Conformal deformations of the Riemannian metrics and homogeneous Riemannian spaces

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Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. In this paper we investigate one-dimensional sectional curvatures of Riemannian manifolds, conformal deformations of the Riemannian metrics and the structure of locally conformally homogeneous Riemannian manifolds. We prove that the nonnegativity of the one-dimensional sectional curvature of a homogeneous Riemannian space attracts nonnegativity of the Ricci curvature and we show that the inverse is incorrect with the help of the theorems O. Kowalski-S. Nikčević [K-N], D. Alekseevsky-B. Kimelfeld [A-K]. The criterion for existence of the left-invariant Riemannian metrics of positive one-dimensional sectional curvature on Lie groups is presented. Classification of the conformally deformed homogeneous Riemannian metrics of positive sectional curvature on homogeneous spaces is obtained. The notion of locally conformally homogeneous Riemannian spaces is introduced. It is proved that each such space is either conformally flat or conformally equivalent to a locally homogeneous Riemannian space.

Keywords: conformal deformations, Riemannian metrics, homogeneous Riemannian spaces

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1. Preliminaries

Let ∇ be the Levi-Chivita connection of the Riemannian metric $ds^2 = g_{ij}dx^i dx^j$ on a manifold M^n, R_{ijks} is the curvature tensor, R_{ij} is the Ricci tensor, $\mathrm{Ric}(\xi) = R_{ij}\xi^i\xi^j$ is the Ricci curvature in direction of a unit vector ξ, R is the scalar curvature of the metric ds^2.

At research of Riemannian manifolds, an important role is played by a tensor which is defined with the help of the formula

$$A_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{Rg_{ij}}{2(n-1)}\right), \tag{1}$$

where R_{ij} denotes the Ricci tensor and R the scalar curvature.

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It represents an integer part from division of the Riemannian curvature tensor by the metric tensor with respect to the Kulkarni-Nomizu product ([B]). Using the tensor A_{ij}, the curvature tensor can be presented in the form

$$R_{lkij} = W_{lkij} + g_{lj}A_{ki} + g_{ki}A_{lj} - g_{li}A_{kj} - g_{kj}A_{li},$$

where W_{lkij} is the conformal Weyl tensor.

Definition 1.1. *The one-dimensional sectional curvature in the tangent direction ξ is defined as the value*

$$A(\xi) = \frac{A_{ij}\xi^i\xi^j}{g_{ij}\xi^i\xi^j},$$

where ξ^i is an arbitrary tangent vector generating the direction ξ.

The sectional curvature along a tangent 2-plane can be expressed as

$$K(\xi\wedge\eta) = \frac{R_{ijkl}\xi^i\eta^j\xi^k\eta^l}{g_{ik}\xi^i\xi^k g_{jl}\eta^j\eta^l} = \frac{W_{ijkl}\xi^i\eta^j\xi^k\eta^l}{g_{ik}\xi^i\xi^k g_{jl}\eta^j\eta^l} + \frac{A_{ij}\xi^i\xi^j}{g_{ik}\xi^i\xi^k} + \frac{A_{kl}\eta^k\eta^l}{g_{jl}\eta^j\eta^l},$$

where ξ, η form an orthonormal basis of the 2-plane. In particular, for the conformally flat metric, or for a three-dimensional Riemannian manifold, this formula has a more simple form

$$K(\xi\wedge\eta) = \frac{A_{ij}\xi^i\xi^j}{g_{ik}\xi^i\xi^k} + \frac{A_{kl}\eta^k\eta^l}{g_{jl}\eta^j\eta^l}.$$

In these notations we have the following result:

Theorem 1.1. *Let (M^n, ds^2) be a Riemannian manifold. Then the following statements are true:*

(i) *if the one-dimensional sectional curvature $A(\xi)$ is nonnegative everywhere on (M^n, ds^2), then the Ricci curvature is nonnegative everywhere on (M^n, ds^2). Moreover, if at some point $p \in M^n$ there is a vector $\eta \in T_pM^n$ such that $\mathrm{Ric}(\eta) = \sum R_{ij}\eta^i\eta^j = 0$, then the Ricci curvature at this point is equal to zero;*

(ii) *there are Riemannian manifolds of nonnegative Ricci curvature and sign-changing one-dimensional sectional curvature.*

Proof: At an arbitrary point of M^n we shall consider the orthonormal basis $\{\xi_1,\dots,\xi_n\}$ for which the Ricci quadratic form is diagonalized. Let $r_1, r_2, \dots, r_n$ be the principal Ricci curvatures; then the condition of nonnegativity of the one-dimensional sectional curvature is equivalent to the system of inequalities:

$$\begin{cases} r_1 - \frac{\sum r_i}{2(n-1)} \geq 0, \\ \dots \\ r_n - \frac{\sum r_i}{2(n-1)} \geq 0. \end{cases}$$

From here it follows that

$$\sum_k \left(r_k - \frac{\sum r_i}{2(n-1)} \right) = \frac{n-2}{2(n-1)} \sum r_i \geq 0,$$

i.e. $\sum r_i \geq 0$, and it signifies that $r_k \geq \frac{\sum r_i}{2(n-1)} \geq 0$. Thus, we see that the Ricci curvature is nonnegative everywhere on (M^n, ds^2).

Let us assume that there are a point $p \in M^n$ and vector $\eta \in T_pM^n$ such that $\mathrm{Ric}(\eta) = 0$. Then $A(\eta) = \frac{1}{n-2}\left(-\frac{R}{2(n-1)}\right) \geq 0$. Hence, the scalar curvature $R = \sum r_i \leq 0$, and as $r_i \geq 0$, $i = 1, \dots, n$, we see that $r_1 = r_2 = \ldots = r_n = 0$.

For the proof of statement (ii) of the theorem, let us consider the case when (M^n, ds^2) is a direct Riemannian product of compact Einstein manifolds: $(M^n, ds^2) = (M_1, ds_1^2) \times \ldots \times (M_k, ds_k^2)$ with Einstein constants $r_1 > 0, \dots,$ $r_k > 0$. Let us consider the principal values of the one-dimensional sectional curvature at an arbitrary point of the manifold: $\frac{1}{n-2}\left(r_i - \frac{\sum r_i}{2(n-1)}\right)$, $i = 1, \dots, n$. If we strongly contract the metric of the factor (M_j, ds_j^2) by a homothety, leaving the metrics on other factors without change, then we see that the one-dimensional sectional curvature is sign-changing and the Ricci curvature is positive. $\square$

Remark 1.1. We note that if for the one-dimensional sectional curvature the inequality

$$A_{ij}\xi^i\xi^j \geq \frac{1}{2}k_0 g_{ij}\xi^i\xi^j \quad \forall \xi \in T_x(M),$$

with a constant k_0 is fulfilled, then for the Ricci curvature the inequality

$$R_{ij}\xi^i\xi^j \geq (n-1)k_0 g_{ij}\xi^i\xi^j \quad \forall \xi \in T_x(M)$$

holds.

Remark 1.2. It is not difficult to see that the condition of constancy of the Ricci curvature (which means $r_1 = r_2 = \cdots = r_n$), or the Einstein condition respectively, implies the constancy of the one-dimensional sectional curvature, i.e.

$$r_1 - \frac{\sum r_i}{2(n-1)} = \cdots = r_n - \frac{\sum r_i}{2(n-1)}$$

holds on (M^n, ds^2).

Let us consider a conformal deformation $d\overline{s}^2 = e^{2\sigma(x)} g_{ij} dx^i dx^j$ of the metric ds^2 on a manifold M^n. Then for such deformation the Weyl tensor is invariant, i.e.

$$\overline{W}_{ijkl} = e^{2\sigma(x)} W_{ijkl}$$

holds on (M^n, ds^2). The tensor A_{ij} will be transformed under the formula

$$\overline{A}_{ij} = A_{ij} - \sigma_{,ij} + \sigma_{,i}\sigma_{,j} - \frac{1}{2}\sigma_{,k}\sigma^k g_{ij} = A_{ij} - B_{ij},$$

where $B_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j} + \frac{1}{2}\sigma_{,k}\sigma^k g_{ij}$ and $\sigma_{,ij}$, $\sigma_{,i}$ are covariant derivatives of the function σ with respect to the initial metric, and the Riemannian curvature of a section under the formula

$$\overline{K}(\xi \wedge \eta) = e^{-2\sigma(x)} \left[K(\xi \wedge \eta) - \frac{B_{ij}\xi^i\xi^j}{g_{ik}\xi^i\xi^k} - \frac{B_{kl}\eta^k\eta^l}{g_{jl}\eta^j\eta^l} \right],$$

where ξ^i, η^j are mutually orthogonal unit vectors.

2. One-dimensional sectional curvature of homogeneous Riemannian manifolds

Everywhere in this paragraph we suppose that $M = G/H$ is a homogeneous space, G is a connected Lie group acting effectively on $M = G/H$ by diffeomorphisms $r(y) : xH \to yxH$, H is a compact connected subgroup of G, g and h are Lie algebras of groups G and H correspondingly, $[\cdot,\cdot]$ is the Lie bracket of algebra g. Let $\mathrm{ad}_\xi : \eta \to [\xi, \eta]$ be an inner automorphism of the algebra g, and let $B(\xi, \eta) = -\operatorname{tr} \mathrm{ad}_\xi \circ \mathrm{ad}_\eta$ be the minus Killing form of g. Under the assumption of compactness and semisimplicity of G the form $B(\xi, \eta)$ is positively defined, and the G-homogeneous Riemannian metric ds_B^2 on the homogeneous space G/H, obtained from $B(\xi, \eta)$ under the natural projection $\pi : G \to G/H$, is called standard. Moreover, if we consider the p-orthogonal complement to h in g with respect to B, then one can identify G-invariant Riemannian metrics on G/H and $\mathrm{Ad}(H)$-invariant scalar products on p. Thus, the sectional curvature, the Ricci curvature and the scalar curvature are easily calculated with the help of the $\mathrm{Ad}(H)$-invariant scalar product on p and the Lie bracket $[\cdot,\cdot]$ of the algebra g ([B]). Hence, for the one-dimensional sectional curvature of G/H, one can obtain an analogous formula, because $A(\xi) = \frac{1}{n-2}\left(\mathrm{Ric}(\xi) - \frac{R}{2(n-1)}\right)$.

Using these notations, we have

Theorem 2.1. *Let $(G/H, ds^2)$ be a homogeneous Riemannian manifold and let o denote the origin of G/H. Then the following statements are true:*

(i) *if the one-dimensional sectional curvature $A(\xi)$ is nonnegative for any vector $\xi \in T_oG/H$, then the Ricci curvature $\mathrm{Ric}(\xi)$ is nonnegative for any vector $\xi \in T_oG/H$. Moreover, if there is a vector $\eta \in T_oG/H$ such that $\mathrm{Ric}(\eta) = \sum R_{ij}\eta^i\eta^j = 0$, then $(G/H, ds^2)$ is isometric to the direct Riemannian product of a flat torus and Euclidean space;*

(ii) *there are homogeneous Riemannian manifolds of nonnegative Ricci curvature and sign-changing one-dimensional sectional curvature.*

Proof: The first part of the statement (i) follows from the first part of the proof of Theorem 1.1. Let us assume that there is a vector $\eta \in T_{\overline{e}}G/H$ such that $\mathrm{Ric}(\eta) = 0$. Then from Theorem 1.1, we have $R_{ij} = 0$ at o, and from the homogeneity it follows that $(G/H, ds^2)$ is Ricci flat. According to the theorem of D. Alekseevsky-B. Kimelfeld [A-K] it is flat, i.e. locally isometric to the Riemannian product of a flat torus and Euclidean space.

To prove the last statement, we consider, for example, a three-dimensional unimodular Lie group G with a left-invariant Riemannian metric ds^2. Let r_1, r_2, r_3 be the principal Ricci curvatures of (G, ds^2); then the conditions of nonnegativity of the Ricci curvature and one-dimensional sectional curvature are equivalent to the systems of inequalities:

$$\begin{cases} r_1 \geq 0 \\ r_2 \geq 0 \\ r_3 \geq 0 \end{cases} \quad \text{for the Ricci curvature,}$$

and

$$\begin{cases} r_1 - \frac{\sum r_i}{4} \geq 0 \\ r_2 - \frac{\sum r_i}{4} \geq 0 \\ r_3 - \frac{\sum r_i}{4} \geq 0 \end{cases} \quad \text{for the one-dimensional sectional curvature.}$$

Obviously, the Ricci curvature is nonnegative and the one-dimensional sectional curvature is sign-changing if and only if the point (r_1, r_2, r_3) lies outside of a three-faced angle bounded by the planes $\alpha : 3r_1 - r_2 - r_3 = 0$, $\beta : -r_1 + 3r_2 - r_3 = 0$, $\gamma : -r_1 - r_2 + 3r_3 = 0$, remaining in the domain of nonnegativity of the Ricci curvature. To complete the proof we apply the theorem of O. Kowalskii-S. Nikčević [K-N]: *Let r_1, r_2, r_3 be real numbers. Then a three-dimensional unimodular Lie group with a left-invariant Riemannian metric and with the principal Ricci curvatures r_1, r_2, r_3 exists if and only if $r_1 r_2 r_3 > 0$ or if at least two of r_i, $i = 1, 2, 3$, are zero.* □

Using Theorem 2.1 and V. Berestovski's theorem [Berest], we obtain the following result:

Theorem 2.2. *Let $(G/H, ds^2)$ be a homogeneous Riemannian manifold of nonnegative one-dimensional sectional curvature which is not isometric to the direct Riemannian product of a flat torus and Euclidean space. Then the following statements are true:*

(i) *the Lie group G is compact and the Levi subgroup LG of G (i.e. maximal connected semisimple subgroup of G) acts transitively on M;*
(ii) *the fundamental group $\pi_1(M)$ is finite.*

Remark 2.1. It is not difficult to construct homogeneous Riemannian manifolds of arbitrary dimension which satisfy to the condition (ii) of Theorem 2.1.

The criterion for existence of left-invariant Riemannian metrics of positive Ricci curvature on Lie groups is well known (see J. Milnor [M, Theorem 2.2]):

Criterion. *A connected Lie group G admits a left-invariant Riemannian metric of positive Ricci curvature if and only if G is compact and its fundamental group $\pi_1(G)$ is finite. In such a case, G also admits a biinvariant metric with the above property.*

In the case of one-dimensional sectional curvature, we have the following result:

Theorem 2.3. *A connected Lie group G admits a left-invariant Riemannian metric of positive one-dimensional sectional curvature if and only if G is compact and its fundamental group $\pi_1(G)$ is finite. In such a case, G also admits a standard metric with the above property.*

Proof: Let us assume that $A(\xi) > 0 \ \ \forall \xi \in T_eG$, then $\mathrm{Ric}(\xi) \geq 0 \ \forall \xi \in T_eG$ according to Theorem 1.1. If $\mathrm{Ric}(\eta) = 0$ for some $\eta \in T_eG$, then (G, ds^2) is flat, therefore one-dimensional sectional curvature is equal to zero and we obtain a contradiction with above assumption. Hence, $\mathrm{Ric}(\xi) > 0 \ \forall \xi \in T_eG$, and from Criterion we see that G is compact, $\pi_1(G)$ is finite.

Conversely, let G be compact, and $\pi_1(G)$ be finite. Since G is connected, the centre of G is trivial and G is semisimple compact connected Lie group. Thus, $G = G_1 \times \ldots \times G_s$is the direct product of compact simple connected Lie groups $G_1, \ldots, G_s$ with Lie algebras $g_1, \ldots, g_s$. Obviously, $B_g = B_{g_1} + B_{g_2} + \ldots + B_{g_s}$, where $g = g_1 \oplus g_2 \oplus \ldots \oplus g_s$. Moreover, $\left(G_i, B_{g_i}\right)$ are standard homogeneous Einstein manifolds with Einstein constants $r_i = \frac{1}{4}$, $i = 1, \ldots, s$ (see, for example, [B]). From here it follows that the principal values of one-dimensional sectional curvature have the form

$$\frac{1}{n-2}\left(r_i - \frac{\sum r_i}{2(n-1)}\right) = \frac{1}{4(n-2)}\left(1 - \frac{n}{2(n-1)}\right) = \frac{1}{8(n-1)}, i = 1, \ldots, s.$$

This completes the proof of Theorem 2.3. □

Remark 2.2. We note that there are biinvariant Riemannian metrics on a connected compact semisimple Lie group $G = G_1 \times \ldots \times G_s$ with positive Ricci curvature and sign-changing one-dimensional sectional curvature. Really, let $(\cdot,\cdot)_{g\times g}$ be a biinvariant Riemannian metric on G, then $(\cdot,\cdot) = \lambda_1 B_{g_1} + \lambda_2 B_{g_2} + \ldots + \lambda_s B_{g_s}$, where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are some positive constants. Further, we consider the principal values of one-dimensional sectional curvature: $\frac{1}{n-2}\left(r_i - \frac{\sum r_i}{2(n-1)}\right)$, $i = 1, \ldots, s$. Obviously, if λ_j tends to zero for some $j \in \{1, \ldots, s\}$ and other constants λ_i, $i \in \{1, \ldots, s\} \setminus \{j\}$, are without change, then one-dimensional sectional curvature is sign-changing and the Ricci curvature is positive.

3. Conformal deformations of the Riemannian metrics with sections of zero curvature on a compact manifold

The following theorem was announced in [RS1].

Theorem 3.1. *Let M^n be a compact manifold with Riemannian metric $ds^2 = g_{ij}dx^i dx^j$. Suppose that there exists a two-dimensional direction of zero sectional curvature at each point $x \in M^n$. Then for any conformal deformation $d\overline{s}^2 = e^{2\sigma(x)} g_{ij}dx^i dx^j$ of the metric ds^2 there are a point $x_0 \in M^n$ and a two-dimensional direction of nonpositive sectional curvature at this point and also a point $x_1 \in M^n$ and a two-dimensional direction of nonnegative sectional curvature at this point.*

Proof: The proof is carried out by contradiction. Suppose that there is a conformal deformation $d\overline{s}^2$ of the initial metric ds^2 such that at each point of M^n the sectional curvature $\overline{K}(\xi \wedge \eta)$ is positive. From here it follows that

$$K(\xi \wedge \eta) - \frac{B_{ij}\xi^i\xi^j}{g_{ik}\xi^i\xi^k} - \frac{B_{kl}\eta^k\eta^l}{g_{jl}\eta^j\eta^l} > 0.$$

Then at the point of minimum of the function σ we have

$$\begin{aligned} \sigma_{,i} &= 0, \\ B_{ij}\xi^i\xi^j &= \sigma_{,ij}\xi^i\xi^j \geq 0, \\ B_{kl}\eta^k\eta^l &= \sigma_{,kl}\eta^k\eta^l \geq 0. \end{aligned}$$

Hence, we see that $K(\xi\wedge\eta) > 0$ for all bivectors $\xi\wedge\eta$. The case of strictly negative curvature is treated analogously. These contradictions prove Theorem 3.1. □

Corollary 3.1. *Suppose that (M^n, ds^2) satisfies the conditions of Theorem 3.1. Then for any metric $d\overline{s}^2$ which is conformally equivalent to the initial metric ds^2 there are a point $x \in M^n$ and a two-dimensional direction $\xi \wedge \eta$ at this point such that $\overline{K}_x(\xi \wedge \eta) = 0$.*

Proof: Let M_0^n be a connected component of M^n. Using Theorem 3.1 we see that there are points $p, q \in M_0^n$ and two-dimensional directions at these points $\pi_p = \xi_p \wedge \eta_p$, $\pi_q = \xi_q \wedge \eta_q$ such that

$$K(\pi_p) \leq 0, \qquad K(\pi_q) \geq 0.$$

Let us consider a continuous curve $x(t)$, $t \in [0,1]$ connecting points p and q in M_0^n and a continuous field of bivectors $\pi_t = \xi_t \wedge \eta_t$ along $x(t)$ such that $\pi_0 = \pi_p$ and $\pi_1 = \pi_q$. Then there exists $\theta \in [0,1]$ such that at the point $x(\theta)$ we have $K(\pi_\theta) = 0$. □

Corollary 3.2. *Let (M^n, ds^2) be a direct Riemannian product of compact Riemannian manifolds. Then for any metric $d\overline{s}^2$ which is conformally equivalent to the initial metric ds^2 there are a point $x_0 \in M^n$ and a two-dimensional direction of nonpositive sectional curvature at this point, and a point $x_1 \in M^n$ and a two-dimensional direction of nonnegative sectional curvature at this point.*

Corollary 3.3. *The metric $d\overline{s}^2$ which is conformally equivalent to the Riemannian metric ds^2 of the direct Riemannian product $M = M_1 \times M_2$ of compact Riemannian manifolds has a point and a two-dimensional direction of zero sectional curvature at this point.*

In the case of homogeneous spaces we have the result:

Theorem 3.2. *If $d\overline{s}^2$ is conformally equivalent to a homogeneous Riemannian metric ds^2 of a simply connected compact homogeneous space G/H and $d\overline{s}^2$ has positive sectional curvature, then G/H is diffeomorphic either to a compact symmetric space of rank one (CROSS), or to one of the Aloff-Berger-Wallach spaces* ([Berger], [W])*:*

$$Sp(2)/SU(2),\ SU(5)/Sp(2)\times S^1,\ SU(3)/S^1,\ SU(3)/T_{\max},$$
$$Sp(3)/Sp(1)^3,\ F_4/Spin(8).$$

Proof: Suppose that $(G/H, d\overline{s}^2)$ has positive sectional curvature. If G/H is not diffeomorphic to a CROSS or to one of the Aloff-Berger-Wallach spaces, then ds^2 admits two-dimensional directions of zero sectional curvature at each point of G/H and according to Corollary 3.1 the metric $d\overline{s}^2$ has a direction of zero sectional curvature at some point of G/H. This contradiction proves Theorem 3.2. □

For the case of Lie groups we have the following theorem:

Theorem 3.3. *If $d\overline{s}^2$ is conformally equivalent to a left-invariant Riemannian metric ds^2 of a compact Lie group G, and $d\overline{s}^2$ has positive sectional curvature, then the Lie group G is locally isomorphic to the group $SU(2)$.*

The proof follows from Theorem 3.2 and Theorem of Wallach [W].

Theorem 3.4. *Let M^n be a compact manifold with Riemannian metric $ds^2 = g_{ij}dx^i dx^j$. Suppose that there exists a one-dimensional direction ξ such that $A(\xi) = 0$ for all points $x \in M^n$. Then for any conformal deformation $d\overline{s}^2 = e^{2\sigma(x)}g_{ij}dx^i dx^j$ there are points $x_0, x_1 \in M^n$ and one-dimensional directions ξ_0, ξ_1 at these points such that the inequalities*

$$\overline{A}_{x_0}(\xi_0) \leq 0, \quad \overline{A}_{x_1}(\xi_1) \geq 0$$

are fulfilled.

The proof of this theorem is similar to the proof of Theorem 3.1.

Theorem 3.5. *Let M^n be a compact manifold with Ricci flat Riemannian metric $ds^2 = g_{ij}dx^i dx^j$. Then for any conformal deformation $d\overline{s}^2 = e^{2\sigma(x)}g_{ij}dx^i dx^j$ there is a point $x_0 \in M^n$ such that the Ricci curvature is nonnegative at this point.*

PROOF: Using conditions of Theorem 3.5 we see that the one-dimensional sectional curvature of (M^n, ds^2) is identically equal to zero. Further, we apply the formula for the conformal deformation of one-dimensional sectional curvature, and we see that at the point of minimum of the function σ the one-dimensional sectional curvature of $d\overline{s}^2$ is nonnegative. Hence, the Ricci curvature of $d\overline{s}^2$ is nonnegative at this point too. $\square$

Remark 3.1. If the Ricci curvature of the initial metric ds^2 is nonnegative and positive at some point, then ds^2 is conformally equivalent to some metric of strictly positive Ricci curvature ([E]).

Theorem 3.6. *Let M^n be a compact manifold with Riemannian metric $ds^2 = g_{ij}dx^i dx^j$. Suppose that there exists a one-dimensional direction ξ such that $A(\xi) = k_0$, for some constant k_0, for all points $x \in M^n$. Then for any conformal deformation $d\overline{s}^2 = e^{2\sigma(x)}g_{ij}dx^i dx^j$ there are points $x_0, x_1 \in M$ and corresponding one-dimensional directions ξ_0, ξ_1 at these points such that the inequalities:*

$$\overline{A}_{x_0}(\xi_0) \leq k_0 e^{-2\sigma(x_0)}, \quad \overline{A}_{x_1}(\xi_1) \geq k_0 e^{-2\sigma(x_1)}$$

are fulfilled.

4. Locally conformally homogeneous Riemannian manifolds

Locally homogeneous Riemannian manifolds were studied by O. Kowalski, F. Tricerri, L. Vanhecke [K], [T-V], [T].

Definition 4.1. A vector field v is called a conformal Killing vector field if and only if

$$v_{i,k} + v_{k,i} = 2wg_{ik}, \tag{2}$$

where $w = v_{k,i}g^{ik}/n$.

The system (2) was studied by many authors ([Y], [C]). In this paper, following [Resh], we find a linear system of equations which is equivalent to the system (2). Further, with the help of this system, we investigate conformal Killing vector fields.

Lemma 4.1. *The equations system* (2) *is equivalent to the linear system*

$$
(3)\qquad
\begin{aligned}
&v_{j,p} = \eta_{jp} + g_{jp} w,\\
&\eta_{ij,p} = v_a R^a{}_{pij} + g_{ip}\zeta_j - g_{jp}\zeta_i,\\
&w_{,p} = \zeta_p,\\
&\varsigma_{j,p} = \eta_{pa} A^a_j + \eta_{ja} A^a_p - A_{jp,b} v^b - 2w A_{jp},
\end{aligned}
$$

where

$$A_{jp} = \frac{1}{n-2}\left(R_{jp} - \frac{R g_{jp}}{2\,(n-1)}\right),$$

$\{v_j\}$ *are covariant components of a vector field* $v(x)$, $\{\eta_{ij}\}$ *is a skew-symmetric covariant tensor,* w *is a function,* $\{\zeta_p\}$ *is a covector field. The integrability conditions of* (3) *have the form*

$$
(4)\qquad
\begin{gathered}
v^a W_{ijsk,a} + 2w W_{ijsk} - \eta_i^{.a} W_{ajsk} - \eta_j^{.a} W_{iask} - \eta_s^{.a} W_{ijak} - \eta_k^{.a} W_{ijsa} = 0,\\
\zeta_a W^a_{jps} - 3w S_{jps} - v^t S_{jps,t} + \eta_j^{.a} S_{aps} + \eta_p^{.a} S_{jas} + \eta_s^{.a} S_{jpa} = 0,
\end{gathered}
$$

where W^a_{jps} *is the Weyl tensor and* $S_{jps} = A_{jp,s} - A_{js,p}$ *is the Schouten-Weyl tensor.*

The proof of this lemma is given in [Y], [RS2].

Remark 4.1. The integrability condition can be written in a more compact form with the help of Lie derivatives (see [Y]):

$$L_v W_{ijks} = 2w W_{ijks}, \qquad L_v S_{ijk} = W^a_{.ijk} w_{;a}.$$

Remark 4.2. In the case $n = 3$, the Weyl tensor is identically equal to zero and therefore the first equality in (4) is fulfilled, and the second equality has the form

$$-3w S_{jps} - v^t S_{jps,t} + \eta_j^{.a} S_{aps} + \eta_p^{.a} S_{jas} + \eta_s^{.a} S_{jpa} = 0.$$

We note that in the case $n \geq 4$ the second equality in (4) follows from first (see, for example, [C]).

Lemma 4.2. *If* $|W| = \mathrm{const} \neq 0$, *then* $w \equiv 0$ *and* $\zeta_s \equiv 0$ (*i.e. the conformal Killing vector field* v *is Killing in this case*).

Proof: Contracting the first equality in (4) with W^{ijsk} we get equality

$$\frac{1}{2} v^a \left(|W|^2\right)_{,a} + 2w\,|W|^2 = 0,$$

and hence the statement of Lemma 4.2 follows. ◻

Lemma 4.3. *Let $\{M, ds^2 = g_{ij}dx^i dx^j\}$ be a Riemannian manifold and $V = \{v^i\}$ be a Killing vector field on (M, ds^2). Then $V = \{v^i\}$ is a conformal Killing vector field on $(M, d\overline{s}^2)$, where $d\overline{s}^2 = e^{2\sigma(x)} g_{ij}dx^i dx^j$.*

PROOF: First at all, we have the equality

$$v_{k,j} = v^i_{,j} g_{ik} = \frac{\partial v^i}{\partial t^j} g_{ik} + v^a \Gamma_{aj,k}.$$

From here it follows that

$$\overline{v}_{k,j} = e^{2\sigma} \left[v_{k,j} + v^a \frac{\partial \sigma}{\partial t^a} g_{kj} + g_{ak} v^a \frac{\partial \sigma}{\partial t^j} - g_{aj} v^a \frac{\partial \sigma}{\partial t^k} \right].$$

Hence, we have

$$\overline{v}_{k,j} + \overline{v}_{j,k} = 2v^a \frac{\partial \sigma}{\partial t^a} \overline{g}_{kj}.$$

□

Definition 4.2. Let $\{M^n, ds^2\}$ be a Riemannian manifold such that for every point $x_0 \in M$ and an arbitrary tangent vector $\vec{v}_0 \in T_{x_0}M$ there is a conformal Killing vector field $v(x)$ in a neighborhood of $x_0 \in M$ such that

$$v(x_0) = \vec{v}_0.$$

Then $\{M^n, ds^2\}$ is called a locally conformally homogeneous Riemannian manifold.

Remark 4.3. Obviously, the conformal deformation of a locally homogeneous Riemannian space gives a locally conformally homogeneous space.

Theorem 4.1. *Let $\{M^n, ds^2\}$ be a locally conformally homogeneous connected Riemannian manifold. Then $\{M^n, ds^2\}$ is either conformally flat, or it is conformally equivalent to a locally homogeneous Riemannian space.*

PROOF: Let us consider the case $\dim M > 3$. Under the conformal deformation the Weyl tensor is invariant, i.e.

$$W^a_{isk} = \overline{W}^a_{isk}, \quad |\overline{W}|^2 = e^{-6\sigma(t)} |W|^2$$

hold on M. Hence, if $|W| \neq 0$, it is possible to choose a function $\sigma(t)$ so that $|\overline{W}| \equiv \text{const} \neq 0$. Using Lemma 4.2, we see that the manifold M is locally homogeneous. In the case when $\dim M = 3$, the Weyl tensor is identically equal to zero. Contracting the second equality in (4) with S^{jps}, we have the following equality:

$$\zeta_a W^a_{jps} S^{jps} - 3w|S|^2 - \frac{1}{2} v^t \left(|S|^2\right)_{,t} = 0.$$

Hence, it follows similarly that either the Schouten-Weyl tensor is identically equal to zero, or with the help of a conformal deformation it is possible to make its norm constant, and the manifold M is locally homogeneous. □

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