear map on V which is defined in equation (2.3.a), we shall let R_a be the associated curvature tensor, and we shall let $\mathcal{J}_a$ be the associated Jacobi operator. We begin the proof of Theorem 2.4 with the following observation:

4.1 Lemma. *We have:*

(1) *R_a is k Osserman for any k and for any a;*

(2) *R_a is Jordan Osserman of type (r,s) if and only if* $\operatorname{rank} R_{\Phi_a}$ *is constant on $Gr_{r,s}(V)$.*

PROOF: It is immediate from the definition that $\Phi_a^2 = 0$. Thus $\operatorname{range}\Phi_a$ is totally isotropic. We use equation (2.2.a) to see that $\mathcal{J}_a(x)y = 3(\Phi_a x, y)\Phi_a x$. Consequently

$$\mathcal{J}_a(v_1)\mathcal{J}_a(v_2)y = 9(\Phi_a v_1, y)(\Phi_a v_1, \Phi_a v_2)\Phi_a v_2 = 0$$

for any vectors v_1 and v_2 in V. Thus $\mathcal{J}_a(\sigma)^2 = 0$ for any non-degenerate subspace σ. This implies that 0 is the only eigenvalue of $\mathcal{J}_a(\sigma)$ and hence R_a is k Osserman for any k; this proves assertion (1).

Since $\mathcal{J}_a(\sigma)^2 = 0$, the Jordan normal form of $\mathcal{J}_a(\sigma)$ is determined by the rank of $\mathcal{J}_a(\sigma)$; assertion (2) now follows directly. □

We will use the following lemma to study the rank of $\mathcal{J}_a(\sigma)$.

4.2 Lemma. *Let $v_1, \ldots, v_k$ be linearly independent vectors in V. Then:*

(1) *there exist vectors $w_1, \ldots, w_k$ in V so that $(v_i, w_j) = \delta_{ij}$;*

(2) *let $Ty := c_1(v_1, y)v_1 + \cdots + c_k(v_k, y)v_k$ define a linear transformation of V, where $c_1, \ldots, c_k$ are non-zero constants. Then the rank of T is k.*

PROOF: Let V^* be the associated dual vector space of linear maps from V to $\mathbb{R}$. We define a linear map $\psi : V \to V^*$ by setting $\psi(w)v = (v, w)$. Since the inner product on V is non-degenerate, ψ is injective. Since $\dim V = \dim V^*$, ψ is bijective. We can extend the collection $\{v_1, \ldots, v_k\}$ to a basis for V; thus without loss of generality, we may assume $k = \dim V$. Let $\{v^1, \ldots, v^k\}$ be the corresponding dual basis for V^*. Set $w_i := \psi^{-1}v_i$. Assertion (1) follows as

$$(v_i, w_j) = \psi(w_j)v_i = v^j \cdot v_i = \delta_{ij}.$$

Let T be the transformation of assertion (2). It is clear from the definition that $\operatorname{range} T \subset \operatorname{span}\{v_1, \ldots, v_k\}$. Since $(v_i, w_j) = \delta_{ij}$, we have $Tw_j = c_j v_j$. Since $c_j \neq 0$, $v_j \in \operatorname{range} T$. It now follows that $\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{range} T$. Thus the rank of T is k. □

We conclude our preparation for the proof of Theorem 2.4 with the following lemma.

4.3 Lemma. *We have:*

(1) *If $2a < p$ and if $1 \le r \le p-1$, then R_a is not Jordan Osserman of type (r, s) for any $0 \le s \le q$.*
(2) *If $2a = p$, if $1 \le r \le p-1$, and if $1 \le s \le q-1$, then R_a is not Jordan Osserman of type (r, s).*
(3) *R_a is Jordan Osserman of type $(p, 0)$.*
(4) *If $2a = p$ and if $1 \le r \le p-1$, then R_a is Jordan Osserman of type $(r, 0)$.*

PROOF: Let $\{e_1^-, \ldots, e_p^-, e_1^+, \ldots, e_q^+\}$ be the orthonormal basis for V used in equation (2.3.a) to define Φ_a. We have $\mathcal{J}_a(e_i^+) = \mathcal{J}_a(e_i^-)$; this vanishes if $i > 2a$. Let

$$\tau := \begin{cases} \operatorname{span}\{e_2^-, \ldots, e_r^-, e_2^+, \ldots, e_s^+\} & \text{if } r \ge 2,\ s \ge 2, \\ \operatorname{span}\{e_2^-, \ldots, e_r^-\} & \text{if } r \ge 2,\ s \le 1, \\ \operatorname{span}\{e_2^+, \ldots, e_s^+\} & \text{if } r \le 1,\ s \ge 2, \\ \{0\} & \text{if } r \le 1,\ s \le 1. \end{cases}$$

To prove assertion (1), we will exhibit two subspaces σ_1 and σ_2 of type (r, s) so that $\operatorname{rank}\{\mathcal{J}_a(\sigma_1)\} \neq \operatorname{rank}\{\mathcal{J}_a(\sigma_2)\}$. Let

$$\sigma_1 := \begin{cases} \tau \oplus \operatorname{span}\{e_1^-, e_1^+\} & \text{if } s \ge 1, \\ \tau \oplus \operatorname{span}\{e_1^-\} & \text{if } s = 0, \end{cases}$$

$$\sigma_2 := \begin{cases} \tau \oplus \operatorname{span}\{e_p^-, e_1^+\} & \text{if } s \ge 1, \\ \tau \oplus \operatorname{span}\{e_p^-\} & \text{if } s = 0. \end{cases}$$

The index '1' does not appear among the indices comprising the basis for τ so there is no 'interaction'. Furthermore, since $2a < p$, $\mathcal{J}_a(e_p^-) = 0$. We have the cancellation $(e_1^-, e_1^-)\mathcal{J}_a(e_1^-) + (e_1^+, e_1^+)\mathcal{J}_a(e_1^+) = 0$. We may now use Lemma 4.2 to compute:

$$\begin{aligned}
\operatorname{rank}\{\mathcal{J}_a(\sigma_1)\} &= \begin{cases} \operatorname{rank}\{\mathcal{J}_a(\tau)\} & \text{if } s \geq 1, \\ \operatorname{rank}\{\mathcal{J}_a(\tau)\} + 1 & \text{if } s = 0, \end{cases} \\
\operatorname{rank}\{\mathcal{J}_a(\sigma_2)\} &= \begin{cases} \operatorname{rank}\{\mathcal{J}_a(\tau)\} + 1 & \text{if } s \geq 1, \\ \operatorname{rank}\{\mathcal{J}_a(\tau)\} & \text{if } s = 0. \end{cases}
\end{aligned}$$

This shows that $\mathcal{J}_a(\sigma_1)$ and $\mathcal{J}_a(\sigma_2)$ have different ranks and thereby completes the proof of assertion (1).

Let $2a = p$, let $1 \leq r \leq p-1$, and let $1 \leq s \leq q-1$. As in the proof of assertion (1), we will construct two subspaces σ_1 and σ_2 of type (r, s) so that $\mathcal{J}_a(\sigma_1)$ and $\mathcal{J}_a(\sigma_2)$ have different ranks. We define

$$\begin{aligned}
\sigma_1 &:= \quad \tau \oplus \operatorname{span}\{e_1^-, e_1^+\}, \qquad \text{and} \\
\sigma_2 &:= \begin{cases} \tau \oplus \operatorname{span}\{e_{r+1}^-, e_1^+\} & \text{if } r \geq s, \\ \tau \oplus \operatorname{span}\{e_1^-, e_{s+1}^+\} & \text{if } r < s. \end{cases}
\end{aligned}$$

Again, note that the index '1' does not appear among the indices comprising the basis for τ. If $r \geq s$ (resp.$r < s$), then the index $r+1$ (resp.$s+1$) does not appear among these indices either. If $s+1 \leq p$, then $\mathcal{J}_a(e_{s+1}^+) \neq 0$; if $s+1 > p$, then $\mathcal{J}_a(e_{s+1}^-) = 0$. Since $r+1 \leq p = 2a$, $\mathcal{J}_a(e_{r+1}^-) \neq 0$. Thus:

$$\begin{aligned}
\operatorname{rank}\{\mathcal{J}_a(\sigma_1)\} &= \quad \operatorname{rank}\{\mathcal{J}_a(\tau)\}, \\
\operatorname{rank}\{\mathcal{J}_a(\sigma_2)\} &= \begin{cases} \operatorname{rank}\{\mathcal{J}_a(\tau)\} + 2 & \text{if } r \geq s, \\ \operatorname{rank}\{\mathcal{J}_a(\tau)\} + 2 & \text{if } r < s < p, \\ \operatorname{rank}\{\mathcal{J}_a(\tau)\} + 1 & \text{if } r < p \leq s. \end{cases}
\end{aligned}$$

Since $\mathcal{J}_a(\sigma_1)$ and $\mathcal{J}_a(\sigma_2)$ have different ranks, R_a is not Jordan Osserman of type (r, s). This establishes assertion (2).

If we can show that $\operatorname{rank}\{\mathcal{J}_a(\sigma)\} = 2a$ for every maximal timelike subspace σ of V, we may then use Lemma 4.1 to show that R_a is Jordan Osserman of type $(p, 0)$ which will prove assertion (3). Since

$$\operatorname{range}\{\mathcal{J}_a(\sigma)\} \subset \{\Phi_a e_1^-, \ldots, \Phi_a e_{2a}^-\}, \qquad \operatorname{rank}\{\mathcal{J}_a(\sigma)\} \leq 2a.$$

We suppose that $\operatorname{rank}\{\mathcal{J}_a(\sigma)\} < 2a$ and argue for a contradiction. Let

$$W := \operatorname{span}\{e_1^-, \ldots, e_{2a}^-\}.$$

As $\dim W = 2a$ and as $\operatorname{rank}\{\mathcal{J}_a(\sigma)\} < 2a$, we have $\ker\{\mathcal{J}_a(\sigma)\} \cap W \neq \{0\}$. Thus we may choose $0 \neq w \in W$ with $\mathcal{J}_a(\sigma)w = 0$. Let $\{v_1, \ldots, v_p\}$ be an orthonormal basis for σ. We use equations (1.1.b) and (2.2.a) to compute:

$$0 = (\mathcal{J}_a(\sigma)w, w) = -3(\Phi_a v_1, w)(\Phi_a v_1, w) - \cdots - 3(\Phi_a v_p, w)(\Phi_a v_p, w).$$

This implies that

$$0 = (\Phi_a v_i, w) = -(v_i, \Phi_a w) \quad \text{for } 1 \leq i \leq p.$$

Consequently, $\Phi_a w \perp \sigma$. Since σ is a maximal timelike subspace, $\Phi_a w$ either vanishes or is spacelike. Since $\operatorname{range} \Phi_a$ is totally isotropic, $\Phi_a w$ is a null vector, not a spacelike vector. Thus we must have that $\Phi_a w = 0$. This is false as Φ_a is injective on W. This contradiction shows that $\operatorname{rank}\{\mathcal{J}_a(\sigma)\} = 2a$ and hence R_a is Jordan Osserman of type $(p, 0)$; assertion (3) is established.

Finally, suppose that $2a = p$. Let $\sigma \in \operatorname{Gr}_{r,0}(V)$. If we can show that the rank of $\mathcal{J}_a(\sigma)$ is r, then it would follow by Lemma 4.1 that R_a is Jordan Osserman of type $(r, 0)$. Let $\{v_1, \ldots, v_r\}$ be an orthonormal basis for σ. We use equations (1.1.b) and (2.2.a) to see that

$$\mathcal{J}_a(\sigma)y = -3\{(y, \Phi_a(v_1))\Phi_a(v_1) + \cdots + (y, \Phi_a(v_r))\Phi_a(v_r)\}.$$

Thus by Lemma 4.2, if $\{\Phi_a(v_1), \ldots, \Phi_a(v_r)\}$ is a linearly independent set, then $\operatorname{rank} \mathcal{J}_a(\sigma) = r$. We suppose the contrary, i.e. that the set $\{\Phi_a(v_1), \ldots, \Phi_a(v_r)\}$ is linearly dependent, and argue for a contradiction. Choose $0 \neq v \in \sigma$ so that $\Phi_a(v) = 0$. We expand:

$$\begin{aligned} v &= c_1^- e_1^- + \cdots + c_p^- e_1^- + c_1^+ e_1^+ + \cdots + c_q^+ e_q^+ \\ \Phi_a v &= (c_1^+ - c_1^-)\Phi_a(e_1^+) + \cdots + (c_p^+ - c_p^-)\Phi_a(e_p^+). \end{aligned}$$

Since $\Phi_a v = 0$, we have $c_i^+ = c_i^-$ for $1 \leq i \leq p$. Thus

$$\begin{aligned} (v, v) &= (c_1^+)^2 - (c_1^-)^2 + \cdots + (c_p^+)^2 - (c_p^-)^2 + (c_{p+1}^+)^2 + \cdots + (c_q^+)^2 \\ &= (c_{p+1}^+)^2 + \cdots + (c_q^+)^2 \geq 0. \end{aligned}$$

Since $v \neq 0$ and since σ is timelike, $(v, v) < 0$. This contradiction shows that $\{\Phi_a v_1, \ldots, \Phi_a v_r\}$ is a linearly independent set. Thus $\operatorname{rank}\{\mathcal{J}_a(\sigma)\} = r$ and hence R_a is Jordan Osserman of type $(r, 0)$. This completes the proof of the last assertion of Lemma 4.3. $\square$

4.4 Remark. We note that we can interchange the roles of spacelike and timelike vectors in Lemma 4.3 by changing the sign of the inner product.

PROOF OF THEOREM 2.4: Suppose that $2a < p \leq q$. We use Lemma 4.3(3) to see that R_a is Jordan Osserman of type $(p, 0)$. Then dually by Theorem 2.2, we have that R_a is Jordan Osserman of type $(0, q)$. By Lemma 4.3 (1), since $2a < p$, R_a is not Jordan Osserman of type (r, s) for $1 \leq r \leq p-1$. Similarly, by Lemma 4.3(1) and Remark 4.4, since $2a < q$, R_a is not Jordan Osserman of type (r, s) for $1 \leq s \leq q-1$. Assertion (1) now follows.

Suppose that $2a = p < q$. We use Lemma 4.3(4) to see that R_a is Jordan Osserman of type $(r, 0)$ if $1 \leq r \leq p-1$. Dually, by Theorem 2.2, R_a is also Jordan Osserman of type $(p-r, q)$. By Lemma 4.3(1) and Remark 4.4, since $2a < q$, R_a is not Jordan Osserman of type (r, s) if $1 \leq s \leq q-1$. This establishes assertion (2).

Suppose finally that $2a = p = q$. Since $2a = p$, we use Lemma 4.3(4) to see that R_a is Jordan Osserman of type $(r, 0)$ if $1 \leq r \leq p-1$. Since $2a = q$, similarly we have, by Remark 4.4, that R_a is Jordan Osserman of type $(0, s)$ if $1 \leq s \leq q-1$. The remaining values $(p-r, q)$ and $(p, q-s)$ then follow dually by Theorem 2.2. If $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$, then R_a is not Jordan Osserman by Lemma 4.3(2). All the assertions of Theorem 2.4 are now proved. $\square$

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