No hedgehog in the product?

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Abstract. Assuming OCA, we shall prove that for some pairs of Fréchet α_4-spaces X, Y, the Fréchetness of the product $X \times Y$ implies that $X \times Y$ is α_4. Assuming MA, we shall construct a pair of spaces satisfying the assumptions of the theorem.

Keywords: Fréchet space, α_4-space, Fréchet fan, (κ, κ)-good set

Classification: Primary 54D55; Secondary 54B10, 54G20, 03E50, 03E65

All spaces are assumed to be Hausdorff. First of all, let us recall two well-known definitions. A topological space is called Fréchet-Urysohn or Fréchet, if its topology is fully described by convergent sequences, i.e., a point $x \in X$ belongs to a closure of a set $M \subseteq X$ iff there is a convergent sequence $\langle x_n : n \in \omega \rangle$ with all values in M and having x as its limit point. We shall frequently identify a convergent sequence with its range, so we shall use to say that $A \subseteq X$ is a convergent sequence with a limit x, if $|A| = \omega$ and for every neighborhood U of x, $|A \setminus U| < \omega$. Using this notation, recall that an α_4-*space* is a space X satisfying: Whenever $x \in X$ and $\{C_n : n \in \omega\}$ is a collection of sequences, all converging to x, then there is another sequence C with $\lim C = x$ and such that $C \cap C_n \neq \emptyset$ for infinitely many $n \in \omega$. The notion of α_4-space was introduced by A.V. Archangel'skii in [Ar]; Fréchet α_4-spaces are sometimes also called strongly Fréchet [Sw] or countably bisequential [Mi]. As a rich source of more information and related items, we recommend to the reader P.J. Nyikos' survey paper [Ny].

The notation used in this paper is fairly standard. The formula $A \subseteq^* B$ ($A =^* B$, resp.) denotes that $A \setminus B$ (the symmetric difference $A \Delta B$, resp.) is finite. The sets A, B are almost disjoint if $A \cap B = {}^*\emptyset$ and a MAD family is a maximal family of pairwise almost disjoint elements.

In 1986, T. Nogura asked the following question:

The research was done during the first author's visit in Department of Mathematics of Trieste University under the support of Italian CNR. Partial supports of GAUK 270/2001 and CEZ:J13/98:113200007 are also gratefully acknowledged. The second author was partially supported by the National Group "Analisi Reale" of the Italian Ministery of the University and Scientific Research at the University of Trieste.

Question [No, Question 3.15]**.** *If X, Y are two Fréchet α_4-spaces and their product $X \times Y$ is Fréchet, is it true then that $X \times Y$ is α_4, too?*

The first named author recently showed that CH implies a negative answer to this question [Si]. In an attempt for showing that the problem is in fact independent of ZFC, we shall discuss in the present paper the structure of possible counterexamples and prove that some of them cannot exist under Open Coloring Axiom.

Usually, no consideration dealing with α_4 property can ignore a countable hedgehog, or briefly a hedgehog. It is a quotient space of $\omega \times (\omega + 1)$ with all limit points (n, ω) identified. Other names of this topological space, appearing in the literature, are S_ω, F_ω, Fréchet fan and sequential fan. Usually, a subset $\{n\} \times \omega$ is called *n-th spine* of the hedgehog and the equivalence class $[(n, \omega)]$ its *body*. — Notice that the word "hedgehog" has a different meaning in the theory of metrizable spaces. In the present paper, however, the metrizable hedgehog will never occur.

It is well-known that a hedgehog is a test space for strong Fréchetness:

Fact 1. *A space is strongly Fréchet if and only if it is Fréchet and does not contain a copy of a hedgehog.*

Indeed, if a Fréchet space X contains a copy of S_ω with a body x, then each spine C_n converges to x. If $C \subseteq X$ is such that $C \cap C_n$ is nonempty for infinitely many $n \in \omega$, select $C' \subseteq C$ such that $C' \cap C_n$ is a one-point set whenever $C \cap C_n \neq \emptyset$. The subspace $\{x\} \cup \bigcup\{C_n : n \in \omega\}$ is homeomorphic to S_ω, hence there must be a neighborhood U of x in X, which is disjoint with C'. This shows that C' does not converge to x and consequently C does not converge to x, too. So X is not α_4.

Suppose X is not α_4 and choose a witness of that: There is a family $\{C_n : n \in \omega\}$ of convergent sequences with a common limit point x, such that no $C \subseteq X$ meeting infinitely many C_n's converges to x. Passing to subsequences, if necessary, we may assume that the sets C_n are pairwise disjoint and that the set $\bigcup\{C_n : n \in \omega\}$ is relatively discrete. For each $n \in \omega$ let $K_n \subseteq C_n$ be a finite set, infinitely many K_n's be non-empty, and put $K = \bigcup\{K_n : n \in \omega\}$. Observe that x does not belong to the closure of K, otherwise the Fréchetness of X would imply the existence of a sequence $C \subseteq K$, which converges to x, but this is impossible by the choice of x and $\{C_n : n \in \omega\}$. So $X \setminus K$ is a neighborhood of x. This however proves that $\{x\} \cup \bigcup\{C_n : n \in \omega\}$ is homeomorphic to S_ω.

If there is a pair P, Q of Fréchet α_4-spaces such that their product is Fréchet, but fails to be α_4, let us call, for the purposes of this paper, such a pair to be *a counterexample*.

Definition. Call a pair of spaces X, Y to be a *standard counterexample*, if both spaces have only one non-isolated point, $X = \omega \cup \{\infty_X\}$, $Y = \omega \cup \{\infty_Y\}$, the set

ω is the countable set of isolated points, X, Y are both Fréchet and α_4, $X \times Y$ is Fréchet and the diagonal $\{(n,n) : n \in \omega\} \cup \{(\infty_X, \infty_Y)\}$ is homeomorphic to S_ω.

Lemma 1. *There is a counterexample if and only if there is a standard counterexample.*

Proof: Suppose that the spaces P, Q are α_4, $P, Q, P \times Q$ are Fréchet, but $P \times Q$ is not α_4. By the Fact 1, there is a copy of S_ω in $P \times Q$ with a point (p,q) as its body. Denote by $S(n)$ its n-th spine, $S(n) = \{(p^n_k, q^n_k) : k \in \omega\}$. Each sequence $\langle p^n_k : k \in \omega\rangle$ converges in P to a point p; if it is eventually constant, then the eventual value must be p. This, however, may happen for at most finitely many n's, otherwise the collection $\{\langle q^n_k : k \in \omega\rangle : n \in \omega$ & for all but finitely many k's, $p^n_k = p\}$ would be a hedgehog in Q, which is impossible, because Q is α_4. Symmetric argument applies for $\langle q^n_k : k \in \omega\rangle$. We can therefore assume without any loss of generality that all sequences $\langle p^n_k : k \in \omega\rangle$, $\langle q^n_k : k \in \omega\rangle$ $(n \in \omega)$ are one-to-one (discard, if necessary, finitely many n's and pass to subsequences then).

Knowing that all sequences $\langle p^n_k : k \in \omega\rangle, \langle q^n_k : k \in \omega\rangle$ are one-to-one, we may, in fact, assume again without a loss of generality that $\{p^n_k : k \in \omega\} \cap \{p^m_k : k \in \omega\} = \emptyset = \{q^n_k : k \in \omega\} \cap \{q^m_k : k \in \omega\}$ for distinct $n, m \in \omega$.

Finally, use the fact that both spaces are Hausdorff. Choose for each point p^n_k its neighborhood $U(p^n_k)$ and a neighborhood $V(n,k)$ of the point p such that $U(p^n_k) \cap V(n,k) = \emptyset$. Proceeding by a simple induction and using the fact that every neighborhood of the point p contains a cofinite part of every set $\{p^n_k : k \in \omega\}$, one can find for every $n \in \omega$ an infinite set A_n such that the set $\{p^n_k : n \in \omega \,\&\, k \in A_n\}$ is discrete in itself. Turning attention to $\{q^n_k : k \in A_n\}$ $(n \in \omega)$, apply the same reasoning to get infinite sets $B_n \subseteq A_n$ such that the set $\{q^n_k : n \in \omega \,\&\, k \in B_n\}$ is discrete in itself as well.

We clearly have that $\{(p^n_k, q^n_k) : k \in B_n\} \subseteq S(n)$, therefore $\{\{(p^n_k, q^n_k) : k \in B_n\} : n \in \omega\} \cup \{(p,q)\}$ is another hedgehog in the product $P \times Q$. Because both Fréchetness and the α_4 property are hereditary, the subspaces $X' = \{p^n_k : n \in \omega \,\&\, k \in B_n\} \cup \{p\} \subseteq P$ and $Y' = \{q^n_k : n \in \omega \,\&\, k \in B_n\} \cup \{q\} \subseteq Q$ may be obviously identified with a standard counterexample. — The opposite implication is trivial. $\square$

It is also well known that the topology of a Fréchet space with only one non-isolated point has two equivalent descriptions, by means of a neighborhood filter and by means of an almost disjoint family of convergent sequences. The precise statement is as follows.

Fact 2. *Let $\omega \cup \{\infty\}$ be a Fréchet space having all points except ∞ isolated. Denote by $\mathcal{F}$ the filter $\{U \cap \omega : U$ is a neighborhood of the point $\infty\}$.*

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an arbitrary family satisfying

(i) *for every $A \in \mathcal{A}$, A converges to ∞;*

(ii) *for distinct* $A, A' \in \mathcal{A}$, $A \cap A'$ *is finite;*

(iii) $\mathcal{A}$ *is a maximal family satisfying* (i) *and* (ii).

Then $\mathcal{F} = \{M \subseteq \omega :\ \text{for each } A \in \mathcal{A}, A \setminus M \text{ is finite}\}$.

Next, let $\mathcal{A}$ *be an arbitrary non-empty almost disjoint family of infinite subsets of* ω. *Declare* $\mathcal{G} = \{\{\infty\} \cup M : M \subseteq \omega \text{ and for each } A \in \mathcal{A},\ A \setminus M \text{ is finite}\}$ *to be a neighborhood system at* ∞. *Then* $\omega \cup \{\infty\}$ *is a Fréchet space, each sequence* $A \in \mathcal{A}$ *converges to* ∞ *and every sequence converging to* ∞ *meets some* $A \in \mathcal{A}$ *in an infinite set.*

An easy proof may be found e.g. in [Si].

We shall proceed now by examining further properties of a standard counterexample. Let us fix the notation: $X = \omega \cup \{\infty_X\}$, $Y = \omega \cup \{\infty_Y\}$ is some standard counterexample. Since the diagonal $\{(n,n) : n \in \omega\} \cup \{(\infty_X, \infty_Y)\}$ is a (copy of) hedgehog, denote by C_n the projection of its n-th spine. Observe that it makes no difference which projection we choose. Notice that each set C_n is infinite, $C_n \cap C_m = \emptyset$ for $n \neq m$ and $\omega = \bigcup\{C_n : n \in \omega\}$.

Clearly, for each $n \in \omega$, C_n converges to ∞_X in X and to ∞_Y in Y. Choose two families $\mathcal{A}$, $\mathcal{B}$ of infinite subsets of ω such that $\mathcal{A}$ ($\mathcal{B}$, resp.) describes the topology of X (of Y, resp.) in the sense of Fact 2. We may and shall assume that $\{C_n : n \in \omega\} \subseteq \mathcal{A} \cap \mathcal{B}$.

Observation 1. *Whenever* $A \in \mathcal{A} \setminus \{C_n : n \in \omega\}$ *and* $B \in \mathcal{B} \setminus \{C_n : n \in \omega\}$, *then* $A \cap B$ *is finite.*

Indeed, if the set $\{(n,n) : n \in A \cap B\}$ were infinite, then the diagonal could not be a hedgehog.

For a function $f \in {}^{\omega}\omega$, put $L(f) = \{k \in \omega : \text{for all } n \in \omega, \text{ if } k \in C_n, \text{ then } k < f(n)\}$ and $T(f) = \omega \setminus L(f)$.

Observation 2. *Whenever* $f \in {}^{\omega}\omega$, *then the set* $\{(k,k) : k \in L(f)\}$ *is closed discrete in* $X \times Y$.

If not, then, since $X \times Y$ is Fréchet, there is a sequence contained in $\{(k,k) : k \in L(f)\}$ and converging to (∞_X, ∞_Y). The existence of such a sequence again contradicts the fact that the diagonal is a hedgehog.

Observation 3. *Whenever* $f \in {}^{\omega}\omega$, *then there are some* $A \in \mathcal{A} \setminus \{C_n : n \in \omega\}$ *and* $B \in \mathcal{B} \setminus \{C_n : n \in \omega\}$ *such that both* $A \cap T(f)$ *and* $B \cap T(f)$ *are infinite sets.*

The family $\{C_n \cap T(f) : n \in \omega\}$ is a collection of countably many sequences converging to ∞_X in X and to ∞_Y in Y. The statement follows from the fact that both spaces are α_4.

Observation 4. *There are two families of sets* $\{P(f) : f \in {}^{\omega}\omega\}$ *and* $\{Q(f) : f \in {}^{\omega}\omega\}$ *with the following properties:*

(i) *for each* $f \in {}^{\omega}\omega$, $P(f) \cup Q(f) \subseteq L(f)$;

(ii) *for each* $f \in {}^{\omega}\omega$, $P(f) \cap Q(f) = \emptyset$;

(iii) $\{\{\infty_X\} \cup T(f) \cup P(f) \ : \ f \in {}^{\omega}\omega\}$ *is a neighborhood basis at* ∞_X *and* $\{\{\infty_Y\} \cup T(f) \cup Q(f) \ : \ f \in {}^{\omega}\omega\}$ *is a neighborhood basis at* ∞_Y.

Proof: Consider the set $M = \{(f, U, V) \ : \ f \in {}^{\omega}\omega$, U is a neighborhood of ∞_X, V is a neighborhood of ∞_Y and $T(f) \subseteq U$, $T(f) \subseteq V\}$. Fix an arbitrary one-to-one mapping $\psi : M \longrightarrow {}^{\omega}\omega$ such that $\psi(f, U, V)(n) \geq f(n)$ for all $n \in \omega$.

Let U be a neighborhood of ∞_X in X and V a neighborhood of ∞_Y in Y. Define the mapping $f = f_{U,V} \in {}^{\omega}\omega$ by the rule $f(n) = \min\{j \in \omega \ :$ if $k \geq j$ and $k \in C_n$, then $k \in U \cap V\}$. Let $g = \psi(f, U, V)$ for the f just defined. Since the set $\{(k, k) : k \in L(g)\}$ is closed discrete in $X \times Y$ by Observation 2, there are neighborhoods U_1 of ∞_X and V_1 of ∞_Y such that $U_1 \cap V_1 \cap L(g) = \emptyset$. Put $P(g) = U \cap U_1 \cap L(g)$, $Q(g) = V \cap V_1 \cap L(g)$.

Notice that $P(g) \cup T(g)$ is a subset of U, because $f \leq g$ and $P(g) \subseteq U$. On the other hand, the set $\{\infty_X\} \cup T(g) \cup P(g)$ contains a neighborhood of ∞_X, namely $U \cap U_1$, so it is a neighborhood as well.

Notice now that (i), (ii) and (iii) hold if we replace everywhere ${}^{\omega}\omega$ by $\{\psi(f_{U,V}, U, V) : U$ is a neighborhood of ∞_X, V is a neighborhood of $\infty_Y\}$. For every remaining $f \in {}^{\omega}\omega$, select an arbitrary $g \geq f$ such that the sets $P(g)$ and $Q(g)$ were defined, then put $P(f) = P(g) \cap L(f)$ and $Q(f) = Q(g) \cap L(f)$. $\square$

For $f \in {}^{\omega}\omega$, denote by $\mathcal{A} \restriction L(f)$ the set $\{A \cap L(f) : A \in \mathcal{A} \ \& \ |A \cap L(f)| = \omega\}$ and analogously for $\mathcal{B} \restriction L(f)$.

Observation 5. *For each* $f \in {}^{\omega}\omega$, $A \in \mathcal{A} \restriction L(f)$, $B \in \mathcal{B} \restriction L(f)$ *we have* $A \subseteq^* P(f)$, $B \subseteq^* Q(f)$.

Indeed, by Observation 4, $\{\infty_X\} \cup T(f) \cup P(f)$ is a neighborhood of ∞_X and A converges to ∞_X. Moreover, belonging to $\mathcal{A} \restriction L(f)$, A is disjoint with $T(f)$.

We complete our list of observations by the trivial

Observation 6. *There is no set* $M \subseteq \omega$ *such that*

(a) $|\{n \in \omega \ : \ |M \cap C_n| = \omega\}| = \omega$;

(b) *for each* $A \in \mathcal{A} \setminus \{C_n : n \in \omega\}$, $A \cap M$ *is finite or for each* $B \in \mathcal{B} \setminus \{C_n : n \in \omega\}$, $B \cap M$ *is finite.*

With an M as in (a), there is an infinite set of disjoint sequences converging to ∞_X in X and to ∞_Y in Y from distinct sets of the form $M \cap C_n$, which shows that X or Y is not α_4, depending on which alternative in (b) actually takes place.

Definition. A pair of spaces X, Y will be called a *strong counterexample*, if it is a standard counterexample which satisfies in addition: If a set $D \subseteq \omega$ is closed discrete in both spaces X, Y, then D is finite.

Let us remark here that a strong counterexample has already been constructed under the assumption of CH [Si]. The forthcoming theorem contrasts to that.

We shall assume a well-known consequence of PFA, Todorcevic's Open Coloring Axiom in the theorem. We will not give a definition of OCA here, the reader may find it in [To]. In fact, a bit weaker statement than OCA will be actually used.

For $f \in {}^{\omega}\omega$, let $U_f = \{(i,j) : j \le f(i)\}$. A *coherent family of functions indexed by* ${}^{\omega}\omega$ is a family $\{g_f : f \in {}^{\omega}\omega\}$ such that

(i) $g_f : U_f \longrightarrow \omega$ for all $f \in {}^{\omega}\omega$,
(ii) $\{x \in U_f \cap U_h : g_f(x) \ne g_h(x)\}$ is finite for all $f, h \in {}^{\omega}\omega$.

Todorcevic proved that OCA implies that every coherent family of functions indexed by ${}^{\omega}\omega$ is trivial, which means that there is some function $g : \omega \times \omega \longrightarrow \omega$ such that for every $f \in {}^{\omega}\omega$, $g_f(x) = g(x)$ for all except finitely many $x \in U_f$ ([To, Theorem 8.7]).

Theorem 1. *Assume OCA. Then there is no strong counterexample.*

PROOF: Assume not and follow with one strong counterexample X, Y.

We shall use the notation introduced in the previous observations. For each $f \in {}^{\omega}\omega$, let F_f be a mapping from the set $L(f)$ into $\{0,1,2\}$ defined by:

$$F_f(n) = \begin{cases} 0 & \text{if } n \in P(f), \\ 1 & \text{if } n \in Q(f), \\ 2 & \text{otherwise.} \end{cases}$$

The family $\{F_f : f \in {}^{\omega}\omega\}$ is coherent, i.e., whenever $f, g \in {}^{\omega}\omega$, then $F_f \restriction (\operatorname{dom} F_f \cap \operatorname{dom} F_g) =^* F_g \restriction (\operatorname{dom} F_f \cap \operatorname{dom} F_g)$. To see this, notice that for every infinite set $M \subseteq \omega$, either $\infty_X \in \overline{M}$ or $\infty_Y \in \overline{M}$, because X, Y is a strong counterexample. Since both spaces are Fréchet, there is a sequence with values in M which converges either in X or in Y. Using Fact 2(iii) and Observation 1, we immediately get that $\mathcal{A} \cup \mathcal{B}$ is a MAD family on ω. Now, let $f, g \in {}^{\omega}\omega$ be arbitrary. Put $H = \{n \in L(f) \cap L(g) : F_f(n) \ne F_g(n)\}$. Suppose for a contradiction that the set H is infinite. Since $\mathcal{A} \cup \mathcal{B}$ is a MAD family, there is some $D \in \mathcal{A} \cup \mathcal{B}$ with $D \cap H$ infinite. Assume that e.g., $D \in \mathcal{A}$. Then $D \cap H \subseteq L(f)$ and by Observation 5, $D \cap H \subseteq^* P(f)$. Similarly, $D \cap H \subseteq^* P(g)$. By the definition of F_f and F_g, $F_f(n) = 0 = F_g(n)$ for all but finitely many points from the infinite set $D \cap H$. This, of course, contradicts our choice of H.

By [To, Theorem 8.7], OCA implies now that there is a mapping $G \in {}^{\omega}\{0,1,2\}$ such that $F_f \subseteq^* G$ for every $f \in {}^{\omega}\omega$.

Consider the set $P = G^{-1}(0)$.

Choose an arbitrary $B \in \mathcal{B} \setminus \{C_n : n \in \omega\}$. Since $B \cap C_n$ is finite for all n, there is a function $f \in {}^{\omega}\omega$ such that $B \subseteq L(f)$. By Observation 4, $B \subseteq^* Q(f)$, thus the set $\{n \in B : F_f(n) = 0\}$ is finite. Having $G \restriction L(f) =^* F_f$, we have also that the set $\{n \in B : G(n) = 0\}$ is finite as well. So $B \cap P$ is finite.

An analogous reasoning shows that for every $A \in \mathcal{A} \setminus \{C_n : n \in \omega\}$, $A \subseteq^* P$.

We have just proved that the set P satisfies (b) from Observation 6. Therefore (a) cannot hold for P and we have the following mapping $f \in {}^{\omega}\omega$: $f(n) = 0$ if $P \cap C_n$ is infinite, $f(n) = 1 + \max(C_n \cap P)$ otherwise. By Observation 3, there is some $A \in \mathcal{A} \setminus \{C_n : n \in \omega\}$ with $A \cap T(f)$ infinite.

For this particular A, the difference $A \setminus P$ is infinite, because it contains almost all points from $A \cap T(f)$. This however contradicts the fact that $A \subseteq^* P$ and concludes the proof. □

Let us reformulate the theorem just proved as a positive statement.

Theorem 1. *Assume OCA. Let $X = \omega \cup \{\infty_X\}$ and $Y = \omega \cup \{\infty_Y\}$ be two Fréchet α_4-spaces, each with a unique nonisolated point, and such that every subset of ω, which is closed in both topologies, is finite. If $X \times Y$ is Fréchet, then the diagonal $\{(n, n) : n \in \omega\} \cup \{(\infty_X, \infty_Y)\}$ is α_4.* □

Now, the questions arise. To what extent does OCA trivialize matters? Are the assumptions of the theorem satisfied non-vacuously? We answer these questions by showing an example. It will be constructed under MA, but it makes no harm, since OCA+MA+$\mathfrak{c} = \omega_2$ is known to be consistent.

Strangely enough, it is known that the product of two Fréchet α_4-spaces may fail to be Fréchet, and it is also known that $S_\omega \times X$ is never Fréchet except when X is discrete, but nobody seems to have asked, whether the product of two Fréchet α_4-spaces may be Fréchet, if neither of them is α_3. (Recall that a space X is *an α_3-space*, if for each $x \in X$ and for each countably infinite collection $\{C_n : n \in \omega\}$ of sequences convergent to x, there is a sequence C with $\lim C = x$ and such that $C \cap C_n$ is infinite for infinitely may $n \in \omega$ [Ar].) Our example clears also this point.

Example. *Assume MA. Then there are two spaces X, Y, each with a unique nonisolated point, $X = \omega \cup \{\infty_X\}$ and $Y = \omega \cup \{\infty_Y\}$, with the following properties:*

(1) *X and Y are Fréchet, α_4, but not α_3;*
(2) *if a set $M \subseteq \omega$ is closed in X and also in Y, then M is finite;*
(3) *$X \times Y$ is Fréchet.*

Proof: We shall construct a maximal almost disjoint family $\mathcal{A}$ on ω and its two subcollections $\mathcal{B}$, $\mathcal{C}$, with $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$. The space X will be described by $\mathcal{B}$ and Y by $\mathcal{C}$ according to Fact 2. Observe that the maximality of $\mathcal{A}$ will imply (2). By Fact 2, both spaces will be Fréchet.

We shall proceed by transfinite recursion to $\mathfrak{c}$. Enumerate all subsets of $\omega \times \omega$ as $\{F_\xi : \xi < \mathfrak{c}\}$ with cofinally repetitions, that means that for every $F \subseteq \omega \times \omega$, the set $\{\xi < \mathfrak{c} : F = F_\xi\}$ is cofinal in $\mathfrak{c}$.

Start: Choose an arbitrary infinite partition of ω into infinite pieces, say $\mathcal{R} = \{R_n : n \in \omega\}$. Define $\mathcal{B}_0 = \{R_{2n} : n \in \omega\}$, $\mathcal{C}_0 = \{R_{2n+1} : n \in \omega\}$ and $\mathcal{A}_0 = \mathcal{B}_0 \cup \mathcal{C}_0$. Observe that $|\mathcal{A}_0| \leq \omega$.

For $\xi \leq \mathfrak{c}$, ξ limit, we take unions of everything done so far: $\mathcal{A}_\xi = \bigcup_{\eta<\xi} \mathcal{A}_\eta$, $\mathcal{B}_\xi = \bigcup_{\eta<\xi} \mathcal{B}_\eta$ and $\mathcal{C}_\xi = \bigcup_{\eta<\xi} \mathcal{C}_\eta$.

Suppose $\xi = \eta + 1$ and $\mathcal{A}_\eta = \mathcal{B}_\eta \cup \mathcal{C}_\eta$ are known. As a permanent assumption we have that $|\mathcal{A}_\eta| \leq \omega \cdot |\eta + 1|$.

We shall consider three cases.

Case 1. There is a one-to-one mapping $f \subseteq F_\eta$ and an infinite set $B \subseteq \operatorname{dom} f$ such that $B \cap f[B] = \emptyset$ and both sets $B, f[B]$ are almost disjoint from all members of $\mathcal{A}_\eta$. Fix one witness f_η and B_η of this case and let $\mathcal{B}_\xi = \mathcal{B}_\eta \cup \{B_\eta\}$, $\mathcal{C}_\xi = \mathcal{C}_\eta \cup \{f_\eta[B_\eta]\}$ and $\mathcal{A}_\xi = \mathcal{A}_\eta \cup \{B_\eta, f_\eta[B_\eta]\}$.

Case 2. Not Case 1, but there is a mapping $g \subseteq F_\eta$ and an infinite set $A \subseteq \operatorname{dom} g$ such that A is almost disjoint from all members of $\mathcal{A}_\eta$ and $g \restriction A = \mathrm{id}_A$. Similarly, select and fix some g_η, A_η with the property and put $\mathcal{A}_\xi = \mathcal{A}_\eta \cup \{A_\eta\}$, $\mathcal{B}_\xi = \mathcal{B}_\eta \cup \{A_\eta\}$ and $\mathcal{C}_\xi = \mathcal{C}_\eta \cup \{A_\eta\}$.

Case 3. Neither Case 1 nor Case 2. Let us relax: $\mathcal{A}_\xi = \mathcal{A}_\eta$, $\mathcal{B}_\xi = \mathcal{B}_\eta$, $\mathcal{C}_\xi = \mathcal{C}_\eta$.

The recursive definitions are complete. It remains to put $\mathcal{A} = \mathcal{A}_\mathfrak{c}$, $\mathcal{B} = \mathcal{B}_\mathfrak{c}$, $\mathcal{C} = \mathcal{C}_\mathfrak{c}$.

Let us verify first that $\mathcal{A}$ is a MAD family on ω. Almost disjointness is a straightforward consequence of the recursive definitions. For maximality, choose an arbitrary infinite set $M \subseteq \omega$ and let η be the first occurrence of $\mathrm{id}_M = F_\eta$. When we passed from η to $\eta + 1$, Case 1 could not happen from obvious reasons. The failure of Case 2 implies that there already is some $A \in \mathcal{A}_\eta$ with $|A \cap M| = \omega$. If Case 2 holds, then $A_\eta \subseteq M$ and $A_\eta \in \mathcal{A}_{\eta+1} \subseteq \mathcal{A}$. The maximality of $\mathcal{A}$ follows.

Now, let us prove a simple statement with a frequent future use.

Claim. *Let $\{A_n : n \in \omega\}$ be an infinite subset of $\mathcal{A}$ and let $M \subseteq \omega$ satisfy $|M \cap A_n| = \omega$ for all $n \in \omega$. Then there are sets $B \in \mathcal{B} \setminus \mathcal{C}$ and $C \in \mathcal{C} \setminus \mathcal{B}$ with $B \subseteq M, C \subseteq M$.*

Proof of the claim: Put $H_n = (M \cap A_n) \setminus \bigcup_{i<n} A_i$. All sets H_n are infinite and pairwise disjoint; let H be their union, $H = \bigcup\{H_n : n \in \omega\}$. Split each H_n into two disjoint infinite parts, $H_n = H_n^0 \cup H_n^1$, and choose a one-to-one mapping $f : H \longrightarrow H$ such that for each n, $f[H_n^0] = H_n^1$. In particular, the domain and the range of f are both subsets of M. Since the mapping f is listed cofinally many times, there is some $\eta < \mathfrak{c}$ such that $\{A_n : n \in \omega\} \subseteq \mathcal{A}_\eta$ and $f = F_\eta$.

The set $\mathcal{A}' = \mathcal{A}_\eta \setminus \{A_n : n \in \omega\}$ is of size $\leq |\eta + 1| \cdot \omega < \mathfrak{c}$ and whenever $\mathcal{D}$ is a finite subfamily of $\mathcal{A}'$, then $|H_n^1 \setminus \bigcup \mathcal{D}| = \omega$ for each $n \in \omega$. By Solovay's lemma (i.e., the Principle S_κ from [MS]), there is a set G^1 such that $|G^1 \cap A| < \omega$ for each $A \in \mathcal{A}'$ and $|G^1 \cap H_n^1| = \omega$ for each $n \in \omega$. Put $G_n^0 = H_n^0 \cap f^{-1}[G^1]$. Applying Solovay's lemma once more, now to the families $\mathcal{A}'$ and $\{G_n^0 : n \in \omega\}$, we obtain a set Z such that $|Z \cap A| < \omega$ for all $A \in \mathcal{A}'$, $|Z \cap G_n^0| = \omega$ for all $n \in \omega$. Whenever $B \subseteq Z$ is such that $B \cap G_n^0$ is nonvoid finite for each $n \in \omega$, then B is infinite, $B \subseteq \operatorname{dom} f$, $B \cap f[B] = \emptyset$ and B as well as $f[B]$ is almost disjoint from each $A \in \mathcal{A}_\eta$.

We have just verified that the assumptions of Case 1 were satisfied in the $(\eta+1)$-st step of the recursion. Thus there is some $g_\eta \subseteq F_\eta = f$ and an infinite set B_η satisfying the assumptions of Case 1 with $B_\eta \in \mathcal{B}_{\eta+1}$ and $g_\eta[B_\eta] \in \mathcal{C}_{\eta+1}$. By our choice of f, the sets B_η and $g_\eta[B_\eta]$ have the required properties. The claim is proved. — Notice that we used the equality $\mathfrak{p} = \mathfrak{c}$ instead of the full strength of MA, and a minor modification of the proof could show the claim under $\mathfrak{b} = \mathfrak{c}$.

Notice two instant consequences of Claim:

If $T \subseteq \omega$ is a sequence in X converging to ∞_X, then the set $\{B \in \mathcal{B} : B \cap T$ is infinite$\}$ contains only finitely many members, and similarly for $\mathcal{C}$ and Y. Indeed, if not, apply the Claim to an arbitrary infinite collection $\{A_n : n \in \omega\} \subseteq \{B \in \mathcal{B} : |B \cap T| = \omega\}$ and T in role of M, we get some $C \in \mathcal{C} \setminus \mathcal{B}$ with $C \subseteq T$. The set $X \setminus C$ is then a neighborhood of ∞_X omitting infinitely many points from T, so T does not converge to ∞_X, a contradiction.

The family

$$\big\{\{\infty_X\} \cup \omega \setminus (n \cup \bigcup \mathcal{C}') : \mathcal{C}' \subset \mathcal{C} \setminus \mathcal{B},\, \mathcal{C}' \text{ is finite}, n \in \omega\big\}$$

is a neighborhood basis at ∞_X in X, and analogously for Y. Clearly any set of the above form is a neighborhood of ∞_X. Let U be an arbitrary neighborhood of ∞_X and consider the difference $M = \omega \setminus U$. Since there is no $B \in \mathcal{B}$ with $B \subseteq M$, Claim implies that the set $\mathcal{A}_1 = \{A \in \mathcal{A} : |A \cap M| = \omega\}$ is finite. Clearly $\mathcal{A}_1 \subseteq \mathcal{C} \setminus \mathcal{B}$. As $\mathcal{A}$ is a MAD family, $M \setminus \bigcup \mathcal{A}_1$ is finite.

Let us prove that X is not α_3: Consider the family $\{T_n : n \in \omega\}$, where $T_n = R_{2n}$. Each T_n is a convergent sequence and $T_n \in \mathcal{B} \subseteq \mathcal{A}$. If $M \cap T_n$ is infinite for infinitely many n's, then, by Claim, there is $C \in \mathcal{C} \setminus \mathcal{B}$ with $C \subseteq M$. So M does not converge to ∞_X and thus X is not α_3.

Let us show that X is α_4: Choose an arbitrary family $\{T_n : n \in \omega\}$ of sequences converging to ∞_X. If there is some $B \in \mathcal{B}$ such that $B \cap T_n$ is infinite for infinitely many T_n's, we are done. Otherwise pick $B_0 \in \mathcal{B}$ such that $B_0 \cap T_0$ is infinite, put $n(0) = 0$ and proceed by an induction: Suppose $T_{n(k)}$ and $B_{n(k)}$ are known. Then the family $\mathcal{B}_k = \{B \in \mathcal{B} :$ for some $i \le k, B \cap T_{n(i)}$ is infinite$\}$ is finite by Claim and, since no $B \in \mathcal{B}$ meets infinitely many T_n's in an infinite set, there is some $T_{n(k+1)}$ with $T_{n(k+1)} \cap B$ finite for all $B \in \mathcal{B}_k$. Pick an arbitrary $B \in \mathcal{B}$ such that $|B \cap T_{n(k+1)}| = \omega$ and denote this B as $B_{n(k+1)}$. After completing the induction, the set $M = \bigcup\{T_{n(k)} \cap B_{n(k)} : k \in \omega\}$ contains some $\tilde{B} \in \mathcal{B}$ by Claim. The sequence $\tilde{B}$ converges to ∞_X and meets infinitely many members from $\{T_n : n \in \omega\}$. So X is α_4.

It remains to verify that the product $X \times Y$ is Fréchet. Only one case is non-trivial: Find a sequence converging to the point (∞_X, ∞_Y) from a set $M \subseteq \omega \times \omega$ with $(\infty_X, \infty_Y) \in \overline{M}$.

We may encounter an easy situation when there is some $B \in \mathcal{B}$ such that

$$(\infty_X, \infty_Y) \in \overline{M \cap (B \times \omega)}.$$

The subspace $B \cup \{\infty_X\}$ is a convergent sequence with a limit point, hence compact and metrizable. To get a desired sequence converging to (∞_X, ∞_Y) we use the well known fact that the product of a compact metric space and a Fréchet α_4-space is Fréchet ([Ar]). The same reasoning applies if

$$(\infty_X, \infty_Y) \in \overline{M \cap (\omega \times C)}$$

for some $C \in \mathcal{C}$.

So we shall assume for the rest of this proof that

$$(\infty_X, \infty_Y) \notin \overline{M \cap (\bigcup \mathcal{B}' \times \omega) \cup (\omega \times \bigcup \mathcal{C}')}$$

for arbitrary finite $\mathcal{B}' \subseteq \mathcal{B}$, $\mathcal{C}' \subseteq \mathcal{C}$.

Thus, for each finite $\mathcal{B}' \subseteq \mathcal{B}$, $\mathcal{C}' \subseteq \mathcal{C}$ and for each neighborhood U of ∞_X and V of ∞_Y, $(M \setminus ((\bigcup \mathcal{B}' \times \omega) \cup (\omega \times \bigcup \mathcal{C}'))) \cap (U \times V)$ is infinite. According to the description of neighborhood bases given above, it means that

$$M \setminus (((n \cup \bigcup \mathcal{A}') \times \omega) \cup (\omega \times (n \cup \bigcup \mathcal{A}')))$$

is infinite, whenever $\mathcal{A}'$ is a finite subfamily of $\mathcal{A}$ and $n \in \omega$.

There is some $\eta < \mathfrak{c}$ such that $M = F_\eta$. The family $\mathcal{A}_\eta$ is of size $< \mathfrak{c}$, so $\mathfrak{p} = \mathfrak{c}$ applies: There is a set $W \subseteq M$ such that W is infinite, $W \cap ((n \times \omega) \cup (\omega \times n))$ is finite for all $n \in \omega$ and $W \cap ((\bigcup \mathcal{A}' \times \omega) \cup (\omega \times \bigcup \mathcal{A}'))$ is finite for each finite $\mathcal{A}' \subseteq \mathcal{A}_\eta$. In particular, $W \cap ((A \times \omega) \cup (\omega \times A))$ is finite for every $A \in \mathcal{A}_\eta$.

Find a strictly increasing function $h \subseteq W$ with an infinite domain. This is clearly possible, because W is infinite, but $W \cap ((n \times \omega) \cup (\omega \times n))$ is always finite.

If the set $\{n \in \operatorname{dom} h : h(n) = n\}$ is finite, then it is again easy to select an infinite $f \subseteq h$ with $\operatorname{dom} f \cap f[\operatorname{dom} f] = \emptyset$. Since $f \subseteq W$, both sets $\operatorname{dom} f$ and $f[\operatorname{dom} f]$ are almost disjoint from all members of $\mathcal{A}_\eta$. So Case 1 had to be applied in the corresponding step of the recursion. Thus there is a mapping $f_\eta \subseteq F_\eta = M$ and a set B_η with $B_\eta \in \mathcal{B}_{\eta+1}$, $f_\eta[B_\eta] \in \mathcal{C}_{\eta+1}$. The sequence $\{(n, f_\eta(n)) : n \in B_\eta\}$ takes all values in M and converges to (∞_X, ∞_Y).

If the set $\{n \in \operatorname{dom} h : h(n) = n\}$ is infinite, define g to be an identity mapping on that infinite set. If Case 1 fails, then this mapping shows that Case 2 must hold. So for the set $A_\eta \in \mathcal{A}_{\eta+1}$ we have that $\{(n, n) : n \in A_\eta\}$ converges to (∞_X, ∞_Y). $\square$

Our last goal is to show the connection between Nogura's question and the existence of a $(\mathfrak{c}, \mathfrak{c})$-good set in the product of two hedgehogs with $\mathfrak{c}$ spines. Before doing so, we recall a few definitions.

Suppose κ to be an infinite cardinal. Consider κ as a discrete topological space. *A κ-hedgehog F_κ* is a quotient space of $\kappa \times (\omega + 1)$ with all points (α, ω) identified to a single point ∞. (Here, the standard name in the literature is the κ-fan; we used the term "hedgehog" to stay in an agreement with the preceding.)

Suppose that a set Z is a subset of $(\kappa \times \omega) \times (\kappa \times \omega) \subseteq F_\kappa \times F_\kappa$. Call a set Z to be *(κ, κ)-good*, if the following are satisfied:

(a) $(\infty, \infty) \in \overline{Z}$;

(b) $(\forall E \in [\kappa]^{<\kappa})(\infty, \infty) \notin \overline{Z \cap ((\kappa \times \omega) \times (E \times \omega))} \cup \overline{Z \cap ((E \times \omega) \times (\kappa \times \omega))}$;

Our definition of a (κ, κ)-good set agrees with the one given in [BL]; and coincides with the notion of (κ, κ)-good set of type κ from [LL].

Finally, notice that a neighborhood of ∞ in F_κ is any set of the form

$$V_\varphi = \{\infty\} \cup \{(\alpha, k) : \varphi(\alpha) < k < \omega\},$$

with $\varphi \in {}^\kappa\omega$. We shall follow the commonly adopted convention and pretend that sets

$$\{(\infty, \infty)\} \cup \{((\alpha, k), (\beta, j)) : \varphi(\alpha) < k, \varphi(\beta) < j\}, \text{ where } \varphi \in {}^\kappa\omega,$$

constitute a neighborhood base at (∞, ∞) in $F_\kappa \times F_\kappa$.

We are ready now to state the next result.

Theorem 2. *Assume $\mathfrak{b} = \mathfrak{c}$. If there is a counterexample, then there is a $(\mathfrak{c}, \mathfrak{c})$-good set in $F_\mathfrak{c} \times F_\mathfrak{c}$.*

Proof: By Observation 1, we are allowed to assume that we are given a standard counterexample. We shall again tacitly use a notation from the observations. Enumerate the almost disjoint family $\mathcal{A} \setminus \{C_n : n \in \omega\}$ as $\{A_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{B} \setminus \{C_n : n \in \omega\} = \{B_\alpha : \alpha < \mathfrak{c}\}$. By Observation 1, there is no loss of generality if we assume that $A_\alpha \cap B_\alpha = \emptyset$ for each $\alpha < \mathfrak{c}$.

Define a subset Z of $(\mathfrak{c} \times \omega) \times (\mathfrak{c} \times \omega)$ by

$$Z = \{((\alpha, k), (\beta, k)) : \alpha < \mathfrak{c},\ \beta < \mathfrak{c},\ k \in A_\alpha \cap B_\beta\}.$$

(a) Let us show that $(\infty, \infty) \in \overline{Z}$. Suppose not and choose a mapping $\varphi \in {}^\mathfrak{c}\omega$ such that $(V_\varphi \times V_\varphi) \cap Z = \emptyset$. Put $M = \bigcup\{A_\alpha \setminus (1 + \varphi(\alpha)) : \alpha < \mathfrak{c}\}$. We shall reach the contradiction by showing that this set M has both properties from Observation 6.

Suppose that the set M meets only finitely many sets C_n in an infinite set and define a mapping $f : \omega \longrightarrow \omega$ by the rule $f(n) = 0$ if $M \cap C_n$ is infinite and $f(n) = 1 + \max(M \cap C_n)$ if $M \cap C_n$ is finite. By Observation 3, some $A \in \mathcal{A}$ meets the set $T(f)$ in an infinite set. We have $A = A_\alpha$ for some $\alpha < \mathfrak{c}$. Both sets $\bigcup\{A_\alpha \cap C_n : C_n \cap M \text{ is infinite}\}$ and $\varphi(\alpha)$ are finite, so there is some $k \in A_\alpha \cap T(f)$

not belonging to any of them. Let $n \in \omega$ be such that $k \in C_n$. Then $k \geq f(n)$, since $k \in T(f)$ and $k < f(n)$ since $k \in A_\alpha \setminus (1+\varphi(\alpha)) \subseteq M$. This contradiction shows that (a) from Observation 6 holds true.

If $\beta < \mathfrak{c}$ is arbitrary and $k \in M \cap B_\beta \setminus (1+\varphi(\beta))$, then there is some $\alpha < \mathfrak{c}$ such that $k \in A_\alpha \setminus (1+\varphi(\alpha))$. This means that $k \in A_\alpha \cap B_\beta$, hence $((\alpha,k),(\beta,k)) \in Z$, but it also means that $(\alpha,k) \in V_\varphi$, $(\beta,k) \in V_\varphi$, which contradicts the assumption $Z \cap (V_\varphi \times V_\varphi) = \emptyset$. Consequently, $B_\beta \cap M \subseteq (1+\varphi(\beta))$, so $B_\beta \cap M =^* \emptyset$.

Since $\beta < \mathfrak{c}$ was arbitrary, we have verified that the set M satisfies also (b) from Observation 6. But this is absurd, because a pair X, Y is a standard counterexample.

(b) Let us verify that $(\forall E \in [\mathfrak{c}]^{<\mathfrak{c}})\ (\infty,\infty) \notin \overline{Z \cap ((E \times \omega) \times (\mathfrak{c} \times \omega))}$.

The assumption $\mathfrak{b} = \mathfrak{c}$ implies that $\mathfrak{c}$ is regular, so it is enough to show that for all $\gamma < \mathfrak{c}$ there is a neighborhood V_φ such that $(V_\varphi \times V_\varphi) \cap Z \cap ((\gamma \times \omega) \times (\mathfrak{c} \times \omega)) = \emptyset$. Consider the family $\{A_\alpha : \alpha < \gamma\}$. Since all sets A_α are almost disjoint from all sets C_n, $n \in \omega$, and since we have $\gamma < \mathfrak{b}$, there is a function $f \in {}^\omega\omega$ such that $A_\alpha \subseteq^* L(f)$ for all $\alpha < \gamma$. By Observations 5 and 6, $A_\alpha \subseteq^* P(f)$ whenever $\alpha < \gamma$, and $B_\alpha \cap P(f) =^* \emptyset$ for all $\alpha < \mathfrak{c}$.

Define a mapping $\varphi : \mathfrak{c} \longrightarrow \omega$ by $\varphi(\alpha) = \min\{n : (A_\alpha \setminus P(f)) \cup (B_\alpha \cap P(f)) \subseteq n\}$ for $\alpha < \gamma$, and by $\varphi(\alpha) = \min\{n : B_\alpha \cap P(f) \subseteq n\}$ for $\gamma \leq \alpha < \mathfrak{c}$.

It remains to verify that $(V_\varphi \times V_\varphi) \cap Z \cap ((\gamma \times \omega) \times (\mathfrak{c} \times \omega)) = \emptyset$. For $\alpha < \gamma$ and $\beta < \mathfrak{c}$, if $((\alpha,k),(\beta,k)) \in V_\varphi \times V_\varphi$ and $k \notin A_\alpha$, then $((\alpha,k),(\beta,k)) \notin Z$. On the other hand, if $k \in A_\alpha$, then $k \in P(f)$ because $k \in A_\alpha \setminus \varphi(\alpha)$, so $k \notin B_\beta \setminus \varphi(\beta)$, also, $k \notin \varphi(\beta)$ because $(\beta,k) \in V_\varphi$, and hence $k \notin B_\beta$, and so, again, $((\alpha,k),(\beta,k)) \notin Z$.

We shall not repeat the argument to show that
$(\infty,\infty) \notin \overline{Z \cap ((\mathfrak{c} \times \omega) \times (E \times \omega))}$.

So the set Z is $(\mathfrak{c},\mathfrak{c})$-good. □

The reformulation of Theorem 2 reads as follows.

Theorem 2. *Assume $\mathfrak{b} = \mathfrak{c}$ + there is no $(\mathfrak{c},\mathfrak{c})$-good set in $F_\mathfrak{c} \times F_\mathfrak{c}$. If X, Y are Fréchet α_4-spaces and $X \times Y$ is Fréchet, then $X \times Y$ is α_4.* □

Unfortunately, we do not know the consistency of the assumptions. By [LL], adding weakly compact many Cohen reals gives a model of ZFC without $(\mathfrak{c},\mathfrak{c})$-good sets. But adding $\kappa \geq \omega_1$ Cohen reals makes $\mathfrak{b} = \omega_1$.

Concluding remarks. By authors' opinion, there is only one way how to prove that Nogura's conjecture may be consistently true: Find a set with the properties (a) and (b) from Observation 6, killing thus a possible standard counterexample. It is not necessary to separate $\mathcal{A} \setminus \{C_n : n \in \omega\}$ from $\mathcal{B} \setminus \{C_n : n \in \omega\}$ as we already did in the proof. An additional assumption of having a counterexample strong allowed us to reach a coherent situation; such a luck is hard to expect in general case, even under OCA.

Concerning the example, one may ask a question if it can be produced in ZFC. Let us remark that any example satisfying (1), (2), (3) implies the existence of a completely separable MAD family on ω. Completely separable MAD families were introduced in [ES] and they are known to exist in various models of set theory, but the existence of just one in ZFC is still an open problem.

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(*Received* December 18, 2000, *revised* July 21, 2001)