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Curvature homogeneous spaces whose curvature tensors have large symmetries

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Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. We study curvature homogeneous spaces or locally homogeneous spaces whose curvature tensors are invariant by the action of "large" Lie subalgebras $\mathfrak{h}$ of $\mathfrak{so}(n)$. In this paper we deal with the cases of $\mathfrak{h} = \mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$ $(2 \leq r \leq n-r)$, $\mathfrak{so}(n-2)$, and the Lie algebras of Lie groups acting transitively on spheres, and classify such curvature homogeneous spaces or locally homogeneous spaces.

Keywords: locally homogeneous spaces, curvature homogeneous spaces, totally geodesic foliations

Classification: 53C30, 53B20

1. Introduction

In this paper we discuss the relation between several kinds of homogeneity of a Riemannian manifold and the curvature tensor. A Riemannian manifold M is said to be (*globally*) *homogeneous* if for any two points $p, q \in M$ there exists an isometry of M which maps p to q. On the other hand, it is called *locally homogeneous* if for any two points $p, q \in M$ there exist a neighborhood U of p and a neighborhood V of q and an isometry of U onto V which maps p to q. The notion of a curvature homogeneous space was introduced by I.M. Singer [13] in his theory of infinitesimally homogeneous spaces, which gives a sufficient condition of a Riemannian manifold to be homogeneous or locally homogeneous. Now we review his theory. Given a Riemannian manifold $(M, \langle , \rangle)$, we denote by R and $\nabla^i R$ the curvature tensor and its i-th covariant derivative. We consider the following condition:

$P(l)$: for every $p, q \in M$ there exists a linear isometry $\phi : T_pM \to T_qM$ such that

$$\phi^*(\nabla^i R)_q = (\nabla^i R)_p \quad i = 0, 1, \ldots, l.$$

If M is locally homogeneous, then M obviously satisfies $P(l)$ for any l. It is enough to take the differential of a local isometry which maps p to q as ϕ. In particular, M is called to be *curvature homogeneous* if M satisfies $P(0)$.

We denote by $\mathfrak{so}(T_pM)$ the Lie algebra of endomorphisms of T_pM which are skew-symmetric with respect to $\langle,\rangle$. For a non-negative integer l, we define a Lie subalgebra $\mathfrak{g}_l(p)$ of $\mathfrak{so}(T_pM)$ by

$$\mathfrak{g}_l(p) = \{A \in \mathfrak{so}(T_pM) \mid A \cdot (\nabla^i R)_p = 0, \quad i = 0, 1, \ldots, l\},$$

where A acts as a derivation on the tensor algebra on T_pM. Since $\mathfrak{g}_l(p) \supseteq \mathfrak{g}_{l+1}(p)$, there exists the first integer $k(p)$ such that $\mathfrak{g}_{k(p)}(p) = \mathfrak{g}_{k(p)+1}(p)$. Namely we have

$$\mathfrak{so}(T_pM) \supseteq \mathfrak{g}_0(p) \supsetneq \mathfrak{g}_1(p) \supsetneq \mathfrak{g}_2(p) \supsetneq \cdots \supsetneq \mathfrak{g}_{k(p)}(p) = \mathfrak{g}_{k(p)+1}(p).$$

Following Singer, we say that $(M, \langle,\rangle)$ is *infinitesimally homogeneous* if M satisfies $P(k(p)+1)$ for some point $p \in M$. If M satisfies $P(l)$, then the linear isometry ϕ induces a Lie algebra isomorphism of $\mathfrak{g}_i(p)$ to $\mathfrak{g}_i(q)$ for $i = 0, 1, \ldots, l$. Therefore, if M is infinitesimally homogeneous, $k(q)$ does not depend on $q \in M$. We put $k_M = k(p)$ and call it *the Singer invariant* of an infinitesimally homogeneous space M.

If M is locally homogeneous, then M evidently satisfies $P(l)$ for any l and in particular M is infinitesimally homogeneous. Singer proved the converse ([13]) (see also L. Nicolodi and F. Tricerri [10]): A connected infinitesimally homogeneous space M is locally homogeneous. As a global version, he proved that a connected, simply connected, complete infinitesimally homogeneous space M is homogeneous. Among others he also posed the following question: Do there exist curvature homogeneous spaces which are not locally homogeneous? At present, many such curvature homogeneous spaces are known. E. Boeckx, O. Kowalski, and L. Vanhecke [1, Chapter 12] give a good survey on recent developments of this subject. Various classification problems of curvature homogeneous spaces naturally arise. You can find many interesting problems in [1]. In this paper we discuss the following problem:

Problem. *Classify curvature homogeneous spaces whose $\mathfrak{g}_0$ are large, where $\mathfrak{g}_0$ denotes the Lie subalgebra of $\mathfrak{so}(T_pM)$ defined as previously.*

What does "large" mean? Now we consider three kinds of candidates.

(I) Lie algebras of large dimensions.

It is known that a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{so}(n)$ with $\dim \mathfrak{h} > \frac{1}{2}(n-3)(n-4)+3$ is conjugate to $\mathfrak{so}(n-1)$, $\mathfrak{so}(n-2) \oplus \mathfrak{so}(2)$, $\mathfrak{so}(n-2)$ with a few exceptions for low dimensions (< 14). See D. Montgomery and H. Samelson [9] and M. Obata [11].

(II) Lie algebras of Lie groups acting transitively on spheres.

They are classified by D. Montgomery and H. Samelson [9] and A. Borel [2]. We will show the list in a table after.

(III) Maximal subalgebras.

(i) A. Borel and J. De Siebenthal [3] classified Lie subalgebras of maximal rank in compact simple Lie algebras. By their classification, it follows that maximal subalgebras of maximal rank in $\mathfrak{so}(2n+1)$ are $\mathfrak{so}(2r) \oplus \mathfrak{so}(2(n-r)+1)$ $(1 \leq r \leq n-1)$, $\mathfrak{so}(2n)$ and that those in $\mathfrak{so}(2n)$ are $\mathfrak{so}(2r) \oplus \mathfrak{so}(2(n-r))$ $(1 \leq r \leq n-1)$, $\mathfrak{u}(n)$.

(ii) Let $N = G/K$ be a compact, simply connected, effective, irreducible symmetric space and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition. We denote by π the isotropy representation of $\mathfrak{k}$ on $\mathfrak{p}$ and by $\pi(\mathfrak{k})^{\perp}$ the orthogonal complement of $\pi(\mathfrak{k})$ in $\mathfrak{so}(\mathfrak{p})$. It is shown by M. Wang and W. Ziller [17, Theorem 3.1(c)] that if K is simple, the representation of $\mathfrak{k}$ on $\pi(\mathfrak{k})^{\perp}$ is irreducible. This implies that $\pi(\mathfrak{k})$ is a maximal Lie subalgebra in $\mathfrak{so}(\mathfrak{p})$.

The case of $\mathfrak{so}(n-1)$ has been already investigated in Y. Kiyota and K. Tsukada [8, Corollary 2.3]. That is, an $n(\geq 4)$-dimensional curvature homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(n-1)$ is locally isometric to a Riemannian product $\mathbb{R} \times M^{n-1}(c)$ $(c \neq 0)$, where $M^{n-1}(c)$ denotes an $n-1$-dimensional real space form of constant curvature c. In Section 2, we will deal with the case of $\mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$ $(2 \leq r \leq n-r)$ and $\mathfrak{so}(n-2)$ and classify the curvature homogeneous spaces or locally homogeneous spaces whose $\mathfrak{g}_0$ are conjugate to the above Lie subalgebras of $\mathfrak{so}(n)$ (Theorem 2.1, Corollary 2.2, Theorems 2.5 and 2.6). In Section 3, we study the case of Lie algebras of Lie groups acting transitively on spheres and show Theorem 3.1. For the case of (III)(ii) we have no complete answer and only give a comment in the final section.

I am very interested in the Singer invariant of homogeneous spaces. At our knowledge, there are only a few homogeneous spaces whose Singer invariants are known and their Singer invariants are all at most 1 (cf. Tsukada [16]). So we would like to know if there exists a homogeneous space whose Singer invariant is not less than 2. This motivates our research of the problem above. However, unfortunately we could not find such homogeneous spaces in this paper.

2. The case of $\mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$ and $\mathfrak{so}(n-2)$

Let $\mathbb{R}^n$ be an n-dimensional vector space with a usual inner product $\langle , \rangle$ and $\mathcal{R}$ be the space of algebraic curvature tensors on $\mathbb{R}^n$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{so}(n)$ which corresponds to a closed subgroup of $O(n)$. We denote by $\mathcal{R}^{\mathfrak{h}}$ the subspase of $\mathcal{R}$ which consists of curvature tensors invariant by $\mathfrak{h}$ and by $M^n(c)$ an n-dimensional real space form of constant sectional curvature c.

Theorem 2.1. *Let M be an $n(\geq 5)$-dimensional curvature homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$ $(2 \leq r \leq n-r)$. Then M is locally isometric either to a Riemannian product $M^r(\alpha) \times M^{n-r}(\beta)$ (either α or β is not zero) or to a curvature homogeneous semi-symmetric space modeled on the curvature tensor of $M^2(\alpha) \times \mathbb{R}^{n-2}$ $(\alpha \neq 0)$.*

Curvature homogeneous semi-symmetric spaces modeled on the curvature tensor of $M^2(\alpha) \times \mathbb{R}^{n-2}$ $(\alpha \neq 0)$ are completely classified by Boeckx, Kowalski, and Vanhecke [1, Chapter 4]. Moreover they proved that locally homogeneous semi-symmetric spaces are locally symmetric ([1, Proposition 4.23]). So we obtain the following.

Corollary 2.2. *Let M be an $n(\geq 5)$-dimensional locally homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$ $(2 \leq r \leq n-r)$. Then M is locally isometric to a Riemannian product $M^r(\alpha) \times M^{n-r}(\beta)$ (either α or β is not zero).*

Proof of Theorem 2.1: We put $\mathfrak{h} = \mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$. Let $\mathbb{R}^n = V_1 \oplus V_2$ be an orthogonal decomposition into $\mathfrak{h}$-invariant subspaces V_1 and V_2 of dimensions r and $n-r$, respectively. First we determine $\mathfrak{h}$-invariant curvature tensors. For $R \in \mathcal{R}^{\mathfrak{h}}$, there exist $\alpha, \beta, \delta \in \mathbb{R}$ such that R is expressed as follows:

$$\begin{aligned} R(x,y)z &= \alpha\{\langle y,z\rangle x - \langle x,z\rangle y\}, \\ R(x,u)y &= \beta\{-\langle x,y\rangle u\}, \\ R(x,u)v &= \beta\{\langle u,v\rangle x\}, \\ R(u,v)w &= \delta\{\langle v,w\rangle u - \langle u,w\rangle v\}, \\ \text{the others} &= 0, \end{aligned}$$

for $x, y, z \in V_1$ and $u, v, w \in V_2$.
In particular, $\dim \mathcal{R}^{\mathfrak{h}} = 3$. $\mathfrak{h} = \mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$ is a maximal subalgebra of $\mathfrak{so}(n)$, that is, the only Lie subalgebra of $\mathfrak{so}(n)$ which contains $\mathfrak{h}$ properly is $\mathfrak{so}(n)$. We remark that $R \in \mathcal{R}^{\mathfrak{h}}$ which has the form above is invariant by $\mathfrak{so}(n)$ if and only if $\alpha = \beta = \delta$. So we assume that $\alpha \neq \beta$ or $\beta \neq \delta$.

Let M be an $n(\geq 5)$-dimensional curvature homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(r) \oplus \mathfrak{so}(n-r)$ $(2 \leq r \leq n-r)$. Since our assertion is local, we may assume that M is connected and simply connected. Let $O(M)$ be an orthonormal frame bundle over M with the projection $\pi : O(M) \to M$. By the curvature homogeneity, $\{u \in O(M) \mid u^* R_{\pi(u)} = R\}$ is a closed submanifold of $O(M)$, where R in the right hand side is a curvature tensor of $\mathcal{R}^{\mathfrak{h}}$ which has the form above. We denote by P its connected component. Then P is a principal subbundle of $O(M)$ with structure group $H = SO(r) \times SO(n-r)$. We can define an r-dimensional distribution $\mathcal{D}_1$ and an $(n-r)$-dimensional distribution $\mathcal{D}_2$ on M by $(\mathcal{D}_1)_{\pi(u)} = u(V_1)$ and $(\mathcal{D}_2)_{\pi(u)} = u(V_2)$ for $u \in P$. Next we will define a new connection. Let ω be the Riemannian connection form on $O(M)$ with the corresponding covariant derivative ∇. Let $\mathfrak{so}(n) = \mathfrak{h} + \mathfrak{h}^{\perp}$ be an orthogonal decomposition with respect to an $\mathrm{Ad}(SO(n))$-invariant inner product on $\mathfrak{so}(n)$ and we denote by $\omega_{\mathfrak{h}}$ the $\mathfrak{h}$-component of ω. Putting $\widetilde{\omega} = \omega_{\mathfrak{h}}$ on P, we obtain a connection form $\widetilde{\omega}$ on P. We denote by $\widetilde{\nabla}$ the corresponding covariant derivative. Then $\widetilde{\nabla} R = 0$. Moreover $\mathcal{D}_1$ and $\mathcal{D}_2$ are parallel with respect to $\widetilde{\nabla}$. (These arguments are due to Singer [13]).

We define a tensor field $\widetilde{S}$ of type $(1,2)$ by $\widetilde{S} = \nabla - \widetilde{\nabla}$ and put $S = u^*\widetilde{S}$ at $u \in P$. Then S is a tensor of type $(1,2)$ on $\mathbb{R}^n$ and it is viewed as a linear map of $\mathbb{R}^n$ to $\mathfrak{h}^\perp$. S is characterized by

$$(u^*(\nabla R)_{\pi(u)})(X;Y,Z)W = (S_X \cdot R)(Y,Z)W \quad \text{for} \quad X,Y,Z,W \in \mathbb{R}^n \quad \text{at} \quad u \in P.$$

By the Bianchi's second identity, we have

$$(2.1) \qquad (S_X \cdot R)(Y,Z)W + (S_Y \cdot R)(Z,X)W + (S_Z \cdot R)(X,Y)W = 0.$$

This system of linear equations with respect to S is crucial for our proof. This constrains S. Let $\{e_1, \dots, e_r\}$ and $\{e_{r+1}, \dots, e_n\}$ be orthonormal bases of V_1 and V_2, respectively. We use the following range of indices: $A, B, C, \dots \in \{1, \cdots, n\}$, $i, j, k, \dots \in \{1, \cdots, r\}$, $a, b, c, \dots \in \{r+1, \cdots, n\}$. We define $S_{AB}^{\ \ C}$ by $S_{e_A}(e_B) = \sum_{C=1}^n S_{AB}^{\ \ C} e_C$. Since S is $\mathfrak{h}^\perp$-valued, we have $S_{Ai}^{\ \ j} = 0$, $S_{Aa}^{\ \ b} = 0$.

Lemma 2.3. *By* (2.1) *we have the following:*

(1) $(\alpha - \beta)\{S_{jh}^{\ \ a}\delta_{kl} - S_{jl}^{\ \ a}\delta_{kh} - S_{kh}^{\ \ a}\delta_{jl} + S_{kl}^{\ \ a}\delta_{jh}\} = 0$,
(2) $(\alpha - \beta)\{S_{aj}^{\ \ b}\delta_{kh} - S_{ak}^{\ \ b}\delta_{jh}\} = 0$,
(3) $(\delta - \beta)\{S_{ij}^{\ \ a}\delta_{bc} - S_{ij}^{\ \ b}\delta_{ac}\} = 0$,
(4) $(\delta - \beta)\{-S_{ai}^{\ \ c}\delta_{bd} + S_{ai}^{\ \ d}\delta_{bc} + S_{bi}^{\ \ c}\delta_{ad} - S_{bi}^{\ \ d}\delta_{ac}\} = 0$.

Lemma 2.4. (1) *If $r \geq 3$ or $r = 2$ and $\beta \neq \delta$, then S vanishes.*
(2) *If $r = 2$ and $\beta = \delta$ and $\alpha \neq \beta$, then we have*

$$S_{ai}^{\ \ b} = 0 \quad (i = 1,2, \ a,b = 3, \cdots, n), \qquad S_{11}^{\ \ a} + S_{22}^{\ \ a} = 0 \quad (a = 3, \cdots, n).$$

Proof of Lemma 2.4: We apply the identities of Lemma 2.3. First we assume that $\alpha \neq \beta$. Then by Lemma 2.3(2), $S_{aj}^{\ \ b} = 0$. When $r \geq 3$, we take mutually different integers j, h, k $(1 \leq j, h, k \leq r)$ and put $l = k$ in Lemma 2.3(1). Then $S_{jh}^{\ \ a} = 0$. Putting $j = h \neq k = l$ in Lemma 2.3(1), we have $S_{jj}^{\ \ a} + S_{kk}^{\ \ a} = 0$. For mutually different integers i, j, k, $S_{ii}^{\ \ a} = -S_{jj}^{\ \ a} = S_{kk}^{\ \ a} = -S_{ii}^{\ \ a}$. Therefore $S_{ii}^{\ \ a} = 0$. Consequently S vanishes. When $r = 2$, we have only the equation $S_{11}^{\ \ a} + S_{22}^{\ \ a} = 0$. Next we assume that $\delta \neq \beta$. We note that by our assumption, $n - r \geq 3$. Therefore, by a similar argument to the above, we see that S vanishes. $\square$

We continue the proof of Theorem 2.1. We consider the following two cases:

Case 1. $\widetilde{S} = 0$ at every point of M.

Case 2. $\widetilde{S} \neq 0$ at some point of M.

Case 1. $\widetilde{S}$ vanishes on M and hence we have $\widetilde{\nabla} = \nabla$. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are parallel with respect to the connection $\widetilde{\nabla}$, they are parallel with respect to the Riemannian connection ∇, too. In particular $\beta = \langle R(x,v)v, x\rangle = 0$ for $x \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$. M is locally isometric to a Riemannian product $M^r(\alpha) \times M^{n-r}(\delta)$, where $M^r(\alpha)$ and $M^{n-r}(\delta)$ denote the real space forms of constant sectional curvatures α and δ, respectively and either α or δ is not zero.

Case 2. By Lemma 2.4, this case occurs only when $r = 2$, $\beta = \delta$ and $\alpha \neq \beta$. We take a local orthonormal frame field $\{E_1, \cdots, E_n\}$ of M which gives a section of P. Then $\{E_1, E_2\}$ and $\{E_3, \cdots, E_n\}$ are local orthonormal frame fields of the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. $\mathcal{D}_2$ is parallel with respect to $\widetilde{\nabla}$. This, with Lemma 2.4, implies that $\langle \nabla_{E_a} E_b, E_i\rangle = \langle \widetilde{\nabla}_{E_a} E_b + \widetilde{S}_{E_a}(E_b), E_i\rangle = 0$ for $a, b = 3, \cdots, n$, $i = 1, 2$. This means that the distribution $\mathcal{D}_2$ is completely integrable and its integral submanifolds are totally geodesic. For this totally geodesic foliation, we define the conullity operator C as a smooth section of $\mathrm{Hom}(\mathcal{D}_2, \mathrm{End}(\mathcal{D}_1))$ (cf. D. Ferus [5]). We denote by $\mu : TM \to \mathcal{D}_1$ the orthogonal projection. Define a linear operator C of $(\mathcal{D}_2)_p$ into $\mathrm{End}((\mathcal{D}_1)_p)$ by

$$C_u x = -\mu(\nabla_x U) \quad \text{for} \quad x \in (\mathcal{D}_1)_p, u \in (\mathcal{D}_2)_p,$$

where U is a local vector field of $\mathcal{D}_2$ on M around p with $U_p = u$. Then we have

$$C_{E_a} E_i = -\mu(\nabla_{E_i} E_a) = -\mu(\widetilde{\nabla}_{E_i} E_a + \widetilde{S}_{E_i}(E_a)) = \sum_{j=1}^{2} S_{ij}{}^a E_j.$$

By Lemma 2.4, $\mathrm{tr}\, C_{E_a} = 0$ for $a = 3, \cdots, n$. Therefore for any $u \in \mathcal{D}_2$, $\mathrm{tr}\, C_u = 0$. We will show that $\det C_u = \beta$ for an arbitrary unit vector $u \in (\mathcal{D}_2)_p$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a unit speed geodesic such that $\gamma(0) = p$ and $\dot\gamma(0) = u$. Then γ is a curve in the integral submanifold of $\mathcal{D}_2$ through p. Let $\{\tilde{E}_1, \tilde{E}_2\}$ be a parallel orthonormal frame field of $\mathcal{D}_1$ along γ. We denote by $\widetilde{C} = (\tilde{c}_{ij}(t))$ the 2×2-matrix which represents the conullity operator $C_{\dot\gamma(t)}$ with respect to $\{\tilde{E}_1, \tilde{E}_2\}$. Let $R_{\dot\gamma(t)}$ be the Jacobi operator defined by $R_{\dot\gamma(t)}(x) = R(x, \dot\gamma(t))\dot\gamma(t)$. Then, because of the form of the curvature tensor, we have $R_{\dot\gamma(t)}(\mathcal{D}_1) \subset \mathcal{D}_1$ and $R_{\dot\gamma(t)} = \beta\, \mathrm{Id}$ on $\mathcal{D}_1$. We denote by $\widetilde{R}$ the 2×2-matrix which represents the Jacobi operator $R_{\dot\gamma(t)}$ with respect to $\{\tilde{E}_1, \tilde{E}_2\}$. Then it is known that the following identity holds (cf. Ferus [5]):

$$\frac{d\widetilde{C}}{dt} = \widetilde{C}^2 + \widetilde{R}. \tag{2.2}$$

Since $\mathrm{tr}\, \widetilde{C} = 0$ on $(-\varepsilon, \varepsilon)$, we have $\mathrm{tr}\left(\frac{d\widetilde{C}}{dt}\right) = \frac{d}{dt}\mathrm{tr}\, \widetilde{C} = 0$ and $\mathrm{tr}\, \widetilde{C}^2 = -2\det \widetilde{C}$. On the other hand $\mathrm{tr}\, \widetilde{R} = 2\beta$. By (2.2) it follows that $\det \widetilde{C} = \beta$. Therefore we see that $\det C_u = \beta$ for an arbitrary unit vector $u \in (\mathcal{D}_2)_p$. Then applying the similar argument to Tsukada [15, Corollary 3.4], we can show that $\beta = 0$. Consequently in this case, M is a curvature homogeneous space which has the same curvature tensor as $M^2(\alpha) \times \mathbb{R}^{n-2}$ $(\alpha \neq 0)$. $\square$

Next we construct an example of an n-dimensional homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(n-2)$. Let $\mathfrak{g}$ be an n-dimensional vector space with an inner product $\langle , \rangle$ and $\{e_1, e_2, \cdots, e_n\}$ be its orthonormal basis. On $\mathfrak{g}$ we define a bracket operation $[,]$ by

$$\begin{aligned} [e_1, e_2] &= \phi e_2 \\ [e_1, e_a] &= \lambda e_a \qquad \text{for } \ a = 3, \cdots, n \\ [e_i, e_j] &= 0 \qquad\ \ \text{for } \ i, j = 2, \cdots, n, \end{aligned}$$

where $\lambda, \phi \in \mathbb{R}$ such that $\lambda\phi \neq 0$ and $\lambda \neq \phi$. Then $\mathfrak{g}$ becomes a solvable Lie algebra. It is known as a semi-direct sum of $\mathbb{R}$ and the abelian ideal $\mathbb{R}^{n-1}$. It is easily seen that $\mathfrak{a}(\mathfrak{g}) = \mathfrak{so}(\mathfrak{g}, \langle , \rangle) \cap \mathrm{Der}(\mathfrak{g})$ is isomorphic to $\mathfrak{so}(n-2)$ (we use the assumption $\lambda\phi \neq 0$, $\lambda \neq \phi$). Let G be a connected simply connected Lie group with Lie algebra $\mathfrak{g}$. We induce a left invariant metric g on G associated to the inner product $\langle , \rangle$ on $\mathfrak{g}$.

By straightforward computations, we get the curvature tensor of (G, g). Under the identification of T_eG with $\mathfrak{g}$,

$$\begin{aligned} \langle R(e_1, e_2)e_2, e_1\rangle &= -\phi^2, \\ \langle R(e_a, e_1)e_1, e_b\rangle &= -\lambda^2\delta_{ab}, \\ \langle R(e_a, e_2)e_2, e_b\rangle &= -\lambda\phi\delta_{ab}, \\ \langle R(e_a, e_b)e_c, e_d\rangle &= -\lambda^2\{\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}\}, \\ \text{the others } &= 0, \end{aligned}$$

where $a, b, c, d = 3, \cdots, n$. By this, it follows that $\mathfrak{g}_0 = \mathfrak{so}(n-2) = \mathfrak{a}(\mathfrak{g})$. In particular the Singer invariant of (G, g) is 0.

Theorem 2.5. *Let M be an $n(\geq 5)$-dimensional curvature homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(n-2)$. Then M has the same curvature tensor as the above example (G, g) of suitable parameters λ, ϕ.*

This theorem does not give a complete answer to our problem. At present we cannot show whether or not there exist curvature homogeneous (not locally homogeneous) spaces which have the same curvature tensor as the above example (G, g). If such curvature homogeneous spaces exist, they have an interesting geometrical property, that is, they admit a codimension-one totally geodesic foliation whose leaves have constant negative sectional curvature (we will prove it later).

For locally homogeneous spaces we have the following.

Theorem 2.6. *Let M be an $n(\geq 5)$-dimensional locally homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(n-2)$. Then M is locally isometric to the above example (G, g) of suitable parameters λ, ϕ.*

In the remainder of this section we will prove the above theorems. We put $\mathfrak{h} = \mathfrak{so}(n-2)$. Let $\mathbb{R}^n = V_1 \oplus V_2$ be an orthogonal decomposition, where V_1 is a 2-dimensional subspace on which $\mathfrak{h}$ acts trivially. For $R \in \mathcal{R}^{\mathfrak{h}}$, there exist $\alpha, \beta_1, \beta_2, \beta_{12}, \delta \in \mathbb{R}$ such that R is expressed as follows:

$$\begin{aligned}
\langle R(e_1, e_2)e_2, e_1\rangle &= \alpha, \\
\langle R(u, e_1)e_1, v\rangle &= \beta_1\langle u, v\rangle, \\
\langle R(u, e_2)e_2, v\rangle &= \beta_2\langle u, v\rangle, \\
\langle R(u, e_1)e_2, v\rangle &= \langle R(u, e_2)e_1, v\rangle = \beta_{12}\langle u, v\rangle, \\
\langle R(u_1, u_2)u_3, u_4\rangle &= \delta\{\langle u_2, u_3\rangle\langle u_1, u_4\rangle - \langle u_1, u_3\rangle\langle u_2, u_4\rangle\}, \\
\text{the others } &= 0,
\end{aligned}$$

where $\{e_1, e_2\}$ is an orthonormal basis of V_1 and $u, v, u_i \in V_2$. In particular $\dim \mathcal{R}^{\mathfrak{h}} = 5$. Moreover we can choose an orthonormal basis $\{e_1, e_2\}$ of V_1 such that $\beta_{12} = 0$. It is easily seen that $\beta_1 = \beta_2$ if and only if R is invariant by the action of $\mathfrak{so}(2) \oplus \mathfrak{so}(n-2)$, where $\mathfrak{so}(2)$ acts on V_1. So we assume that $\beta_1 \neq \beta_2$. Under this assumption, R is invariant by the action of a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{so}(n)$ which is isomorphic to $\mathfrak{so}(n-1)$ and contains $\mathfrak{so}(n-2)$ if and only if $\alpha = \beta_1, \beta_2 = \delta$ or $\alpha = \beta_2, \beta_1 = \delta$. In the former case, $\mathbb{R}e_2 \oplus V_2$ is invariant by $\mathfrak{k}$ and in the latter case, $\mathbb{R}e_1 \oplus V_2$ is invariant by $\mathfrak{k}$.

Let M be an $n(\geq 5)$-dimensional simply connected curvature homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{h} = \mathfrak{so}(n-2)$. We trace the same way as the proof of Theorem 2.1. We construct a principal subbundle P of $O(M)$ over M with the structure group $H = SO(n-2)$. We can define unit vector fields E_1, E_2 and an $(n-2)$-dimensional distribution $\mathcal{D}$ on M by $E_i = u(e_i)$ $(i = 1, 2)$, $\mathcal{D} = u(V_2)$ for $u \in P$. We define a new connection $\widetilde{\omega}$ on P which is induced from the Riemannian connection ω. The unit vector fields E_1 and E_2 are parallel and the distribution $\mathcal{D}$ is parallel with respect to the covariant derivative $\widetilde{\nabla}$ corresponding to $\widetilde{\omega}$. For each $u \in P$, we define a tensor $S \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{h}^{\perp})$ which corresponds to the difference between ω and $\widetilde{\omega}$. Let $\{e_1, e_2\}$ be the orthonormal basis of V_1 which has been already fixed and $\{e_3, \cdots, e_n\}$ be an orthonormal basis of V_2.

Lemma 2.7. *By* (2.1) *we have the following:*

(1) $(\alpha - \beta_2)S_{11}^{\ a} + (\alpha - \beta_1)S_{22}^{\ a} = 0,$
(2) $(\alpha - \beta_1)S_{a2}^{\ b} + (\beta_1 - \beta_2)S_{11}^{\ 2}\delta_{ab} = 0,$
(3) $(\alpha - \beta_2)S_{a1}^{\ b} + (\beta_1 - \beta_2)S_{21}^{\ 2}\delta_{ab} = 0,$
(4) $(\delta - \beta_1)\{S_{11}^{\ a}\delta_{bc} - S_{11}^{\ b}\delta_{ac}\} = 0,$
(5) $(\delta - \beta_2)\{S_{22}^{\ a}\delta_{bc} - S_{22}^{\ b}\delta_{ac}\} = 0,$

(6) $(\beta_1-\beta_2)\{S_{a1}{}^2\delta_{bc}-S_{b1}{}^2\delta_{ac}\}+(\delta-\beta_2)\{S_{12}{}^a\delta_{bc}-S_{12}{}^b\delta_{ac}\}=0$,
(7) $(\beta_1-\beta_2)\{S_{a1}{}^2\delta_{bc}-S_{b1}{}^2\delta_{ac}\}+(\delta-\beta_1)\{S_{21}{}^a\delta_{bc}-S_{21}{}^b\delta_{ac}\}=0$,
(8) $(\delta-\beta_1)\{-S_{a1}{}^c\delta_{bd}+S_{a1}{}^d\delta_{bc}+S_{b1}{}^c\delta_{ad}-S_{b1}{}^d\delta_{ac}\}=0$,
(9) $(\delta-\beta_2)\{-S_{a2}{}^c\delta_{bd}+S_{a2}{}^d\delta_{bc}+S_{b2}{}^c\delta_{ad}-S_{b2}{}^d\delta_{ac}\}=0$,

for $a,b,c,d=3,\cdots,n$.

We consider the following two cases:

Case 1. $\delta\neq\beta_1$ and $\delta\neq\beta_2$.

Case 2. $\delta=\beta_1$ and $\alpha\neq\beta_2$ (since $\beta_1\neq\beta_2$, $\delta\neq\beta_2$ in this case).

We note that interchanging 1 and 2, we can reduce the case of $\delta=\beta_2$ and $\alpha\neq\beta_1$ to Case 2 above.

Case 1. In this case there is no curvature homogeneous space whose curvature tensor has the form above. We will prove it.

Lemma 2.8. *Under the assumption of Case* 1, *we have the following:*

$$S_{11}{}^2=0,\qquad S_{11}{}^a=0,\qquad S_{22}{}^1=0,\qquad S_{22}{}^a=0,$$
$$S_{a1}{}^b=0,\qquad S_{a2}{}^b=0,$$
$$S_{12}{}^a=\frac{\beta_1-\beta_2}{\beta_2-\delta}S_{a1}{}^2,\qquad S_{21}{}^a=\frac{\beta_1-\beta_2}{\beta_1-\delta}S_{a1}{}^2,$$

for $a,b=3,\cdots,n$.

Proof of Lemma 2.8: We apply the identities in Lemma 2.7. Putting $a\neq b=c$ in Lemma 2.7(4) and (5), we have $S_{11}{}^a=S_{22}{}^a=0$. We take mutually different integers a,b,c ($3\leq a,b,c\leq n$) and put $d=b$ in (8) . Then $S_{a1}{}^c=0$ ($a\neq c$). Putting $a=c\neq b=d$ in (8), we have $S_{a1}{}^a+S_{b1}{}^b=0$. Since $n-2\geq 3$, this implies $S_{a1}{}^a=0$. Therefore $S_{a1}{}^b=0$ for any $a,b=3,\cdots,n$. Similarly by (9), we have $S_{a2}{}^b=0$ for any a,b. By (2) and (3), $S_{11}{}^2=S_{22}{}^1=0$. Putting $a\neq b=c$ in (6) and (7) we have

$$(\beta_1-\beta_2)S_{a1}{}^2+(\delta-\beta_2)S_{12}{}^a=0,$$
$$(\beta_1-\beta_2)S_{a1}{}^2+(\delta-\beta_1)S_{21}{}^a=0.$$

By these it follows that $S_{12}{}^a=\frac{\beta_1-\beta_2}{\beta_2-\delta}S_{a1}{}^2$, $S_{21}{}^a=\frac{\beta_1-\beta_2}{\beta_1-\delta}S_{a1}{}^2$. □

We take a local orthonormal frame field $\{E_3,\cdots,E_n\}$ of $\mathcal{D}$. Then a local orthonormal frame field $\{E_1,E_2,E_3,\cdots,E_n\}$ of M gives a section of P. By Lemma 2.8, $\langle\nabla_{E_a}E_b,E_i\rangle=\langle\widetilde{\nabla}_{E_a}E_b+\widetilde{S}_{E_a}(E_b),E_i\rangle=S_{ab}{}^i=0$ for $a,b=3,\cdots,n$, $i=1,2$. This means that the distribution $\mathcal{D}$ is completely integrable and its integral submanifolds are totally geodesic. We trace the same way as Case 2 in the proof of Theorem 2.1. We denote by C the conullity operator of this totally geodesic foliation. It is a section of $\mathrm{Hom}(\mathcal{D}, \mathrm{End}(\mathcal{D}^\perp))$, where $\mathcal{D}^\perp$ is spanned by E_1 and E_2. Similarly to the argument of Case 2 in the proof of Theorem 2.1, we have $C_{E_a}E_i = \sum_{j=1}^2 S_{ij}{}^a E_j$ for $a = 3, \cdots, n$, $i = 1, 2$. By Lemma 2.8, $\operatorname{tr} C_{E_a} = 0$ for $a = 3, \cdots, n$. Therefore for any $u \in \mathcal{D}$, $\operatorname{tr} C_u = 0$. We define a linear form θ on $\mathcal{D}$ by $\theta(u) = \langle \nabla_u E_1, E_2 \rangle = \sum_{a=3}^n u^a S_{a1}{}^2$ for $u = \sum_{a=3}^n u^a E_a \in \mathcal{D}$. Then $\det C_u = -\dfrac{(\beta_1 - \beta_2)^2}{(\beta_2 - \delta)(\beta_1 - \delta)}\theta(u)^2$. We denote by R_u the Jacobi operator associated with a unit vector $u \in \mathcal{D}$. Because of the form of the curvature tensor, we have $R_u(\mathcal{D}^\perp) \subset \mathcal{D}^\perp$ and $\operatorname{tr}(R_u|_{\mathcal{D}^\perp}) = \beta_1 + \beta_2$. Similarly to the argument of Case 2 in the proof of Theorem 2.1, we can prove that $\det C_u = \frac{1}{2}(\beta_1 + \beta_2)$ for an arbitrary unit vector $u \in \mathcal{D}$. Therefore $\frac{1}{2}(\beta_1 + \beta_2) = \det C_u = -\dfrac{(\beta_1 - \beta_2)^2}{(\beta_2 - \delta)(\beta_1 - \delta)}\theta(u)^2$ for an arbitrary unit vector $u \in \mathcal{D}$. By this we see that $\theta = 0$ and $\beta_1 + \beta_2 = 0$. In particular $S_{a1}{}^2 = 0$ for $a = 3, \cdots, n$. This, together with Lemma 2.8, implies that S vanishes on P. This means $\nabla = \widetilde{\nabla}$ on M. In particular E_1 and E_2 are parallel with respect to the Riemannian connection ∇. Hence $R(\cdot, E_i)E_i = 0 \quad i = 1, 2$. So $\beta_1 = \beta_2 = 0$. This contradicts our assumption $\beta_1 \neq \beta_2$.

Case 2. By Lemma 2.7, we have the following:

Lemma 2.9. *Under the assumption of Case* 2*, the tensor S has the following form:*

$$
\begin{aligned}
&S_{11}{}^2 = 0, \qquad S_{1i}{}^a = 0, \qquad S_{22}{}^a = 0,\\
&S_{a1}{}^2 = 0, \qquad S_{a2}{}^b = 0, \qquad S_{a1}{}^b = -\frac{\beta_1 - \beta_2}{\alpha - \beta_2} S_{21}{}^2 \delta_{ab},
\end{aligned}
$$

for $i = 1, 2$, $a, b = 3, \cdots, n$.

We note that $S_{21}{}^a$ $(a = 3, \cdots, n)$ are unknown.

We take a local orthonormal frame field $\{E_3, \cdots, E_n\}$ of $\mathcal{D}$. Since $S_{1a}{}^2 = S_{a1}{}^2 = S_{ab}{}^2 = 0$,

$$\langle \nabla_{E_1} E_a, E_2 \rangle = \langle \nabla_{E_a} E_1, E_2 \rangle = \langle \nabla_{E_a} E_b, E_2 \rangle = 0.$$

This means that the codimension-one distribution $\mathcal{D} + \mathbb{R}E_1$ is completely integrable and its integral submanifolds are totally geodesic. Furthermore these integral submanifolds have constant sectional curvature $\beta_1 (= \delta)$. We put $S_{21}{}^2 = -\phi$,

$\frac{\beta_1-\beta_2}{\alpha-\beta_2}\phi = -\lambda$. Then by Lemma 2.9, we have

$$
\begin{aligned}
&\nabla_{E_1}E_1 = 0, &&\nabla_{E_2}E_1 = -\phi E_2 + \sum_{a=3}^{n} S_{21}{}^{a} E_a,\\
&\nabla_{E_a}E_1 = -\lambda E_a, &&\nabla_{E_1}E_2 = 0,\\
&\nabla_{E_2}E_2 = \phi E_1, &&\nabla_{E_a}E_2 = 0,\\
&\nabla_{E_1}E_b \equiv 0 \pmod{\mathcal{D}}, &&\nabla_{E_2}E_b \equiv -S_{21}{}^{b}E_1 \pmod{\mathcal{D}},\\
&\nabla_{E_a}E_b \equiv \lambda\delta_{ab}E_1 \pmod{\mathcal{D}}.
\end{aligned}
$$

Lemma 2.10. *The above λ and ϕ are constant on the principal fiber bundle P. Moreover we have $\alpha = -\phi^2$, $\beta_1 = \delta = -\lambda^2$, $\beta_2 = -\lambda\phi$ and in particular $\lambda \neq 0$, $\phi \neq 0$, $\lambda \neq \phi$.*

Proof of Lemma 2.10: By straightforward computation, $\langle R(E_a, E_2)E_2, E_a\rangle = -\lambda\phi$ $(a = 3, \cdots, n)$. Hence we have $\beta_2 = -\lambda\phi = \frac{\beta_1-\beta_2}{\alpha-\beta_2}\phi^2$. Both ϕ and λ are determined by α, β_1, and β_2 up to the sign and hence they are constant. Similarly by straightforward computation, $\beta_1 = \langle R(E_a, E_1)E_1, E_a\rangle = -\lambda^2$ and $\alpha = \langle R(E_1, E_2)E_2, E_1\rangle = -\phi^2$. Since $\beta_1 \neq \beta_2$ and $\alpha \neq \beta_2$, we have $\lambda \neq 0$, $\phi \neq 0$ and $\lambda \neq \phi$. □

We define vector fields X and Y of $\mathcal{D}$ as follows: We put $X = \left(\nabla_{E_2}E_1\right)_{\mathcal{D}} = \sum_{a=3}^{n} S_{21}{}^{a}E_a$, where $(\cdot)_{\mathcal{D}}$ denotes the $\mathcal{D}$-component with respect to the orthogonal decomposition $TM = \mathcal{D} + \mathcal{D}^{\perp}$. Furthermore we put $Y = \left(\nabla_{E_2}X\right)_{\mathcal{D}}$.

Lemma 2.11. *X and Y satisfy the following identities:*

(1) $\nabla_{E_1}X = (\lambda + \phi)X$,
(2) $(\nabla_{E_a}X)_{\mathcal{D}} = 0$,
(3) $\nabla_{E_1}Y = (\lambda + 2\phi)Y$,
(4) $(\nabla_{E_a}Y)_{\mathcal{D}} = (2\lambda + \phi)\langle E_a, X\rangle X - \lambda\langle X, X\rangle E_a$

for $a = 3, \cdots, n$.

Proof of Lemma 2.11: By straightforward computation,

$$
\begin{aligned}
0 &= (R(E_2, E_1)E_1)_{\mathcal{D}} = -\nabla_{E_1}X + (\lambda + \phi)X,\\
0 &= (R(E_2, E_a)E_1)_{\mathcal{D}} = -\left(\nabla_{E_a}X\right)_{\mathcal{D}}.
\end{aligned}
$$

Using these identities, we have

$$
\begin{aligned}
0 &= (R(E_2, E_1)X)_{\mathcal{D}} = -\nabla_{E_1}Y + (\lambda + 2\phi)Y,\\
0 &= (R(E_2, E_a)X)_{\mathcal{D}} = -\left(\nabla_{E_a}Y\right)_{\mathcal{D}} + (2\lambda + \phi)\langle E_a, X\rangle X - \lambda\langle X, X\rangle E_a.
\end{aligned}
$$

□

Applying these lemmas, we prove Theorems 2.5, 2.6.

Proof of Theorem 2.5: Let M be an $n(\geq 5)$-dimensional curvature homogeneous space whose $\mathfrak{g}_0$ is conjugate to $\mathfrak{so}(n-2)$. Then by the argument of Case 1, its curvature tensor R has $\delta = \beta_1$ and $\alpha \neq \beta_2$ (or $\delta = \beta_2$ and $\alpha \neq \beta_1$). By Lemma 2.10, it coincides with the curvature tensor of (G, g) with suitable parameters λ, ϕ. After Lemma 2.9, we have shown that such curvature homogeneous space M admits a codimension-one totally geodesic foliation whose leaves have constant sectional curvature $\beta_1(=\delta)$. By Lemma 2.10, β_1 is negative. □

Proof of Theorem 2.6: We will show that if M is locally homogeneous,$S_{21}^{\ a} = 0$ $(a = 3, \cdots, n)$, equivalently $\left(\nabla_{E_2} E_1\right)_{\mathcal{D}} = X = 0$. Suppose that M is locally homogeneous. Since local isometries of M leave the unit vector fields E_1 and E_2 and the distribution $\mathcal{D}$ invariant, they also leave $X = \left(\nabla_{E_2} E_1\right)_{\mathcal{D}}$ and $Y = \left(\nabla_{E_2} X\right)_{\mathcal{D}}$ invariant. In particular $\langle X, X\rangle$ and $\langle Y, Y\rangle$ are constant on M. By Lemma 2.11(1), we have

$$0 = E_1\langle X, X\rangle = 2(\lambda + \phi)\langle X, X\rangle.$$

Suppose that $\langle X, X\rangle \neq 0$. Then $\lambda + \phi = 0$. Similarly,

$$0 = E_1\langle Y, Y\rangle = 2(\lambda + 2\phi)\langle Y, Y\rangle = 2\phi\langle Y, Y\rangle.$$

Since $\phi \neq 0$, $\langle Y, Y\rangle = 0$. Namely Y vanishes. By Lemma 2.11 (4),

$$0 = (2\lambda + \phi)\langle E_a, X\rangle X - \lambda\langle X, X\rangle E_a.$$

We take a unit vector field E_a such that E_a and X are linearly independent. Then we have $\lambda\langle X, X\rangle = 0$. Since $\lambda \neq 0$, $\langle X, X\rangle = 0$. It is a contradiction. Consequently we see that $X = 0$. Equivalently $S_{21}^{\ a} = 0$ $(a = 3, \cdots, n)$. This, together with Lemmas 2.9 and 2.10, implies that S is constant as a $\mathrm{Hom}(\mathbb{R}^n, \mathfrak{h}^{\perp})$-valued function on P and is determined by parameters λ and ϕ. Since the curvature tensor R and S of M coincide with those of (G, g). Therefore M is locally isometric to (G, g). □

3. The case of Lie algebras of Lie groups acting transitively on spheres

In this section, we show the following:

Theorem 3.1. *Let M be an n-dimensional curvature homogeneous space whose $\mathfrak{g}_0$ is conjugate to a Lie subalgebra of $\mathfrak{so}(n)$ which corresponds to a connected closed Lie subgroup of $SO(n)$ acting transitively on S^{n-1}. Then M is locally isometric to a Euclidean space or a Riemannian symmetric space of rank* 1.

Our proof depends on the classification result of such Lie groups by Montgomery and Samelson [9] and Borel [2].

Let $\mathbb{R}^n$ be an n-dimensional vector space with the usual inner product $\langle , \rangle$ and G a compact connected Lie subgroup of $SO(n)$ which acts transitively on S^{n-1}. We show the list in the following table.

	G	H	isotropy repr.	curvature homo.
(1)	$SO(n)$	$SO(n-1)$	irred.	real space forms
(2)	$U(m)$	$U(m-1)$	$\mathfrak{p}_0 \oplus \mathfrak{p}_1$	$\mathbb{C}P^m, \mathbb{C}H^m$
(3)	$SU(m)$	$SU(m-1)$	$\mathfrak{p}_0 \oplus \mathfrak{p}_1$	$\times$
(4)	$Sp(m)Sp(1)$	$Sp(m-1)Sp(1)$	$\mathfrak{p}_1 \oplus \mathfrak{p}_2$	$\mathbb{H}P^m, \mathbb{H}H^m$
(5)	$Sp(m)U(1)$	$Sp(m-1)U(1)$	$\mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$	None
(6)	$Sp(m)$	$Sp(m-1)$	$\mathfrak{p}_0 \oplus \mathfrak{p}_1$	None
(7)	$Spin\ 9$	$Spin\ 7$	$\mathfrak{p}_1 \oplus \mathfrak{p}_2$	$\mathbb{C}ayP^2, \mathbb{C}ayH^2$
(8)	$Spin\ 7$	G_2	irred.	$\times$
(9)	G_2	$SU(3)$	irred.	$\times$

Notations in the table.

G: a compact connected Lie subgroup of $SO(n)$ which acts transitively on S^{n-1},

H: the isotropy subgroup of a point $e \in S^{n-1}$,

isotropy repr.: The isotropy representation of H on $T_eS^{n-1} = \mathfrak{p}$,

curvature homo.: Curvature homogeneous spaces whose $\mathfrak{g}_0$ are conjugate to the Lie algebra of G,

"None" means that there is no curvature homogeneous space,

"$\times$" means that it does not occur as a Lie algebra of the isotropy subgroup under the action of $SO(n)$ on the space $\mathcal{R}$ of curvature tensors.

On the isotropy representations of H, we refer to Ziller [18].

Let $R \in \mathcal{R}$ be a curvature tensor which is invariant by G in the table above. We denote by R_e the symmetric endomorphism of $e^\perp = T_eS^{n-1}$ defined by $x \mapsto R(x,e)e$, which is called the Jacobi operator. Since G acts transitively on S^{n-1}, the eigenvalues of the Jacobi operator R_e do not depend on the choice of $e \in S^{n-1}$.

Case (1), (8), (9): Since the Jacobi operator R_e has the only one eigenvalue, a G-invariant curvature tensor R has constant sectional curvature. In particular the corresponding Lie algebra $\mathfrak{g}_0$ of the curvature tensor R coincides with $\mathfrak{so}(n)$. So the Lie algebras corresponding to *Spin* 7 and G_2 do not occur as a Lie algebra of the isotropy subgroup under the action of $SO(n)$ on $\mathcal{R}$. Obviously a curvature homogeneous space whose curvature tensor has the form above is real space form.

Case (2): The action of $U(m)$ is the isotropy representation of an m-dimensional complex projective space $\mathbb{C}P^m = SU(m+1)/U(m)$ (or its noncompact dual). We denote by R_N the curvature tensor of $\mathbb{C}P^m$. Obviously R_N is $U(m)$-invariant. $H = U(m-1)$ acts trivially on $\mathfrak{p}_0$ ($\dim \mathfrak{p}_0 = 1$) and irreducibly on $\mathfrak{p}_1$. Therefore there exist $\lambda, \mu \in \mathbb{R}$ such that $R_e = \lambda(R_o)_e + \mu(R_N)_e$ and hence $R = \lambda R_o + \mu R_N$, where $R_o(x,y)z = \langle y,z\rangle x - \langle x,z\rangle y$. If $\mu = 0$, then R is $SO(2m)$-invariant. Therefore we assume that $\mu \neq 0$. Let M be a simply connected curvature homogeneous space whose curvature tensor has the form above. Then by the argument of Section 2 the orthonormal frame bundle $O(M)$ over M is reducible to a principal fiber bundle with the structure group $U(m)$. This reduction gives rise to an almost Hermitian structure on M. Then by Theorem 12.7 and its remark (A) of F. Tricerri and L. Vanhecke [14], M is locally isometric to $\mathbb{C}P^m$ or $\mathbb{C}H^m$.

Case (3) ($m \geq 3$): By similar argument to Case (2), we see that $SU(m)$-invariant curvature tensor is $U(m)$-invariant.

Case (4), (5), (6) ($m \geq 2$): We put $\mathfrak{g} = \mathfrak{sp}(m)$. Using the method of Iwahori [7], we see that $\dim \mathcal{R}^{\mathfrak{g}} = 7$. Moreover choosing a basis $\{I, J, K\}$ of the quaternionic structure, we can express $R \in \mathcal{R}^{\mathfrak{g}}$ as follows:

$$R = \lambda_0 R_o + \lambda_1 R_I + \lambda_2 R_J + \lambda_3 R_K \quad (\lambda_i \in \mathbb{R}),$$

where $R_I(x,y)z = \langle Iy,z\rangle Ix - \langle Ix,z\rangle Iy - 2\langle Ix,y\rangle Iz$. On the other hand, curvature homogeneous spaces whose curvature tensors have the form above are classified in P. Gilkey, A. Swann, and L. Vanhecke [6, Theorem 7.1]. Actually they obtained the result under a weaker assumption.

Case (7): The action of *Spin* 9 is the isotropy representation of the Cayley projective plane $\mathbb{C}ayP^2 = F_4/Spin\ 9$ (or its non-compact dual). $H = Spin\ 7$ acts irreducibly on $\mathfrak{p}_i$ $(i = 1,2)$, where $\dim \mathfrak{p}_1 = 7$ and $\dim \mathfrak{p}_2 = 8$. By the same reason as Case (2) a G-invariant curvature tensor R has the form $R = \lambda R_o + \mu R_N$, where R_N denotes the curvature tensor of $\mathbb{C}ayP^2$. Let M be a 16-dimensional curvature homogeneous space whose curvature tensor has the form $R = \lambda R_o + \mu R_N$ ($\mu \neq 0$). Then M satisfies two axioms introduced by Q.S. Chi ([4, Section 2]). Therefore by a theorem of Chi (Theorem 1 in [4]) M is locally isometric to $\mathbb{C}ayP^2$ or $\mathbb{C}ayH^2$. □

4. The case of some maximal subalgebras

We give a comment for the case of III (ii) stated in Section 1. We denote by R_N the curvature tensor of the compact simply connected effective irreducible symmetric space $N = G/K$ with K simple. Then obviously R_N is $\pi(\mathfrak{k})$-invariant, where π denotes the isotropy representation of $\mathfrak{k}$ on $\mathfrak{p}$. Moreover if R is a curvature tensor invariant by $\pi(\mathfrak{k})$, there exist $\lambda, \mu \in \mathbb{R}$ such that $R = \lambda R_o + \mu R_N$, where $R_o(x,y)z = \langle y,z\rangle x - \langle x,z\rangle y$. In particular $\dim \mathcal{R}^{\pi(\mathfrak{k})} = 2$ (cf. [17], proof of Theorem 4.2). On the other hand, F. Podestà and F. Tricerri [12] proved the following.

Theorem 4.1. *Let N be an Einstein symmetric space of dimension $n(\geq 4)$ and not of constant sectional curvature. Then there is no curvature homogeneous space whose curvature tensor has the form $\lambda R_o + \mu R_N$, $\lambda > 0$, $\mu \neq 0$.*

The case $\lambda < 0$ remains open.

For our argument, the equation (2.1) is crucial. We would like to know a unified method of treating (2.1).

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