Tightness of compact spaces is preserved by the t-equivalence relation

Oleg Okunev

Abstract. We prove that if there is an open mapping from a subspace of $C_p(X)$ onto $C_p(Y)$, then Y is a countable union of images of closed subspaces of finite powers of X under finite-valued upper semicontinuous mappings. This allows, in particular, to prove that if X and Y are t-equivalent compact spaces, then X and Y have the same tightness, and that, assuming $2^{\mathfrak{t}} > \mathfrak{c}$, if X and Y are t-equivalent compact spaces and X is sequential, then Y is sequential.

Keywords: function spaces, topology of pointwise convergence, tightness

Classification: 54B10, 54D20, 54A25, 54D55

All spaces below are assumed to be Tychonoff (that is, completely regular Hausdorff). We study the spaces $C_p(X, Z)$ of all continuous functions on a space X with the values in a space Z equipped with the topology of pointwise convergence (see [Arh3] for a thorough presentation of the theory of spaces of functions equipped with this topology). The space $C_p(X, \mathbb{R})$ is denoted by $C_p(X)$, and $C_p^*(X)$ denotes the subspace of $C_p(X)$ consisting of all bounded functions; in all cases we denote by 0 the zero constant function on X. We say that Y is a *t-image* of X if $C_p(Y)$ is homeomorphic to a subspace (not necessarily linear) of $C_p(X)$. Every continuous image of a space is its t-image by virtue of the dual mapping between the function spaces (see [Arh3]). Two spaces X and Y are called *t-equivalent* if the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic, and *l-equivalent* if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Of course, if two spaces are t-equivalent, then each of them is a t-image of the other; simple examples show that the converse is not true. Note also that the spaces $C_p(X, [0, 1])$ and $C_p^*(X)$ contain homeomorphic copies of $C_p(X)$, and their homeomorphic copies are contained in $C_p(X)$. It follows that if one of the spaces $C_p(Y)$, $C_p^*(Y)$, $C_p(Y, [-1, 1])$, admits a homeomorphic embedding in $C_p(X)$, $C_p^*(X)$, or $C_p(X, [-1, 1])$, then Y is a t-image of X.

We denote by $t(X)$ and $l(X)$ the tightness and the Lindelöf number of a space X (see e.g. [Eng]); we put $l^*(X) = \sup\{\, l(X^n) : n \in \mathbb{N}\,\}$ and $t^*(X) = \{\, t(X^n) : n \in \mathbb{N}\,\}$. All cardinals are assumed to be infinite; ω is the set of all naturals, and $\mathbb{N} = \omega \setminus \{0\}$. The cardinal $\mathfrak{t}$ is the minimum cardinality of a tower of infinite subsets in ω (see [vDo]), and $\mathfrak{c} = 2^\omega$.

For a set-valued mapping $p\colon X \to Y$ and a set $A \subset X$, we define *the image of* A, $p(A)$ as the union $\bigcup\{\, p(x) : x \in A \,\}$. We say that a set-valued mapping $p\colon X \to Y$ is *onto* if $p(X) = Y$. A set-valued mapping $p\colon X \to Y$ is called *compact-valued* (*finite-valued*) if for every $x \in X$ the set $p(x)$ is compact (finite), and *upper semicontinuous* if for every closed set $F \subset Y$, the preimage $p^{-1}(F) = \{\, x \in X : p(x) \cap F \neq \emptyset \,\}$ is closed. We do not require $p(x) \neq \emptyset$ for every $x \in X$; this is slightly different from the common usage of the term, but is more convenient in the context of this article. Note that for every upper-semicontinuous mapping $p\colon X \to Y$ the set $p^{-1}(Y)$ of all points of X with nonempty images is closed in X, and every closed subspace of X is an image of X under a finite-valued upper semicontinuous mapping (the one identical on the subspace, and with empty images of the points of the complement), so "an image of X under an upper semicontinuous mapping" in this article is the same as "an image of a closed subspace of X under an upper semicontinuous mapping" in the traditional sense. It is easy to verify that a set-valued mapping from a space X is compact-valued upper semicontinuous if and only if it is the composition of the inverse of a perfect mapping (onto a closed subspace of X) and a continuous mapping; in particular, this implies the standard fact that we often use in this article: *Upper semicontinuous compact-valued mappings preserve compactness and do not raise the Lindelöf number*.

A set-valued mapping $p\colon X \to Y$ is called *upper semicontinuous at a point* $x_0 \in X$ if for every open neighborhood V of $p(x_0)$ in Y, there is a neighborhood U of x_0 in X such that $p(U) \subset V$. It is easy to verify that p is upper semicontinuous if and only if it is upper semicontinuous at every point of X.

In [Ok1] the author proved that if there is an open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$, then Y is a countable union of continuous images of closed subspaces of products of finite powers of X and a compact space — in other words, Y is a countable union of images of finite powers of X under compact-valued upper semicontinuous mappings. In this article we refine this result by showing that Y is a countable union of images of finite powers of X under finite-valued upper semicontinuous mappings; this allows to prove that if X is compact, then the tightness of every compact subspace of Y does not exceed the tightness of X. In particular, the tightness in compact spaces is not increased by t-images, which gives a positive answer to Problem 32 (1057) in [Arh2] (the question first appeared in [Tk1] and was repeated in [Tk2].) We also prove that if X and Y are compact, X is sequential, and Y is a t-image of X, then Y is a countable union of sequential compact subspaces, which consistently implies that Y is sequential. Note that neither tightness, nor sequentiality are preserved by the relation of t-equivalence without the assumption of compactness ([Ok2]).

1. Statements

1.1 Theorem. *Let X and Y be spaces, and assume that there is a continuous* *open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$. Then there is a sequence of finite-valued upper semicontinuous mappings $T_k\colon X^k \to Y$, $k \in \mathbb{N}$, such that $Y = \bigcup\{\, T_k(X^k) : k \in \mathbb{N}\,\}$.*

1.2 Proposition. *Let τ be a cardinal, Z a space, K a compact space, and $p\colon Z \to K$ a compact-valued upper semicontinuous mapping such that $p(Z) = K$. If $l(Z)t(Z) \le \tau$ and $t(p(z)) \le \tau$ for every $z \in Z$, then $t(K) \le \tau$.*

1.3 Theorem. *If there is a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$ (in particular, if Y is a t-image of X), then for every compact subspace K of Y, $t(K) \le t^*(X)l^*(X)$. In particular, if X is compact, then $t(K) \le t(X)$.*

1.4 Corollary. *Let Y be a k-space. If Y is a t-image of a compact space X, then $t(Y) \le t(X)$.*

Indeed, if every compact subspace of a k-space Y has the tightness $\le \tau$, then $t(Y) \le \tau$.

1.5 Corollary. *If X and Y are t-equivalent compact spaces, then $t(X) = t(Y)$.*

The last statement is an answer to Problem 32(1057) in [Arh2].

Remark. The preservation of the tightness of compact spaces by the relation of l-equivalence was proved by Tkachuk in [Tk1].

1.6 Proposition. *Let Z and K be compact spaces, and $p\colon Z \to K$ a finite-valued upper semicontinuous mapping such that $p(Z) = K$. If Z is sequential, then K is sequential.*

1.7 Corollary. *If X and Y are compact spaces, X is sequential, and there is a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$ (in particular, if Y is a t-image of X), then Y is a countable union of sequential compact subspaces. In particular, every countably compact subspace of Y is compact, and if $2^{\mathfrak{t}} > \mathfrak{c}$, then Y is sequential.*

2. The proofs

Proof of Theorem 1.1: Let Φ_0 be a continuous open mapping from a subspace C_0 of $C_p(X)$ onto $C_p(Y)$. Since $C_p(X)$ and $C_p(Y)$ are homogeneous, we may assume without loss of generality that $0 \in C_0$ and $\Phi_0(0) = 0$.

Denote $I = [-1, 1]$. The space $C_p(Y, I)$ is a subspace of $C_p(Y)$; put $C = \Phi_0^{-1}(C_p(Y, I))$ and let $\Phi\colon C \to C_p(Y, I)$ be the restriction of Φ_0. Then Φ is continuous, open, onto $C_p(Y, I)$, and $\Phi(0) = 0$.

Let βY be the Stone-Čech compactification of Y. For every $g \in C_p(Y, I)$ we denote by $\tilde{g}$ the continuous extension of g over βY.

For every $k \in \mathbb{N}$, $\bar{x} = (x_1, \dots, x_k) \in X^k$, $\bar{y} = (y_1, \dots, y_k) \in (\beta Y)^k$ and $\varepsilon > 0$ denote

$$O_X(\bar{x}, \varepsilon) = \{ f \in C : |f(x_1)| < \varepsilon, \dots, |f(x_k)| < \varepsilon \},$$

$$O_Y(\bar{y}, \varepsilon) = \{ g \in C_p(Y, I) : |\tilde{g}(y_1)| < \varepsilon, \dots, |\tilde{g}(y_k)| < \varepsilon \},$$

and

$$\bar{O}_Y(\bar{y}, \varepsilon) = \{ g \in C_p(Y, I) : |\tilde{g}(y_1)| \leq \varepsilon, \dots, |\tilde{g}(y_k)| \leq \varepsilon \}.$$

The sets $O_X(\bar{x}, 1/k)$, $k \in \mathbb{N}$, $\bar{x} \in X^k$ form an open base at 0 of the space C. Similarly, the sets $O_Y(\bar{y}, 1/k)$, $k \in \mathbb{N}$, $\bar{y} \in Y^k$ form an open base at 0 of the space $C_p(Y, I)$ (see e.g. [Arh3]).

For every $k \in \mathbb{N}$ put

$$P_k = \{ y \in \beta Y : \text{there is a point } \bar{x} \in X^k \text{ such that } \Phi(O_X(\bar{x}, 1/k)) \subset \bar{O}_Y(y, 1/2) \}.$$

From the continuity of Φ it follows that $Y \subset \bigcup \{ P_k : k \in \mathbb{N} \}$.

For every $\bar{x} \in X^k$ put

$$T_k(\bar{x}) = \{ y \in \beta Y : \Phi(O_X(\bar{x}, 1/k)) \subset \bar{O}_Y(y, 1/2) \}.$$

Obviously, $T_k(X^k) = P_k$, so $Y \subset \bigcup \{ T_k(X^k) : k \in \mathbb{N} \}$.

Claim 1. *For every $\bar{x} \in X^k$, $T_k(\bar{x})$ is a finite subset of Y.*

Since Φ is open, the set $\Phi(O_X(\bar{x}, 1/k))$ is a neighborhood of 0 in $C_p(Y, I)$. Hence there are points $y_1, \dots, y_m \in Y$ and $\delta > 0$ such that $O_Y(y_1, \dots, y_m, \delta) \subset \Phi(O_X(\bar{x}, 1/k))$. Then $T_k(\bar{x}) \subset \{y_1, \dots, y_m\}$. Indeed, if y is a point of βY distinct from $y_1, \dots, y_m$, then there is a function $g \in C_p(Y, I)$ such that $g(y_i) = 0$, $i = 1, \dots, m$, and $\tilde{g}(y) = 1$. Then $g \in O_Y(y_1, \dots, y_m, \delta)$, and therefore $g \in \Phi(O_X(\bar{x}, 1/k))$. Then there is an $f \in O_X(\bar{x}, 1/k)$ such that $\Phi(f) = g$; then $g = \Phi(f) \notin O_Y(y, 1/2)$, so $y \notin T_k(\bar{x})$.

Thus, we have defined finite-valued mappings $T_k \colon X^k \to Y$ so that $\bigcup \{ T_k(X^k) : k \in \mathbb{N} \} = Y$.

Claim 2. *For every $k \in \mathbb{N}$, the mapping T_k is upper semicontinuous.*

Obviously, it is sufficient to verify that T_k is upper semicontinuous as a mapping to βY.

Let $\bar{x}_0$ be a point of X^k, and let V be an open neighborhood of $T_k(\bar{x}_0)$ in βY. For every $y \in \beta Y \setminus V$ choose a function $f_y \in O(\bar{x}_0, 1/k)$ so that $\tilde{g}_y(y) > 1/2$ where

$g_y = \Phi(f_y)$, and put $F_y = \tilde{g}_y^{-1}([-1/2, 1/2])$. Then F_y is closed in βY and $y \notin F_y$, so

$$\bigcap\{\, F_y : y \in \beta Y \setminus V \,\} \subset V.$$

By the compactness of βY, there is a finite set $y_1, \ldots, y_m$ in $\beta Y \setminus V$ such that

$$F_{y_1} \cap \cdots \cap F_{y_m} \subset V.$$

Put

$$U = \{\, (x_1, \ldots, x_k) \in X^k : |f_{y_i}(x_j)| < 1/k, \quad i \leq m, \quad j \leq k \,\}.$$

Then U is a neighborhood of $\bar{x}_0$ in X^k, and $T_k(U) \subset V$. Indeed, if $\bar{x} \in U$ and $y \notin V$, then $y \notin F_{y_i}$ for some $i \leq m$, so $f_{y_i} \in O(\bar{x}, 1/k)$ and $g_{y_i} = \Phi(f_{y_i}) \notin \bar{O}_Y(y, 1/2)$, so $y \notin T_k(\bar{x})$.

This concludes the proof of Theorem 1.1. $\square$

Remark. The above proof may be easily (almost literally) modified to prove the following:

2.1 Theorem. *Let X and Y be spaces such that* $\operatorname{ind} Y = 0$, *and assume that there is a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y, 2)$. Then there is a sequence of finite-valued upper semicontinuous mappings $T_k\colon X^k \to Y$, $k \in \mathbb{N}$, such that $Y = \bigcup\{\, T_k(X^k) : k \in \mathbb{N} \,\}$.*

Proof of Proposition 1.2: Let

$$\Gamma = \{\, (z, y) \in Z \times K : y \in p(z) \,\}.$$

Then Γ is closed in $Z \times K$. Indeed, if $(z_0, y_0) \notin \Gamma$, then y_0 and $p(z_0)$ have disjoint neighborhoods V and W in K; put $U = \{\, z \in Z : p(z) \subset W \,\}$. Then $U \times V$ is a neighborhood of (z_0, y_0) disjoint from Γ.

Let $\pi_Z\colon Z \times K \to Z$, $\pi_K\colon Z \times K \to K$ be the projections. Since K is compact, the projection π_Z is perfect, so its restriction $h = \pi_Z | \Gamma$ is perfect. In particular, this implies $l(\Gamma) \leq \tau$. Obviously, for every $z \in Z$, π_K maps $h^{-1}(z)$ homeomorphically onto $p(z)$, so $h\colon \Gamma \to Z$ is a closed mapping whose all fibers have the tightness $\leq \tau$. By Theorem 4.5 in [Arh1], $t(\Gamma) \leq \tau$. The statement of the proposition now follows from the next well-known fact (apparently, first discovered by Tkachenko; see also Theorem 1 in [Ra]):

2.2 Proposition. *Let K be a compact space, and suppose there is a continuous mapping p from a space Γ onto K. Then $t(K) \leq l(\Gamma)t(\Gamma)$.*

$\square$

Proof of Theorem 1.3: Let Φ be a continuous open mapping of a subspace of $C_p(X)$ onto $C_p(Y)$, and let $r\colon C_p(Y) \to C_p(K)$ be the restriction mapping; since

K is compact, r is open and onto $C_p(K)$. Hence, the composition $r \circ \Phi$ is an open mapping of a subspace of $C_p(X)$ onto $C_p(K)$.

Let $T_k \colon X^k \to K$, $k \in \mathbb{N}$, be as in Theorem 1.1. Put $M = \bigoplus_{k \in \mathbb{N}} X^k$, and define a mapping $T \colon M \to K$ by the rule: $T(\bar{x}) = T_k(\bar{x})$ if $\bar{x} \in X^k$. Obviously, T is finite-valued and upper semicontinuous. By Proposition 1.2, $t(K) \le l(M)t(M) = l^*(X)t^*(X)$.

If X is compact, then $l^*(X)t^*(X) = t(X)$ [Mal], so $t(K) \le t(X)$. □

Proof of Proposition 1.6: Let Γ, π_Z, π_K and $h = \pi_Z | \Gamma$ be as in the proof of Proposition 1.2. Since Z is compact, π_K is perfect, and its restriction h to the closed set Γ is closed. Thus, it is sufficient to verify that Γ is sequential.

Let A be a non-closed set in Γ; we will prove that A is not sequentially closed. Let $a_0 \in \Gamma \setminus A$ be a limit point of A and $b_0 = h(a_0)$. Fix a closed neighborhood W of a_0 in Γ so that $\{a_0\} = W \cap h^{-1}(b_0)$, and put $A_0 = W \cap A$. Then $h_0 = h|W$ is closed and has finite fibers, and a_0 is a limit point of A_0. The point b_0 is a limit point of $B = h(A_0)$ and is not in B, so B is not closed in Z. Since Z is sequential, there is a sequence $\{ z_n : n \in \omega \}$ in B that converges to a point $b_1 \in Z \setminus B$. The set $M = h_0^{-1}(\{ z_n : n \in \omega \}) \cup h_0^{-1}(b_1)$ is a countable compact subspace of W, and $h(M \cap A) = \{ z_n : n \in \omega \}$ is not compact. It follows that $M \cap A$ is not compact, and hence A is not sequentially closed. □

Proof of Corollary 1.7: The first statement follows immediately from Theorem 1.1 and Proposition 1.6. Let $Y = \bigcup \{ Y_n : n \in \mathbb{N} \}$ where each Y_n is compact and sequential. If A is a countably compact subspace of Y, then for each $n \in \mathbb{N}$, $A \cap Y_n$ is countably compact, and therefore is closed in Y_n. It follows that A is σ-compact, so it is compact. This proves the second statement. The last statement follows from the fact that $2^{\mathfrak{t}} > \mathfrak{c}$ implies that a compact space is sequential if and only if every its countably compact subspace is closed (Corollary 6.4 in [vDo]). □

Remark. The sequentiality of a compact space that is a countable union of sequential compact subspaces was proved under the assumption of Martin's Axiom or $\mathfrak{c} < 2^{\omega_1}$ in [Ra]. Both assumptions are stronger that $2^{\mathfrak{t}} > \mathfrak{c}$.

3. Some open problems

It is shown in [Ok2] that there are l-equivalent spaces X and Y such that X is bisequential and the tightness of Y is uncountable. The example, however, relies heavily on the non-normality of the space X, so the following questions appear very interesting.

3.1 *Problem.* Let X and Y be t-equivalent normal spaces. Is it true that $t(X) = t(Y)$?

3.2 *Problem.* Let X and Y be l-equivalent normal spaces. Is it true that $t(X) = t(Y)$?

From Theorem 2.2 follows that if X is σ-compact and all finite powers of X have tightness $\leq \tau$, then every compact subspace in Y has the tightness $\leq \tau$. The following version of Problem 1.1 remains open; it also appears more natural, because compactness is not preserved by t-equivalence [GH], while σ-compactness is [Ok1].

3.3 *Problem.* Let X and Y be t-equivalent σ-compact spaces. Is it true that $t(X) = t(Y)$?

3.4 *Problem.* Let X and Y be l-equivalent σ-compact spaces. Is it true that $t(X) = t(Y)$?

Note that the tightness is not preserved by t-images in the class of σ-compact spaces. Indeed, there are σ-compact spaces of uncountable tightness in which all compact subspaces are Fréchet — for example, consider the subspace X of I^{ω_1} consisting of the σ-product with the center at 0 and the point whose all coordinates are equal to 1. This space is obviously a continuous image (and hence a t-image) of a countable direct sum of Eberlein compact spaces. Furthermore, using the construction as in Theorem III.1.11 in [Arh3] one can show that X is a t-image of an Eberlein (hence, Fréchet) compact space.

A positive answer to the next question, suggested by Reznichenko, would be a big improvement of Corollary 1.5.

3.5 *Problem.* Let X be a compact space. Is it true that $t(K) \leq t(X)$ for every compact subspace K of $C_p(C_p(X))$?

The proof of the preservation of the tightness of compact spaces by the relation of l-equivalence given in [Tk1] in fact shows that if X is compact, then $t(K) \leq t(X)$ for every compact set K in the subspace $L_p(X)$ of $C_p(C_p(X))$ consisting of all linear continuous functions on $C_p(X)$.

Corollary 1.7 leaves open the next question:

3.6 *Problem.* Let X and Y be t-equivalent (or l-equivalent) compact spaces. Is it true in ZFC that the sequentiality of X implies the sequentiality of Y?

Clearly, the answer is positive if it is true in ZFC that every compact space, which is a union of a countable family of sequential closed subspaces, is sequential.

The following interesting question was suggested by the referee:

3.7 *Problem.* Let X and Y be t-equivalent (or l-equivalent) compact spaces. Is it true that the orders of sequentiality of X and Y coincide?

In particular, it is unknown whether the Fréchet property is preserved by l-equivalence within the class of compact spaces (Problem 33 (1058) in [Arh2]).

References

[Arh1] Arhangel'skii A.V., *The spectrum of frequencies of a topological space and the product operation*, Trudy Moskov. Mat. Obshch. **40** (1979), 171–206 (Russian); English translation: Trans. Moscow Math. Soc. **40** (1981), no. 2, 169–199.

[Arh2] Arhangel'skii A.V., *Problems in C_p-theory*, Open Problems in Topology (J. van Mill and G.M. Reed, eds.), North-Holland, 1990, pp. 603–615.

[Arh3] Arhangel'skii A.V., *Topological Function Spaces*, Kluwer Acad. Publ., Dordrecht, 1992.

[vDo] van Douwen E.K., *The Integers and Topology*, Handbook of Set-Theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 111–167.

[Eng] Engelking R., *General Topology*, PWN, Warszawa, 1977.

[GH] Gul'ko S.P., Khmyleva T.E., *Compactness is not preserved by the relation of t-equivalence*, Matematicheskie Zametki **39** (1986), no. 6, 895–903 (Russian); English translation: Math. Notes **39** (1986), no. 5–6, 484–488.

[Mal] Malykhin V.I., *On tightness and the Suslin number in* $\exp X$ *and in a product of spaces*, Dokl. Akad. Nauk SSSR **203** (1972), 1001–1003 (Russian); English translation: Soviet Math. Dokl. **13** (1972), 496–499.

[Ok1] Okunev O., *Weak topology of a dual space and a t-equivalence relation*, Matematicheskie Zametki **46** (1989), no. 1, 53–59 (Russian); English translation: Math. Notes **46** (1989), no. 1–2, 534–536.

[Ok2] Okunev O., *A method for constructing examples of M-equivalent spaces*, Topology Appl. **36** (1990), 157–171; *Correction*, Topology Appl. **49** (1993), 191–192.

[Ra] Ranchin D., *Tightness, sequentiality and closed coverings*, Dokl. AN SSSR **32** (1977), 1015–1018 (Russian); English translation: Soviet Math. Dokl. **18** (1977), no. 1, 196–199.

[Tk1] Tkachuk V.V., *Duality with respect to the functor C_p and cardinal invariants of the type of the Souslin number*, Matematicheskie Zametki **37** (1985), no. 3, 441–445 (Russian); English translation: Math. Notes, **37** (1985), no. 3, 247–252.

[Tk2] Tkachuk V.V., *Some non-multiplicative properties are l-invariant*, Comment. Math. Univ. Carolinae **38** (1997), no. 1, 169–175.

Facultad de Ciencias, Universidad Nacional Autonóma de México, Ciudad Universitaria, México D.F., 04510, México

E-mail: oleg@servidor.unam.mx

(*Received* September 17, 2001)