Comment.Math.Univ.Carolin. 43,2 (2002)319–333

Products of Lindelöf T_2-spaces are Lindelöf — in some models of ZF

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Abstract. The stability of the Lindelöf property under the formation of products and of sums is investigated in **ZF** (= Zermelo-Fraenkel set theory without **AC**, the axiom of choice). It is

- not surprising that countable summability of the Lindelöf property requires some weak choice principle,
- highly surprising, however, that productivity of the Lindelöf property is guaranteed by a drastic failure of **AC**,
- amusing that finite summability of the Lindelöf property takes place if either some weak choice principle holds or if **AC** fails drastically.

Main results:

1. Lindelöf = compact for T_1-spaces iff $\mathbf{CC}(\mathbb{R})$, the axiom of countable choice for subsets of the reals, fails.
2. Lindelöf T_1-spaces are finitely productive iff $\mathbf{CC}(\mathbb{R})$ fails.
3. Lindelöf T_2-spaces are productive iff $\mathbf{CC}(\mathbb{R})$ fails and **BPI**, the Boolean prime ideal theorem, holds.
4. Arbitrary products and countable sums of compact T_1-spaces are Lindelöf iff **AC** holds.
5. Lindelöf spaces are countably summable iff **CC**, the axiom of countable choice, holds.
6. Lindelöf spaces are finitely summable iff either **CC** holds or $\mathbf{CC}(\mathbb{R})$ fails.
7. Lindelöf T_2-spaces are T_3 spaces iff $\mathbf{CC}(\mathbb{R})$ fails.
8. Totally disconnected Lindelöf T_2-spaces are zerodimensional iff $\mathbf{CC}(\mathbb{R})$ fails.

Keywords: axiom of choice, axiom of countable choice, Lindelöf space, compact space, product, sum

Classification: 03E25, 54A35, 54B10, 54D20, 54D30

1. Introduction

Ordinarily topology is dealt with in the setting of **ZFC**, i.e., Zermelo-Fraenkel set theory including **AC**, the *axiom of choice*. Although **AC** is neither evidently true nor evidently false, this adherence to **AC** seems to be based on a general belief that adoption of **AC** enables topologists to prove more and better theorems[1]. Aside from the trivial observation that no theorem **T** in **ZFC** is lost in **ZF** (Zermelo-Fraenkel set theory without **AC**), — it simply turns into the implication **AC** $\Longrightarrow$ **T** which often enough can be even improved to an equivalence **WC** $\Longleftrightarrow$ **T** for a suitable weak form **WC** of **AC**, — it may be possible that certain desirable topological results hold only under assumptions that are incompatible with **AC**. That some measure theoretic results of this kind do in fact exist has been shown convincingly by means of the *axiom of determinateness* (see, e.g., [22]). However, the latter, though inconsistent with **AC**, still implies **CC**($\mathbb{R}$), a weak form of **AC**, stating that for each sequence (X_n) of non-empty sets X_n of real numbers the product $\prod X_n$ is not empty. In this paper we will go even further and present some surprising results about Lindelöf spaces under assumptions that are inconsistent even with **CC**($\mathbb{R}$). In fact we will show that the equality

$$\textit{Lindelöf} = \textit{compact}$$

holds for all T_1-spaces if and only if **CC**($\mathbb{R}$) fails. Even more striking, perhaps, is the observation that there are models of **ZF** in which arbitrary products of Hausdorff Lindelöf spaces are Lindelöf, whereas under **AC**, as is well known, even the product of two Hausdorff Lindelöf spaces may fail badly to be Lindelöf.

In the following we list some familiar concepts and known results. These, as everything else in this paper, take place in the setting of **ZF**.

1.1 Definitions. 1. A topological space is called *Lindelöf* (resp. *compact*) if each of its open covers contains an at most countable (resp. finite) subcover.

2. **CC**, the *axiom of countable choice*, states that for each sequence (X_n) of non-empty sets X_n the product $\prod X_n$ is not empty.
3. **CC**($\mathbb{R}$) states that for each sequence (X_n) of non-empty subsets X_n of the set $\mathbb{R}$ of real numbers the product $\prod X_n$ is not empty.
4. **CMC**, the *axiom of countable multiple choice*, states that for each sequence (X_n) of non-empty sets there exists a sequence (F_n) of non-empty, finite subsets F_n of X_n.
5. **BPI**, the *Boolean prime ideal theorem*, states that every non-trivial Boolean algebra contains a prime ideal (equivalently: for each set X, every filter on X can be extended to an ultrafilter on X).

[1]The feeling that topology without **AC** is a painful undertaking is aptly expressed by such titles as *Horrors of topology without AC* (van Douwen [27]), *Continuing horrors of topology without choice* (Good and Tree [10]) and *Disasters in topology without the axiom of choice* (Keremedis [18]).

1.2 Theorem ([17]). *Equivalent are:*

1. **AC**,
2. *products of compact T_1-spaces are compact.*

1.3 Theorem (see [24] and, e.g., [12]). *Equivalent are:*

1. **BPI**,
2. *products of compact T_2-spaces are compact.*

1.4 Theorem ([14]). *Equivalent are:*

1. **CC**($\mathbb{R}$),
2. $\mathbb{N}$*, the discrete space of natural numbers, is Lindelöf,*
3. *every topological space with a countable base is Lindelöf.*

1.5 Remarks. 1. Notice that the following proper implications hold in **ZF**:

$$\mathbf{AC} \Longrightarrow \mathbf{BPI}, \quad \mathbf{AC} \Longrightarrow \mathbf{CC} \Longrightarrow \mathbf{CC}(\mathbb{R}).$$

It is not known whether the implication

$$\mathbf{CC} \Longrightarrow \mathbf{CMC}$$

is proper or an equivalence. See [15].

2. For some countable and finitary modifications of Theorem 1.2 see [13].
3. Observe further that **CC**($\mathbb{R}$) is equivalent to:
   (∗) For every sequence (X_n) of non-empty sets with $|\cup X_n| \leq 2^{\aleph_0}$ the product $\prod X_n$ is not empty,

   but strictly weaker than:
   (∗∗) For every sequence (X_n) of non-empty sets X_n with $|X_n| \leq 2^{\aleph_0}$ for each n the product $\prod X_n$ is not empty.
   See [15].
4. As is well known (see, e.g., [7]) the Lindelöf property occupies a prominent place in **ZFC**-topology. On one hand
   (a) all compact spaces (more generally: all σ-compact spaces[2]) and all separable metrizable spaces[3] (more generally: all separable paracompact T_3-spaces) are Lindelöf.

   and on the other hand
   (b) all Lindelöf T_3-spaces are paracompact and realcompact.

   Moreover,

[2]Obviously, in **ZF** all compact spaces are Lindelöf. However, all σ-compact spaces are Lindelöf iff **CC** holds. See [3].

[3]Separable metrizable spaces are Lindelöf iff **CC**($\mathbb{R}$) holds. See [1]; cf. also [10] and [14].

(c) continuous images, closed subspaces, and countable sums of Lindelöf spaces are Lindelöf.

But unfortunately even finite products of Lindelöf spaces may fail to be Lindelöf and thus, whereas compact T_2-spaces form an epireflective subcategory of **Haus**, the category of T_2-spaces, Lindelöf T_2-spaces fail drastically to be epireflective in **Haus**.

2. Lindelöf = compact

2.1 Theorem. *Equivalent are:*

(a) *Lindelöf = compact for T_1-spaces,*
(b) *Lindelöf = compact for subspaces of $\mathbb{R}$,*
(c) $\mathbf{CC}(\mathbb{R})$ *fails.*

Proof: (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c). (b) implies that $\mathbb{N}$ is not Lindelöf. Thus, by Theorem 1.4, $\mathbf{CC}(\mathbb{R})$ fails.

(c) $\Longrightarrow$ (a). We need only show that failure of (a) implies $\mathbf{CC}(\mathbb{R})$. So let X be a non-compact, Lindelöf T_1-space. Let $\mathfrak{C}$ be an open cover of X that has no finite subcover. Since X is Lindelöf we may assume $\mathfrak{C}$ to be countable. By forming finite unions and deleting superfluous members we obtain an open cover $\mathfrak{B} = \{B_n \mid n \in \mathbb{N}\}$ of X such that

- $B_m \subset B_n$ for $m < n$ and
- $C_n = B_n \setminus \bigcup_{m<n} B_m \neq \emptyset$ for each $n \in \mathbb{N}$.

Define for each $n \in \mathbb{N}$ and each $x \in C_n$ the set

$$A(n,x) = B_n \setminus \{x\}$$

and consider the open cover

$$\mathfrak{A} = \{A(n,x) \mid n \in \mathbb{N} \text{ and } x \in C_n\}$$

of X. Then there exist unique maps $\alpha\colon \mathfrak{A} \longrightarrow \mathbb{N}$ and $\beta\colon \mathfrak{A} \longrightarrow X$ such that $A = A(\alpha(A), \beta(A))$ for each $A \in \mathfrak{A}$.

Since X is Lindelöf $\mathfrak{A}$ has a countable subcover $\{A_n \mid n \in \mathbb{N}\}$. The set $M = \{\alpha[A_n] \mid n \in \mathbb{N}\}$ is an unbounded, thus countable, subset of $\mathbb{N}$. For each $m \in M$ define $x_m = \beta(A_{\min\{n\in\mathbb{N} \mid \alpha(A_n)=m\}})$. Then $x_m \in C_m$. The subspace Y of X with underlying set $\{x_m \mid m \in M\}$ is countable and discrete, since for each $m \in M$

(a) the set $\{x_n \mid n \leq m\} = B_m \cap Y$ is open in Y,
(b) the set $\{x_n \mid n < m\}$ is closed in Y as a finite subset of a T_1-space,

and thus

(c) $\{x_m\}$ is open in Y.

Consequently Y is homeomorphic to $\mathbb{N}$. As a closed subspace of X, Y is Lindelöf. Thus $\mathbb{N}$ is Lindelöf, and therefore Theorem 1.4 implies that $\mathbf{CC}(\mathbb{R})$ holds. □

2.2 Remarks. 1. There exist models of **ZF** in which $\mathbf{CC}(\mathbb{R})$ fails, e.g., Cohen's original model (M1 in [15]).

2. As a possible alternative to **AC**, Alonzo Church [6] introduced his postulate:

$$\mathbf{C}\text{: } \omega_1 \textit{ is a countable union of countable sets}^4$$

and demonstrated that **C** implies the failure of $\mathbf{CC}(\mathbb{R})$.
Going one step further[5], Specker [26] introduced the condition

$$\mathbf{H}\text{: } \mathbb{R} \textit{ is a countable union of countable sets}$$

and demonstrated that **H** implies **C**.
Feferman and Levy [8] (compare also [16]) constructed a model (called M9 in [15]) of **ZF** that satisfies **H**, hence **C**, hence the negation of $\mathbf{CC}(\mathbb{R})$. Thus Theorem 2.1 implies that in M9 the equation

$$\text{Lindelöf} = \text{compact}$$

holds for all T_1-spaces.

3. The above theorem cannot be extended to T_0-spaces, since the space $\mathbb{N}_l = (\mathbb{N}, \tau_l)$ where τ_l, the *lower topology* on $\mathbb{N}$, consists of all subsets of $\mathbb{N}$ that

[4] Church's postulate **C** is equivalent to the statement

$$\omega_1 \textit{ is weakly Lindelöf}$$

see Definition 8.1 below, [10, Corollary 3.7] and form 34 as well as note 107 in [15] — where in each case *Lindelöf* should be replaced by *weakly Lindelöf*.
Observe that ω_1 is never Lindelöf. If it were, the open cover $\{[0,\alpha] \mid \alpha < \omega_1\}$ would have a countable subcover. Thus **C** would hold. This would imply on one hand (via [6]) that $\mathbf{CC}(\mathbb{R})$ fails and on the other hand (via Theorem 4.1 and the fact that $\mathbb{N}$ is homeomorphic to a closed subspace of ω_1) that $\mathbf{CC}(\mathbb{R})$ holds — a contradiction!

[5] As another strengthening of **C** (unrelated to **H**) Specker [26] introduced the condition

$$\textit{cof } \aleph_\alpha = \aleph_0 \;\; \textit{for each ordinal } \alpha,$$

equivalently:

$$\text{each } \aleph_\alpha \textit{ is weakly Lindelöf},$$

and, — assuming the consistency of the existence of arbitrary large strongly compact cardinals in **ZFC**, — Gitik [9] constructed a model of **ZF** (called M17 in [15]) that satisfies this condition.

contain with any n each $m \in \mathbb{N}$ with $m \leq n$, is a non-compact, Lindelöf T_0-space.

4. As shown in [14], Theorem 1.4 can be enriched by adding the following equivalent conditions:
   (d) $\mathbb{Q}$ is a Lindelöf space,
   (e) $\mathbb{R}$ is a Lindelöf space.

   Moreover, Lindelöf's original result [21] may be added:
   (f) $\mathbb{R}^n$ is hereditarily Lindelöf for any n.

   By [1] we may add:
   (g) Every separable pseudometric space is Lindelöf.

   By [23] we may add further:
   (h) The classical Ascoli Theorem.

   In view of Theorem 2.1 the following equivalent conditions can be added as well:
   (i) there exists a non-compact Lindelöf T_1-space,
   (j) there exists a non-compact Lindelöf subspace of $\mathbb{R}$,
   (k) there exists an unbounded Lindelöf subspace of $\mathbb{R}$,
   (l) there exists a non-closed Lindelöf subspace of $\mathbb{R}$.
5. Theorem 2.1 implies further the following result of Gonçalo Gutierres [11] that triggered the present investigations:
   $(*)$ every unbounded Lindelöf subspace of $\mathbb{R}$ contains an unbounded sequence.

   Recall that the condition
   $(**)$ every unbounded subset of $\mathbb{R}$ contains an unbounded sequence

   is equivalent to $\mathbf{CC}(\mathbb{R})$. See [14].
6. Under the assumption
   $(*)$ There exists an infinite, Dedekind-finite subset of $\mathbb{R}$,

   Brunner ([4], see also [5]) has shown that a wide class of spaces, including $\mathbb{R}$, have dense, Dedekind-finite subsets. Moreover, he demonstrated that every Lindelöf T_3-space with a dense, Dedekind-finite subset is compact. In view of the fact that T_3 properly implies T_1 and that $(*)$ properly implies that $\mathbf{CC}(\mathbb{R})$ fails (equivalently: $\mathbf{CC}(\mathbb{R})$ properly implies that Dedekind-finite subsets of $\mathbb{R}$ are finite — in Sageev's model [25] (called M6 in [15]) Dedekind-finite subsets of $\mathbb{R}$ are finite but $\mathbf{CC}(\mathbb{R})$ fails), Theorem 2.1 may be considered as a natural (in a way ultimate) extension of Brunner's result.
7. If a class $\mathfrak{C}$ of subspaces of $\mathbb{R}$ is called a *Lindelöf-class* provided that there exists a model of **ZF** in which the members of C are precisely the Lindelöf subspaces of $\mathbb{R}$, then — by Theorem 2.1 above — there exist precisely two Lindelöf classes, namely

(a) the class of all subspaces of $\mathbb{R}$ (exactly if $\mathbf{CC}(\mathbb{R})$ holds),
(b) the class of all compact (= closed and bounded) subspaces of $\mathbb{R}$ (exactly if $\mathbf{CC}(\mathbb{R})$ fails).

8. Comparison of Theorems 1.4 and 2.1 shows that $\mathbf{CC}(\mathbb{R})$ as well as its negation can be considered as axioms that guarantee certain *positive* topological results. Generally, the axiom of choice, **AC** and its variants **CC** and $\mathbf{CC}(\mathbb{R})$, being of the form $\forall x\ \exists y\ P(x,y)$, are regarded as conditions that *guarantee the existence* of certain desirable entities. However, their negations, being of the form $\exists x\ \forall y\ Q(x,y)$, can equally well be regarded as such existence guaranteeing conditions.

3. Products of Lindelöf spaces

3.1 Lemma (see, e.g. [7]). *$\mathbb{N}^{\mathbb{R}}$ is not Lindelöf.*

PROOF: Let $\mathcal{P}_2\mathbb{R}$ be the set of all subsets of $\mathbb{R}$ with exactly two elements. For $D = \{a,b\}$ in $\mathcal{P}_2\mathbb{R}$ define

$$C_D = \{(n_x) \in \mathbb{N}^{\mathbb{R}} \mid n_a = n_b\}.$$

Since $\mathbb{R}$ is uncountable, the set $\mathfrak{C} = \{C_D \mid D \in \mathcal{P}_2\mathbb{R}\}$ is an open cover of $\mathbb{N}^{\mathbb{R}}$. For each sequence (D_n) in $\mathcal{P}_2\mathbb{R}$ the set $\{C_{D_n} \mid n \in \mathbb{N}\}$ does not cover $\mathbb{N}^{\mathbb{R}}$, since $D = \bigcup_n D_n$ is at most countable, hence there exists an injective map $\phi\colon D \longrightarrow \mathbb{N}$, and thus the point (n_x) of $\mathbb{N}^{\mathbb{R}}$, defined by

$$n_x = \begin{cases} \phi(x), & \text{if } x \in D \\ 0, & \text{otherwise} \end{cases}$$

does not belong to $\bigcup_n C_{D_n}$. Consequently $\mathbb{N}^{\mathbb{R}}$ is not Lindelöf. □

3.2 Theorem. *Equivalent are:*

(a) *products of Lindelöf T_2-spaces are Lindelöf,*
(b) **BPI** *holds and* $\mathbf{CC}(\mathbb{R})$ *fails.*

PROOF: (a) $\Longrightarrow$ (b). Since, by Lemma 3.1, $\mathbb{N}^{\mathbb{R}}$ is not Lindelöf, $\mathbb{N}$ must fail to be Lindelöf, too. Thus, by Theorem 1.4, $\mathbf{CC}(\mathbb{R})$ must fail. Hence, by Theorem 2.1 the Lindelöf T_2-spaces are precisely the compact T_2-spaces. By Theorem 1.3, **BPI** holds.

(b) $\Longrightarrow$ (a). Vice versa, the failure of $\mathbf{CC}(\mathbb{R})$ implies, by Theorem 2.1, that the Lindelöf T_2-spaces are precisely the compact T_2-spaces. Hence, by Theorem 1.3, **BPI** implies that (a) holds. □

3.3 Remarks. 1. There are models of **ZF** in which **BPI** holds and $\mathbf{CC}(\mathbb{R})$ fails. In fact, this is the case in Cohen's original model (M1 in [15]).

2. In **ZF** the Lindelöf-property is closed-hereditary. Thus in any model of **ZF** in which **BPI** holds and $\mathbf{CC}(\mathbb{R})$ fails, Lindelöf T_2-spaces form an *epireflective subcategory* of the category **Haus** of T_2-spaces, and the Lindelöf-reflection of a T_2-space coincides with its Čech-Stone-compactification, in particular
$$\mathbb{N} \hookrightarrow \beta\mathbb{N}$$
is the Lindelöf-reflection of $\mathbb{N}$ — somewhat surprising, perhaps.
3. There is no model of **ZF** in which products of Lindelöf T_1-spaces are Lindelöf. This can be seen as follows: By Theorem 3.2, in such a model, $\mathbf{CC}(\mathbb{R})$ must fail and products of compact T_1-spaces must be compact. Hence, by Theorem 1.2, **AC** must hold. But **AC** and not $\mathbf{CC}(\mathbb{R})$ is obviously inconsistent.
For T_0-spaces the failure of the Lindelöf property to be productive is even more severe: In **ZF** the space $\mathbb{N}_l$, defined in 2.2(3), is Lindelöf, but the product space $\mathbb{N}_l^{\mathbb{R}}$ fails to be so ([2]).

Next we turn our attention to finite productivity of the Lindelöf property.

3.4 Definition. The *Sorgenfrey line* **S** is the topological space that has $\mathbb{R}$ as underlying set and the collection of intervals of the form
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
as a base for the topology τ_S.

3.5 Lemma (see, e.g., [7]). $\mathbf{S}^2$ *is not Lindelöf.*

Proof: Define

$$\begin{aligned} C &= \{(x, y) \in \mathbb{R}^2 \mid y < -x\} \quad \text{and} \\ C_a &= \{(x, y) \in \mathbb{R}^2 \mid a \leq x \ \text{ and } \ -a \leq y\} \ \text{ for each } \ a \in \mathbb{R}. \end{aligned}$$

Then $\mathfrak{C} = \{C\} \cup \{C_a \mid a \in \mathbb{R}\}$ is an uncountable open cover of $\mathbf{S}^2$, but no proper subset of $\mathfrak{C}$ covers $\mathbf{S}^2$. □

3.6 Proposition. *Equivalent are:*

(1) **S**, *the Sorgenfrey line, is Lindelöf,*
(2) $\mathbf{CC}(\mathbb{R})$.

Proof: (1) $\Longrightarrow$ (2). If **S** is Lindelöf, then its closed subspace $\mathbb{N}$ is Lindelöf. Thus $\mathbf{CC}(\mathbb{R})$ follows by Theorem 1.4.

(2) $\Longrightarrow$ (1). First, we show that

(∗) $|\mathbb{R}| = |\tau_S|$, i.e., there is a bijection between $\mathbb{R}$ and the topology τ_S of **S**.

Obviously $|\mathbb{R}| \leq |\tau_S|$, since the map $\varphi\colon \mathbb{R} \longrightarrow \tau_S$, defined by $\varphi(a) = [a, a+1)$ is injective.

Next, let A be an element of τ_S. Let τ be the ordinary topology of $\mathbb{R}$, let $r\colon \mathbb{N} \longrightarrow \mathbb{Q}$ be a bijection, let B_A be the interior of A with respect to τ, and consider $C_A = A \backslash B_A$. Define maps α_A and β_A as follows:

$$\begin{array}{rcll} \alpha_A & : & C_A & \longrightarrow \mathbb{N} \\ & & c & \longmapsto \min\{n \in \mathbb{N} \mid (c, c+r(n)) \cap C_A = \emptyset\}. \end{array}$$

$$\begin{array}{rcll} \beta_A & : & \mathbb{N} & \longrightarrow \mathbb{R} \cup \{\infty\} \\ & & n & \longmapsto \begin{cases} c, & \text{if } \alpha_A(c) = n \text{ for some } c \\ \infty, & \text{otherwise.} \end{cases} \end{array}$$

Then α_A is injective. Thus C_A is at most countable and β_A is well-defined. Moreover, $A \cup \{\infty\} = B_A \cup \beta_A[\mathbb{N}]$. Thus the map

$$\begin{array}{rcll} \gamma & : & \tau_S & \longrightarrow \tau \times (\mathbb{R} \cup \{\infty\})^{\mathbb{N}} \\ & & A & \longmapsto (B_A, \beta_A) \end{array}$$

is injective. Consequently:

$$|\tau_S| \leq |\tau| \cdot |\mathbb{R} \cup \{\infty\}|^{\omega} = 2^{\omega} \cdot (2^{\omega})^{\omega} = 2^{\omega} \cdot 2^{(\omega^2)} = 2^{\omega} \cdot 2^{\omega} = 2^{\omega+\omega} = 2^{\omega} = |\mathbb{R}|.$$

Thus $|\mathbb{R}| \leq |\tau_S| \leq |\mathbb{R}|$. By Bernstein's Theorem this implies $|\mathbb{R}| = |\tau_S|$. Consequently, (2) is equivalent to the statement:

($**$) For any sequence $(\mathfrak{C}_n)$ of non-empty subsets $\mathfrak{C}_n$ of τ_S the product $\prod_n \mathfrak{C}_n$ is not empty.

Finally, consider an open cover $\mathfrak{A}$ of $\mathbf{S}$. Define $X = \cup\{B_A \mid A \in \mathfrak{A}\}$. Then the subspace $\mathbf{X}$ of $\mathbb{R}$ with underlying set X has a countable base. Since $\{B_A \mid A \in \mathfrak{A}\}$ is an open cover of $\mathbf{X}$, condition (1) implies via Theorem 1.4 that $\{B_A \mid A \in \mathfrak{A}\}$ has an at most countable subcover $\mathfrak{B}$. Moreover, as in the first part of this proof one can construct an injective map from $\mathbb{R} \backslash X$ into $\mathbb{N}$. Thus $\mathfrak{C} = \{\{x\} \mid x \in \mathbb{R} \backslash X\} \cup \mathfrak{B}$ is a countable refinement of $\mathfrak{A}$, say $\mathfrak{C} = \{C_n \mid n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ the set $\mathfrak{C}_n = \{A \in \mathfrak{A} \mid C_n \subset A\}$ is a non-empty subset of τ_S. Consequently, ($**$) implies that there exists a sequence (A_n) in $\mathfrak{A}$ with $C_n \subset A_n$ for each $n \in \mathbb{N}$. The set $\{A_n \mid n \in \mathbb{N}\}$ is an at most countable subcover of $\mathfrak{A}$. Thus $\mathbf{S}$ is Lindelöf. □

3.7 Theorem. *Equivalent are:*

(1) *finite products of Lindelöf T_1-spaces are Lindelöf,*
(2) $\mathbf{CC}(\mathbb{R})$ *fails.*

Proof: (1) $\Longrightarrow$ (2). Immediate from Lemma 3.5 and Proposition 3.6.

(2) $\Longrightarrow$ (1). Immediate from Theorem 2.1 and the fact that finite products of compact spaces are compact (see, e.g., [12]). $\square$

For the proof of the following result we will draw on Theorem 4.3 from the next paragraph:

3.8 Theorem. *Equivalent are:*

(1) *products of Lindelöf T_1-spaces with compact T_1-spaces are Lindelöf,*
(2) $\mathbf{CC}(\mathbb{R})$ *implies* $\mathbf{CC}$.

Proof: (1) $\Longrightarrow$ (2). Let X be a Lindelöf T_1-space, and let Y be a compact T_1-space.

We want to show that the sum $X+Y$ is Lindelöf. If X or Y is empty this is obvious. Otherwise, let (x_0, y_0) be a fixed element of $X \times Y$ and let Z be the discrete space with underlying set $\{0,1\}$. Then $X+Y$ is homeomorphic to the closed subspace of $X \times (Y \times Z)$, determined by the set $(X \times \{(y_0, 0)\}) \cup (\{x_0\} \times Y \times \{1\})$. By (1), $X \times (Y \times Z)$ and hence $X+Y$ are Lindelöf. Thus (2) holds by Theorem 4.3.

(2) $\Longrightarrow$ (1). If $\mathbf{CC}$ holds, then the familiar proof of (1) in $\mathbf{ZFC}$ works as well. If $\mathbf{CC}(\mathbb{R})$ fails, then (1) follows from Theorem 3.7. $\square$

4. Sums of Lindelöf spaces

4.1 Theorem. *Equivalent are:*

(1) *countable sums of Lindelöf spaces are Lindelöf,*
(2) *countable sums of compact T_2-spaces are Lindelöf,*
(3) $\mathbb{N}+X$ *is Lindelöf for each compact T_2-space X,*
(4) $\mathbf{CC}$.

Proof: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Obvious.

(3) $\Longrightarrow$ (4). By (3), $\mathbb{N}$ is Lindelöf.

To show $\mathbf{CC}$, let (X_n) be a sequence of non-empty sets. Let $X = \bigcup_n X_n \cup \{\infty\}$ the Alexandroff-one-point-compactification of the discrete space $\bigcup_n X_n$. By (3) the sum $Y = \mathbb{N}+X$ is Lindelöf. Consider the open cover

$$\mathfrak{C} = \{X\} \cup \{\{n, x\} \mid n \in \mathbb{N} \text{ and } x \in X_n\}$$

of Y. This contains a countable subcover, say $\{C_n \mid n \in \mathbb{N}\}$, of Y. For each $n \in \mathbb{N}$, define

$$n^* = \min\{m \in \mathbb{N} \mid n \in C_m\}.$$

Then $C_{n^*} = \{n, x_n\}$ for a unique element x_n of X_n.

Thus $(x_n) \in \prod X_n$. Consequently $\mathbf{CC}$ holds.

(4) $\Longrightarrow$ (1). The familiar proof of (1) works under $\mathbf{CC}$. $\square$

4.2 Remark. The equivalence of the above conditions (1) and (4) and many related results have been established in [19].

4.3 Theorem. *Equivalent are:*

(1) *finite sums of Lindelöf T_1-spaces are Lindelöf,*
(2) *$X+Y$ is Lindelöf for each Lindelöf T_2-space X and each compact T_2-space Y,*
(3) **CC**($\mathbb{R}$) *implies* **CC**.

Proof: (1) $\Longrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (3). If **CC**($\mathbb{R}$) holds then, by Theorem 1.4, $\mathbb{N}$ is Lindelöf. Thus condition (3) of Theorem 4.1 is satisfied. Consequently **CC** holds.

(3) $\Longrightarrow$ (1). If **CC** holds, then the familiar proof of (1) works. If **CC** fails then, by (3), **CC**($\mathbb{R}$) fails, too. Thus, by Theorem 2.1, Lindelöf = compact for T_1-spaces. Since finite sums of compact spaces are compact, (1) follows. □

5. Products and sums of compact T_1-spaces

5.1 Theorem. *Equivalent are:*

(1) (a) *products of compact T_1-spaces are Lindelöf*

and

(b) *countable sums of compact T_1-spaces are Lindelöf,*

(2) **AC**.

Proof: (1) $\Longrightarrow$ (2). By Theorem 4.1, condition (b) implies **CC**. Assume that **AC** fails. Then there exists a family $(X_i)_{i\in I}$ of non-empty sets with $\prod_{i\in I} X_i = \emptyset$. Let T be the topological space with underlying set $\{\infty\}$, where $\infty \notin \bigcup_{i\in I} X_i$. Supply each X_i with the cofinite topology and form the sum $Y_i = X_i + T$. Then each Y_i is a compact T_1-space and thus $Y = \prod_{i\in I} Y_i$ is Lindelöf. Denote, for each $i \in I$, the i-th projection by $\pi_i: Y \longrightarrow Y_i$. Since $\prod_{i\in I} X_i = \emptyset$, the collection $\mathfrak{A} = \{\pi_i^{-1}(\infty) \mid i \in I\}$ is an open cover of Y. Since Y is Lindelöf there exists an at most countable subset K of I such that $\{\pi_j^{-1}(\infty) \mid j \in J\}$ covers Y. This implies $\prod_{j\in J} X_j = \emptyset$ which, in view of **CC**, is impossible.

(2) $\Longrightarrow$ (1) is well known (see, e.g., [7]). □

6. Separation axioms for Lindelöf spaces

In **ZFC** Lindelöf T_3-spaces are paracompact and thus normal (see, e.g., [7]). This remains true in **ZF**. However, for some models of **ZF** we have more:

6.1 Theorem. *Equivalent are:*

(1) *every Lindelöf T_2-space is a T_3-space,*
(2) $\mathbf{CC}(\mathbb{R})$ *fails.*

Proof: (1) $\Longrightarrow$ (2). Let τ be the familiar topology of the reals. Consider the set $A = \mathbb{R}\backslash\{\frac{1}{n} \mid n \in \mathbb{N}^+\}$. Then $\tau \cup \{A\}$ is a subbase for a topology σ on $\mathbb{R}$. The space $X = (\mathbb{R}, \sigma)$ is T_2-space that fails to be a T_3-space. Thus (1) implies that X is not Lindelöf. Since X has a countable base, Theorem 1.4 implies that $\mathbf{CC}(\mathbb{R})$ fails.

(2) $\Longrightarrow$ (1). If $\mathbf{CC}(\mathbb{R})$ fails then Theorem 2.1 implies that Lindelöf = compact for T_1-spaces. Since compact T_2-spaces are T_3-spaces (see, e.g., [12]), (1) follows. □

7. Disconnected Lindelöf spaces

In **ZFC** zerodimensional Lindelöf spaces are strongly zerodimensional (see, e.g., [7]). This remains true in **ZF**. However, for some models of **ZF** we have more:

7.1 Theorem. *Equivalent are:*

1. *totally disconnected Lindelöf T_2-spaces are zerodimensional,*
2. $\mathbf{CC}(\mathbb{R})$ *fails.*

Proof: (1) $\Longrightarrow$ (2). Erdös has constructed (see [7, 6.2.19]) a totally disconnected, non zerodimensional T_2-space X with a countable base. Iff $\mathbf{CC}(\mathbb{R})$ holds then, by Theorem 1.4, X is Lindelöf, thus (1) fails.

(2) $\Longrightarrow$ (1). If $\mathbf{CC}(\mathbb{R})$ fails then, by Theorem 2.1, every totally disconnected Lindelöf T_2-space is compact, thus (see, e.g., [7]) zerodimensional. □

8. The Lindelöf concept

8.1 Definition (cf. [3]). A topological space X is called

- *s-Lindelöf* (= super Lindelöf) if for every extension Y of X each open cover of X in Y contains an at most countable subcover of X,
- *w-Lindelöf* (= weakly Lindelöf) if every open cover of X has an at most countable open refinement,
- *vw-Lindelöf* (= very weakly Lindelöf) if every open cover of X has an at most countable refinement.

In **ZF** the implications

$$s\text{-Lindelöf} \Rightarrow \text{Lindelöf} \Rightarrow \text{w-Lindelöf} \Rightarrow \text{vw-Lindelöf}$$

are proper. In **ZFC**, however, they are equivalences.

8.2 Proposition. *Equivalent are:*

1. *Lindelöf = s-Lindelöf for T_1-spaces,*
2. $\mathbf{CC}(\mathbb{R})$ *implies* $\mathbf{CC}$.

Proof: (1) $\Longrightarrow$ (2). If $\mathbf{CC}(\mathbb{R})$ holds, then $\mathbb{N}$ is Lindelöf. If (X_n) is a sequence of non-empty sets, consider the discrete space Y with underlying set the disjoint union of $\mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} X_n$ as an extension of $\mathbb{N}$. Then $\mathfrak{U} = \{\{n,x\} \mid n \in \mathbb{N}$ and $x \in X_n\}$ covers $\mathbb{N}$ and thus contains a countable cover of $\mathbb{N}$. This produces a choice-function for the sequence (X_n).

(2) $\Longrightarrow$ (1). If $\mathbf{CC}$ holds the familiar proof works. If $\mathbf{CC}(\mathbb{R})$ fails, then — by Theorem 2.1 — Lindelöf = compact for T_1-spaces. Thus (1) follows from the fact that the axiom of choice for finite families holds in $\mathbf{ZF}$. □

8.3 Proposition. *Equivalent are:*

(1) *Lindelöf = w-Lindelöf,*
(2) $\mathbf{CC}$.

Proof: (1) $\Longrightarrow$ (2). For every compact T_2-space X, the sum $\mathbb{N}+X$ is w-Lindelöf. Thus (2) follows from Theorem 4.1.

(2) $\Longrightarrow$ (1). Obvious. □

8.4 Remark. Related results have been obtained in [20].

8.5 Proposition. *Equivalent are:*

(1) *w-Lindelöf = vw-Lindelöf,*
(2) $\mathbf{CMC}$.

Proof: (1) $\Longrightarrow$ (2). Let (X_n) be a sequence of non-empty sets.

Assume, without loss of generality, that $X = \bigcup_n X_n \cup \mathbb{N} \cup \{\infty\}$ is a union of pairwise disjoint sets. Define

$$\tau = \{A \subset X \mid (\infty \in A \Rightarrow \bigcup_n X_n \subset A) \text{ and } (n \in A \Rightarrow X_n \backslash A \text{ finite})\}.$$

Then the space (X,τ) is vw-Lindelöf since the countable cover $\{\{n\} \mid n \in \mathbb{N}\} \cup \{X\backslash\mathbb{N}\}$ refines every open cover of (X,τ). Thus, by (1), (X,τ) is w-Lindelöf. Consequently, the open cover

$$\mathfrak{C} = \{X\backslash\mathbb{N}\} \cup \{(\{n\}\cup X_n)\backslash F \mid n \in \mathbb{N},\ F \text{ a finite non-empty subset of } X_n\}$$

of (X,τ) has an open refinement of the form $\{C_n \mid n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ define $n^* = \min\{m \in \mathbb{N} \mid n \in C_m\}$. Then $F_n = X_n\backslash C_{n^*}$ is a non-empty, finite subset of X_n. Thus (2) holds.

(2) $\Longrightarrow$ (1). Let X be vw-Lindelöf and let $\mathfrak{C}$ be an open cover of X. Then there exists a refinement $\{A_n \mid n \in \mathbb{N}\}$ of $\mathfrak{C}$. For each $n \in \mathbb{N}$ the set $X_n = \{C \in \mathfrak{C} \mid$

$A_n \subset C\}$ is not empty. Thus, by (2), there exists a sequence (F_n) of non-empty, finite subsets F_n of X_n. Thus $\mathfrak{C}$ is refined by the open cover $\{\cap F_n \mid n \in \mathbb{N}\}$. Consequently (1) holds. $\square$

8.6 Corollary ([3], see also [5]). *Equivalent are:*

(1) *Lindelöf = vw-Lindelöf,*
(2) **CC**.

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(*Received* June 18, 2001, *revised* December 11, 2001)