Comment.Math.Univ.Carolin. 43,2 (2002)379–384

A new rank formula for idempotent matrices with applications

Yongge Tian, George P.H. Styan

Abstract. It is shown that

$$\mathrm{rank}(P^*AQ) = \mathrm{rank}(P^*A) + \mathrm{rank}(AQ) - \mathrm{rank}(A),$$

where A is idempotent, $[P, Q]$ has full row rank and $P^*Q = 0$. Some applications of the rank formula to generalized inverses of matrices are also presented.

Keywords: Drazin inverse, group inverse, idempotent matrix, inner inverse, rank, tripotent matrix

Classification: 15A03, 15A09

A complex square matrix A is said to be idempotent, or a projector, whenever $A^2 = A$; when A is idempotent, and Hermitian (or real symmetric), it is often called an orthogonal projector, otherwise an oblique projector. Projectors are closely linked to generalized inverses of matrices. For example, for a given matrix A the product $P_A = AA^+$ is well known as the orthogonal projector on the range (column space) of A, where A^+ is the Moore-Penrose inverse of A; which is the unique solution of the following four Penrose equations

$$\text{(i) } AA^+A = A, \quad \text{(ii) } A^+AA^+ = A^+, \quad \text{(iii) } (AA^+)^* = AA^+, \quad \text{(iv) } (A^+A)^* = A^+A.$$

In addition, the products $AA^\#$, AA^D and AA^- are also idempotent matrices, where $A^\#$, A^D and A^- are the group inverse, the Drazin inverse, and an inner inverse of A, respectively. In a recent paper by Drury, Liu, Lu, Puntanen and Styan [1], a rank formula for the orthogonal projector P_A is established as follows

$$\mathrm{rank}(P^*AA^+Q) = \mathrm{rank}(AP) + \mathrm{rank}(AQ) - \mathrm{rank}(A), \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is Hermitian nonnegative definite, $P \in \mathbb{C}^{n\times p}$ and $Q \in \mathbb{C}^{n\times q}$ such that $[\,P,\ Q\,]$ has full row rank and $P^*Q = 0$. This formula can help to establish several useful rank equalities for block matrices and orthogonal projectors when X and Y are properly chosen, see Drury *et al.* [1] and Tian [2]. This work motivates us to consider in general the rank of P^*AQ and various related topics, where A is idempotent, $[\,P,\ Q\,]$ has full row rank and $P^*Q = 0$. To do so, we need the following result.

Lemma 1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{k\times l}$ be given. Then*

$$\operatorname{rank}(ABC) = \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B) \tag{2}$$

holds if and only if there are matrices X and Y such that $BCX + YAB = B$.

In fact it is well known that the equation $AX + YB = C$ is consistent if and only if

$$\operatorname{rank}\begin{bmatrix} C & A \\ B & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

Applying this result to the equation $BCX + YAB = B$, we obtain Lemma 1.

Our main results are given below.

Theorem 2. *Let $A \in \mathbb{C}^{m\times m}$ be an idempotent matrix, and let $P \in \mathbb{C}^{m\times p}$ and $Q \in \mathbb{C}^{m\times q}$ be any two matrices such that $[\,P,\ Q\,]$ has full row rank and $P^*Q = 0$. Then*

$$\operatorname{rank}(P^*AQ) = \operatorname{rank}(P^*A) + \operatorname{rank}(AQ) - \operatorname{rank}(A). \tag{3}$$

Proof: Since $[\,P,\ Q\,]$ has full row rank and $P^*Q = 0$, it follows that

$$[\,P,\ Q\,]^+ = \begin{bmatrix} P^+ \\ Q^+ \end{bmatrix} \quad \text{and} \quad [\,P,\ Q\,][\,P,\ Q\,]^+ = PP^+ + QQ^+ = I_m.$$

Let $X = Q^+A$ and $Y = A(P^+)^*$. Then we have

$$AQX + YP^*A = AQQ^+A + A(P^+)^*P^*A = A(\,I_m - PP^+\,)A + APP^+A = A.$$

Applying Lemma 1 to this equality yields (3). $\square$

Now let $P = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$. Then $[\,P,\ Q\,]$ is of full row rank and $P^*Q = 0$. We derive from (3) the following result.

Corollary 3. *Suppose that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ A_{11} \in \mathbb{C}^{m\times m},\ A_{12} \in \mathbb{C}^{m\times k},\ A_{21} \in \mathbb{C}^{k\times m},\ A_{22} \in \mathbb{C}^{k\times k} \tag{4}$$

is an idempotent matrix. Then the rank of A satisfies the following two rank equalities

$$\operatorname{rank}(A) = \operatorname{rank}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} + \operatorname{rank}[\,A_{11},\ A_{12}\,] - \operatorname{rank}(A_{12}), \tag{5}$$

and

$$\text{(6)} \qquad \operatorname{rank}(A) = \operatorname{rank}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + \operatorname{rank}[\, A_{21},\ A_{22} \,] - \operatorname{rank}(A_{21}).$$

Moreover, if

$$\text{(7)} \qquad A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad A_{ij} \in \mathbb{C}^{t_i \times t_j}, \quad 1 \le i, j \le p$$

is idempotent, then the rank of A satisfies the rank equalities

$$\text{(8)} \quad \operatorname{rank}(A) = \operatorname{rank}(Q_{1i}) + \operatorname{rank}(Q_{i+1,p}) - \operatorname{rank}(Q_{i+1,i}), \qquad i = 1, 2, \ldots, p-1,$$

where

$$Q_{ij} = \begin{bmatrix} A_{i1} & \cdots & A_{ij} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pj} \end{bmatrix}, \qquad 1 \le i, j \le p.$$

The rank formulas in (8) are derived from (6). If the matrix A in (4) is an orthogonal projector, then (5) becomes

$$\operatorname{rank}(A) = \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22}) - \operatorname{rank}(A_{12}).$$

If we replace the idempotent matrix A in (5) by the idempotent matrix $I_{m+k} - A$, and note that $\operatorname{rank}(I_{m+k} - A) = m + k - \operatorname{rank}(A)$, then (5) becomes

$$\operatorname{rank}(A) = m + k + \operatorname{rank}(A_{12}) - \operatorname{rank}[\, I_m - A_{11},\ A_{12} \,] - \operatorname{rank}\begin{bmatrix} A_{12} \\ I_k - A_{22} \end{bmatrix}.$$

The Drazin inverse A^D of a square matrix A with $\operatorname{index}(A) = k$ is defined to be the unique solution of the three matrix equations

$$\text{(i)} \quad A^k X A = A^k, \quad \text{(ii)} \quad XAX = X, \quad \text{(iii)} \quad AX = XA.$$

When $\operatorname{index}(A) = 1$, i.e., $\operatorname{rank}(A^2) = \operatorname{rank}(A)$, A^D is called the group inverse of A and denoted by $A^{\#}$. From $A^D A A^D = A^D$ we see that $AA^D AA^D = AA^D$. Thus AA^D is idempotent. In addition, $\operatorname{rank}(A^D) = \operatorname{rank}(AA^D) = \operatorname{rank}(A^k)$. In this case, applying Theorem 2 to $P^* AA^D Q$ and $P^* AA^{\#} Q$, we get the following corollary.

Corollary 4. *Let $A \in \mathbb{C}^{m\times m}$ be given with* $\operatorname{index}(A) = k$, *let $P \in \mathbb{C}^{m\times p}$ and $Q \in \mathbb{C}^{m\times q}$ be any two matrices such that $[\,P,\ Q\,]$ has full row rank and $P^*Q = 0$. Then*

$$\operatorname{rank}(P^*AA^DQ) = \operatorname{rank}(P^*A^k) + \operatorname{rank}(A^kQ) - \operatorname{rank}(A^k). \tag{9}$$

In particular,

$$\operatorname{rank}(P^*AA^{\#}Q) = \operatorname{rank}(P^*A) + \operatorname{rank}(AQ) - \operatorname{rank}(A). \tag{10}$$

Let $P = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$ in (10). We also have the following corollary.

Corollary 5. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m\times m},\ A_{12} \in \mathbb{C}^{m\times k},\ A_{21} \in \mathbb{C}^{k\times m},\ A_{22} \in \mathbb{C}^{k\times k}$$

with $\operatorname{index}(A) = 1$, *and denote by $(AA^{\#})_{12}$ the upper-right $m \times k$ block of the projector $AA^{\#}$. Then the rank of $(AA^{\#})_{12}$ is*

$$\operatorname{rank}[(AA^{\#})_{12}] = \operatorname{rank}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} + \operatorname{rank}[\,A_{11},\ A_{12}\,] - \operatorname{rk}(A). \tag{11}$$

A square matrix A is called tripotent if $A^3 = A$. For the tripotent matrix A, its group inverse is $A^{\#} = A$. Now applying (9) to a tripotent matrix A and noting that

$$(AA^{\#})_{12} = (A^2)_{12} = [\,A_{11},\ A_{12}\,]\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix},$$

we obtain the following result.

Corollary 6. *Suppose that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m\times m},\ A_{12} \in \mathbb{C}^{m\times k},\ A_{21} \in \mathbb{C}^{k\times m},\ A_{22} \in \mathbb{C}^{k\times k}$$

is a tripotent matrix. Then the rank of A satisfies the following two rank equalities

$$\operatorname{rank}(A) = \operatorname{rank}[\,A_{11},\ A_{12}\,] + \operatorname{rank}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} - \operatorname{rank}\left([\,A_{11},\ A_{12}\,]\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}\right), \tag{12}$$

and

$$\operatorname{rank}(A) = \operatorname{rank}[\,A_{21},\ A_{22}\,] + \operatorname{rank}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} - \operatorname{rank}\left([\,A_{21},\ A_{22}\,]\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\right). \tag{13}$$

Finally, we present a result for a triangular inner inverse of an idempotent matrix. We will use the following result due to Tian [3, Corollary 4.3].

Lemma 7. *The block matrix*

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad \text{where } \ A_{ij}\in\mathbb{C}^{s_i\times t_j},\ 1\le i,\ j\le p,$$

has an inner inverse with the upper triangular block form

$$A^{-}=\begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ & S_{22} & \cdots & S_{2p} \\ & & \ddots & \vdots \\ & & & S_{pp} \end{bmatrix}, \quad S_{ij}\in\mathbb{C}^{t_i\times s_j}, \quad 1\le i,\ j\le p$$

if and only if

$$\operatorname{rank}(A)=\operatorname{rank}(Q_{1i})+\operatorname{rank}(Q_{i+1,p})-\operatorname{rank}(Q_{i+1,i}), \qquad i=1,2,\ldots,p-1,$$

where

$$Q_{ij}=\begin{bmatrix} A_{i1} & \cdots & A_{ij} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pj} \end{bmatrix}, \quad 1\le i,\ j\le p.$$

Applying this lemma to the idempotent block matrix A in (7) under (8), we immediately see that

Theorem 8. *If the block matrix*

$$A=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad \text{where } \ A_{ij}\in\mathbb{C}^{t_i\times t_j}, \quad 1\le i,\ j\le p,$$

is idempotent, then it must have an inner inverse with the upper triangular block form

$$A^{-}=\begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ & S_{22} & \cdots & S_{2p} \\ & & \ddots & \vdots \\ & & & S_{pp} \end{bmatrix}, \quad S_{ij}\in\mathbb{C}^{t_i\times t_j}, \quad 1\le i,\ j\le p.$$

In particular, if

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ \text{where } \ A_{11}\in\mathbb{C}^{m\times m}, A_{12}\in\mathbb{C}^{m\times k}, A_{21}\in\mathbb{C}^{k\times m}, A_{22}\in\mathbb{C}^{k\times k},$$

is idempotent, then it must have an inner inverse with the upper triangular block form

$$A^- = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}, \quad G_{11} \in \mathbb{C}^{m\times m},\ G_{12} \in \mathbb{C}^{m\times k},\ \ G_{22} \in \mathbb{C}^{k\times k}.$$

For more rank equalities for idempotent matrices, see the authors' recent paper [4].

Acknowledgment. The research of both authors was supported in part by the Natural Sciences and Engineering Research Council of Canada.

References

[1] Drury S.W., Liu S., Lu C.Y., Puntanen S., Styan G.P.H., *Some comments on several matrix inequalities with applications to canonical correlations: historical background and recent developments*, Report A332 (December 2000), Dept. of Mathematics, Statistics and Philosophy, University of Tampere, Tampere, Finland, 63 pp. To be published in the special issue of *Sankhyā: The Indian Journal of Statistics, Series A* associated with "An International Conference in Honor of Professor C.R. Rao on the Occasion of his 80th Birthday, Statistics: Reflections on the Past and Visions for the Future, The University of Texas at San Antonio, March 2000".

[2] Tian Y., *Two rank equalities associated with blocks of orthogonal projector. Problem* 25–4, Image, The Bulletin of the International Linear Algebra Society **25** (2000), p. 16 [Solutions by J.K. Baksalary & O.M. Baksalary, by H.J. Werner, and by S. Puntanen, G.P.H. Styan & Y. Tian, Image, The Bulletin of the International Linear Algebra Society **26** (2001), 6–9].

[3] Tian Y., *Completing block matrices with maximal and minimal ranks*, Linear Algebra Appl. **321** (2000), 327–345.

[4] Tian Y., Styan, G.P.H., *Some rank equalities for idempotent and involutory matrices*, Linear Algebra Appl. **335** (2001), 101–117.

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L 3N6, Canada

E-mail: ytian@mast.queensu.ca

Department of Mathematics and Statistics, McGill University, Montréal, Québec H3A 2K6, Canada

E-mail: styan@together.net

(*Received* May 25, 2001, *revised* January 28, 2002)