A note on condensations of $C_p(X)$ onto compacta

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Abstract. A condensation is a one-to-one continuous mapping onto. It is shown that the space $C_p(X)$ of real-valued continuous functions on X in the topology of pointwise convergence very often cannot be condensed onto a compact Hausdorff space. In particular, this is so for any non-metrizable Eberlein compactum X (Theorem 19). However, there exists a non-metrizable compactum X such that $C_p(X)$ condenses onto a metrizable compactum (Theorem 10). Several curious open problems are formulated.

Keywords: condensation, compactum, network, Lindelöf space, topology of pointwise convergence, σ-compact space, Eberlein compactum, Corson compactum, Borel set, monolithic space, tightness

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All spaces considered in this paper are assumed to be Tychonoff. *A condensation* is a continuous one-to-one mapping onto. For a space X, $C_p(X)$ denotes the space of continuous real-valued functions on X in the topology of pointwise convergence [2]. Obviously, $C_p(X)$ is compact only if X is empty. However, it is much less clear when the topology of $C_p(X)$ contains a weaker compact topology. Some concrete questions in this direction were formulated in [3]. Note, that S. Banach was one of the first to consider a question of this kind: he asked whether every separable Banach space condenses onto a compact space [6]. Recently it was shown in [4] that, for every metrizable compactum X (even for every σ-compact metrizable space), $C_p(X)$ condenses onto a metrizable compactum. On the other hand, W. Marciszewski [10] established the consistency of the following statement: there exists a subspace X of the space of real numbers such that $C_p(X)$ cannot be condensed onto a σ-compact space.

Below we present examples of compacta X for which $C_p(X)$ cannot be condensed onto a compactum. We also show that there is a non-metrizable compact space X such that $C_p(X)$ condenses onto a metrizable compactum. Several sufficient conditions that guarantee that $C_p(X)$ does not condense onto a compactum are provided.

Our notation and terminology are as in [2], [5], and [9]. In particular, for a space X, we denote by $nw(X)$ the networkweight of X, by $w(X)$ the weight of X, by $d(X)$ the density of X, by $\psi(X)$ the pseudocharacter of X. A space X is said to be *monolithic* [2], if for every subset A of X we have $nw(\overline{A}) \leq |A|$. If A is a subspace of a space X, then $C_{p,A}(X)$ denotes the subspace of $C_p(A)$, consisting of the restrictions to A of continuous real-valued functions on X. A space X is said to be of *point-countable type* if every point of X is contained in a compact subspace of X with a countable base of neighborhoods.

A space X is called a *Lindelöf p-space* if there exists a perfect mapping of X onto a separable metrizable space (for more details on p-spaces, see [5]). A continuous image of a Lindelöf p-space is called a *Lindelöf Σ-space*.

Theorem 1. *Let X be a monolithic Lindelöf Σ-space. Suppose also that $C_p(X)$ condenses onto a compact space F. Then:* (1) $nw(X) < 2^\omega$; (2) $|C_p(X)| \leq 2^\omega$; (3) $|F| \leq 2^\omega$; *and* (4) $w(F) < 2^\omega$.

Lemma 2. *Let X be a monolithic Lindelöf Σ-space. Then, for each $A \subset X$ such that $|A| \leq 2^\omega$, we have $|C_{p,A}(X)| \leq 2^\omega$.*

Proof: Put $Y = \overline{A}$. Then Y is a closed subspace of X and the networkweight of Y is $\leq 2^\omega$, since X is monolithic. Therefore, $nw(C_p(Y)) = nw(Y) \leq 2^\omega$ [2, Theorem 1.1.3]. The space Y^n is Lindelöf, for every $n \in \omega$, since X and Y are Lindelöf Σ-spaces. It follows that the tightness of $C_p(Y)$ is countable [2, Theorem 2.1.1]. From $d(C_p(Y)) \leq nw(C_p(Y)) \leq 2^\omega$ it follows that we can fix a subset B of $C_p(Y)$ such that $\overline{B} = C_p(Y)$ and $|B| \leq 2^\omega$. Now $t(C_p(Y)) \leq \omega$ implies that $C_p(Y) = \bigcup\{\overline{P} : P \subset B, |P| \leq \omega\}$. However, $C_p(Y)$ is monolithic, since Y is a Lindelöf Σ-space [2, Theorems 2.6.21 and 2.6.8]. Therefore, for each countable subset P of $C_p(Y)$, $\overline{P}$ is a space with a countable network. Hence, $|\overline{P}| \leq 2^\omega$, for every countable $P \subset B$. It follows that $|C_p(Y)| \leq 2^\omega$. Clearly, $|C_{p,A}(X)| \leq |C_p(Y)| \leq 2^\omega$. □

Now we need the next result of M.G. Tkačenko [13, Theorem 2]:

Theorem 3. *Let τ be a cardinal number and S a dense subspace of a product $Z = \Pi\{X_\alpha : \alpha \in A\}$, where each X_α has a countable network. Suppose also that, for every $B \subset A$ such that $|B| \leq 2^\tau$, the cardinality of the projection $\pi_B(S)$ of S into the subproduct $Z_B = \Pi\{X_\alpha : \alpha \in B\}$ does not exceed 2^τ. Then, for every compact continuous image F of the space S, the weight of F is smaller than 2^τ.*

Proof of Theorem 1: Lemma 2 shows that $C_p(X)$ satisfies the restrictions on S in Theorem 3 (with ω in the role of τ). It follows that $w(F) < 2^\omega$. Since $C_p(X)$ condenses onto F, we have $\psi(C_p(X)) \leq w(F) < 2^\omega$. Therefore, $d(X) = \psi(C_p(X)) < 2^\omega$ (see [2]). Since X is monolithic, $nw(X) = d(X) < 2^\omega$ [2]. Thus, we have established (1) and (4). Clearly, (3) follows from (2).

Let us prove (2). We have $d(C_p(X)) \leq nw(C_p(X)) = nw(X) \leq 2^\omega$ [2, Theorems 1.1.3 and 1.1.4]. Fix a dense subset M in $C_p(X)$ such that $|M| \leq 2^\omega$. Since X is a Lindelöf Σ-space, we have $C_p(X) = \cup\{\overline{P} : P \subset M, |P| \leq \omega\}$, where each $\overline{P}$ is a space with a countable network and has, therefore, cardinality not greater than 2^ω. Hence, $|C_p(X)| \leq 2^\omega$. □

Corollary 4. *Assume that $2^\omega < 2^{\omega_1}$. Then every monolithic Lindelöf Σ-space X such that $C_p(X)$ condenses onto a compact space F has a countable network* (*and F has a countable network as well*).

PROOF: Indeed, $|F| \leq 2^\omega$, by Theorem 1. Since $2^\omega < 2^{\omega_1}$, it follows, by a well known theorem of E. Čech and B. Pospíšil (see [9]) that the space F satisfies the first axiom of countability at least at one point. Hence, $\psi(C_p(X)) \leq \omega$. This implies that X is separable (see [2, Theorem 1.1.4]). Therefore, since X is monolithic, the space X has a countable network. Hence $C_p(X)$ has a countable network [2, Theorem 1.1.3], which implies that the space F has a countable network, as a continuous image of $C_p(X)$. □

Corollary 5. *Let X be a monolithic Lindelöf p-space such that $C_p(X)$ condenses onto a compact space. Then $w(X) < 2^\omega$.*

PROOF: By Theorem 1, $nw(X) < 2^\omega$. It remains to note that $w(X) = nw(X)$, since X is a p-space [5]. □

Corollary 6. *If X is a monolithic compact space such that $d(X) \geq 2^\omega$, then $C_p(X)$ cannot be condensed onto a compact space.*

PROOF: This follows from Theorem 1, since $d(X) = nw(X)$. □

The assumption, that the space X in Theorem 1 and Corollary 4 is monolithic, cannot be dropped. The next example shows it.

Example 7. Let X be the "two-arrows" space, that is, $X = [0,1] \times \{0,1\}$ with the topology generated by the lexicographic order. The space X is hereditarily separable and first countable. Fix any countable dense subset A of X, and consider the space $C_{p,A}(X)$ of all continuous real-valued functions on X in the topology of pointwise convergence on A. The space $C_{p,A}(X)$ is an $F_{\sigma\delta}$-subset of the complete separable metrizable space $\mathbb{R}^A$ (see Lemma 5.7 in [7]). On the other hand, $C_{p,A}(X)$ is not σ-compact, since A is a non-discrete countable space. Indeed, the next result was established in [2] (Theorem 1.2.2): if Y is dense in X, and $C_{p,Y}(X)$ is σ-compact, then X is pseudocompact, and Y is a P-space. Recall, that *a P-space* is a space in which every G_δ-subset is open. Clearly, every countable P-space is discrete. E.G. Pytkeev showed that every non-σ-compact Borel subset of a separable complete metric space can be condensed onto a (metrizable) compact space [12]. Therefore, $C_{p,A}(X)$ condenses onto a metrizable compact space F. Since $C_p(X)$ obviously condenses onto $C_{p,A}(X)$, it follows that $C_p(X)$ condenses onto the metrizable compactum F. Therefore, X is a non-metrizable compactum for which $C_p(X)$ condenses onto a metrizable compactum. Notice, that X is a Rosenthal compactum (see [8]).

Problem 8. When does a separable compact space X have the following property: for every countable dense subspace A of X, the space $C_{p,A}(X)$ condenses onto a (metrizable) compactum?

Remark 9. W. Marciszewski has informed me that he has an example of a perfectly normal (hence, first countable) hereditarily separable compact space X with a countable dense subspace A such that $C_{p,A}(X)$ cannot be condensed onto a compact space.

Theorem 10. *The space* $C_p(D^c)$ *(where* $c = 2^\omega$*) condenses onto a metrizable compact space.*

PROOF: Let M be the Cantor set, with the usual topology. Then D^M is a compact subspace of $\mathbb{R}^M$, and the set A of continuous mappings of M into the discrete space $D = \{0,1\}$ is a countable dense subset of D^M. Theorem 2.3 from [11] implies that $C_{p,A}(D^M)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^A$. On the other hand, $C_{p,A}(D^M)$ is not σ-compact, by Theorem 1.2.2 from [2]. Now, a result of Pytkeev in [12] implies that $C_{p,A}(D^M)$ condenses onto a metrizable compact space. Since $C_p(D^M)$ condenses onto $C_{p,A}(D^M)$, and D^c is, obviously, homeomorphic to D^M, we are done. □

Problem 11. Is it true that $C_p(D^{\omega_1})$ condenses onto a (metrizable) compact space? onto a σ-compact space?

Of course, under CH the answer is "yes", by Theorem 10. However, curiously enough, the proof of Theorem 10 cannot be adapted to provide an answer to Problem 11 in ZFC. This can be seen from the following statement:

Theorem 12. *The continuum hypothesis CH is equivalent to the following condition:*

(e) *there exists a countable dense subset* A *in* D^{ω_1} *such that* $C_{p,A}(D^{\omega_1})$ *is an* $F_{\sigma\delta}$*-subset of* $\mathbb{R}^A$.

PROOF: Indeed, CH implies condition (e), by Theorem 10. Assume now the negation of CH, and let us show that the negation of (e) holds. Assume the contrary. Then from Corollary 2.4 in [7] it follows that either D^{ω_1} is a Rosenthal compactum or else D^{ω_1} contains a topological copy of $\beta\omega$. However, the second alternative is impossible since the weight of $\beta\omega$ is $c = 2^\omega$, and the weight of D^{ω_1} is ω_1 and $\omega_1 < 2^\omega$, by the assumption. On the other hand, D^{ω_1} is not a Rosenthal compactum, since, for example, the space D^{ω_1} is not Fréchet-Urysohn (see [8]). Thus, the first alternative is also impossible. This contradiction completes the proof. □

In connection with the above argument, notice, that D^c does contain a topological copy of $\beta\omega$ and that in the proof of Theorem 10 we showed that there exists a countable dense subset A in D^c such that $C_{p,A}(D^c)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^A$.

Example 13. Let X be the Σ-product subspace of the product D^c (over zero-point). Then X is a countably compact monolithic Fréchet-Urysohn topological group, and $C_p(X)$ is Lindelöf (see [2]). Let us show that $C_p(X)$ does not condense onto a space of point-countable type (in particular, $C_p(X)$ does not condense onto a compact space or onto a Čech complete space). Compare this statement with Theorem 10 and Problem 11.

Indeed, assume that $C_p(X)$ condenses onto a space F of point countable type. Then from Corollary 1 in [14] it follows that F is first countable at some point. Hence, the pseudocharacter of $C_p(X)$ is countable. It follows that X is separable [2], a contradiction.

Notice that $C_p(D^c)$ condenses onto $C_p(X)$ (by the natural restriction mapping).

The case of monolithic spaces covers the important cases of Eberlein compacta and of Corson compacta [2]. However, we can formulate a very general condition for non-condensability of $C_p(X)$ onto a compactum in the following terms, not using explicitly the notion of monolithicity. The first step in this direction is the following statement.

Proposition 14. *Suppose X is a space such that $2^{d(X)} > |C_p(X)|$. Then $C_p(X)$ cannot be condensed onto a compact space.*

Proof: Suppose that $C_p(X)$ condenses onto a compactum F. Clearly, F is dense in itself. Let τ be the smallest cardinal number such that the character of F at some point $z \in F$ does not exceed τ. By a theorem of Čech and Pospíišil, $|F| \geq 2^\tau$ [9]. From $2^{d(X)} > |C_p(X)| = |F| \geq 2^\tau$ it follows that $\tau < d(X)$. However, $d(X) = \psi(C_p(X)) \leq \tau$ (see [2]). This contradiction completes the proof. □

Theorem 15. *Suppose X is a Lindelöf Σ-space such that $2^{d(X)} > (nw(X))^\omega$. Then $C_p(X)$ cannot be condensed onto a compact space.*

Proof: The space $C_p(X)$ is monolithic, since X is a Lindelöf Σ-space. For the same reason, the tightness of $C_p(X)$ is countable [2]. It follows that $|C_p(X)| \leq (nw(C_p(X)))^\omega$. Since $nw(X) = nw(C_p(X))$ (see [2]), we conclude that $2^{d(X)} > |C_p(X)|$. It remains to apply Proposition 14. □

In connection with Example 7, note the next corollary from Proposition 14.

Corollary 16. *Assume that $2^\omega < 2^{\omega_1}$, and let X be a space such that $C_p(X)$ condenses onto a compact space. Then X is separable if and only if $|C_p(X)| \leq 2^\omega$.*

The next statement also directly follows from Proposition 14.

Corollary 17. *If X is a space such that $d(X) = |C_p(X)|$, then $C_p(X)$ cannot be condensed onto a compact space.*

Example 18. Let X be a well ordered space of cardinality 2^τ, with the topology generated by the well ordering. Then $C_p(X)$ cannot be condensed onto a compact space. Indeed, it is easy to see that, under the assumptions, $d(X) = |C_p(X)|$. Therefore, Corollary 17 is applicable.

One more restriction on the existence of condensations of $C_p(X)$ onto compacta involves the Lindelöf degree of $C_p(X)$.

Theorem 19. *Suppose X is a space such that X^n is Lindelöf, for every $n \in \omega$, and $C_p(X)$ is also Lindelöf. Suppose further that $C_p(X)$ condenses onto a space Y of point countable type (for example, onto a compact space Y). Then Y has a countable network and X is separable.*

PROOF: Let us show that the tightness of Y is countable. Since Y is a space of point countable type, it suffices to verify that the tightness of each compact subspace F of Y is countable (see [5]).

Assume the contrary. Then there exists an uncountable free sequence in F, which obviously implies that there exists an uncountable free sequence in $C_p(X)$. However, this is impossible, since $C_p(X)$ is a Lindelöf space of countable tightness. Hence, the tightness of Y is countable. Now it follows from Theorem 2.20 in [15] that the space Y has a countable network. Hence, $\psi(C_p(X)) \le \psi(Y) \le nw(Y) \le \omega$, which implies that X is separable. □

Corollary 20. *Suppose X is a compact space such that $C_p(X)$ is Lindelöf. Suppose further that $C_p(X)$ condenses onto a space Y of point countable type (for example, onto a compact space Y). Then Y has a countable network and X is separable.*

Corollary 21. *Suppose that X is a monolithic compactum such that $C_p(X)$ is Lindelöf. Then $C_p(X)$ condenses onto a compact space if and only if X is metrizable.*

PROOF: If X is a metrizable compactum, then $C_p(X)$ condenses onto a metrizable compactum [4]. Assume now that $C_p(X)$ condenses onto a metrizable compactum. Then, by Theorem 19, X is separable. Since X is monolithic and compact, it follows that X is metrizable. □

A compact space is said to be a *Corson compactum* if it is homeomorphic to a subspace of a Σ-product of separable metrizable spaces (see [2]).

Corollary 22. *If X is a non-metrizable Corson compactum, then $C_p(X)$ cannot be condensed onto a compact space.*

PROOF: Indeed, this follows from Corollary 21, since every Corson compactum X is monolithic and $C_p(X)$ is Lindelöf (see [2]). □

Example 23. Let A_τ be the Alexandroff one-point compactification of an uncountable discrete space (of the cardinality τ). Then $C_p(X)$ cannot be condensed onto a compact space. Indeed, A_τ is a Corson compactum (even an Eberlein compactum) [2]. Therefore, Corollary 22 is applicable.

Below we denote by $(MA + \neg CH)$ the combination of Martin's Axiom with the negation of Continuum Hypothesis.

Theorem 24. *Assume* $(MA+\neg CH)$, *and suppose that* X *is a compact space such that* $C_p(X)$ *is Lindelöf. Suppose also that* $C_p(X)$ *condenses onto a space of point countable type* (*for example, onto a compact space*). *Then* X *is metrizable.*

Proof: E.A. Reznichenko showed (see [2]) that, under $(MA+\neg CH)$, every separable compact space X with Lindelöf $C_p(X)$ is metrizable. Thus, it remains to apply Corollary 20. □

Problem 25. Can one drop the assumption that $(MA+\neg CH)$ in Theorem 24?

Remark 26. Notice that the restrictions on X in Theorem 19 can be replaced by a more technical assumption that there are no uncountable free sequences in X.

Problem 27. Can the space $C_p(\beta\omega)$ (where $\beta\omega$ is the Stone-Čech compactification of the discrete space ω) be condensed onto a compact space?

Note that $C_p(\beta\omega)$ naturally condenses onto the σ-compact space Z of all bounded real-valued functions on ω (in the topology of pointwise convergence). However, Z cannot be condensed onto a compact space, since Z is the union of a countable family of nowhere dense compacta (the Baire category theorem for compacta works). Problem 27 is obviously related to the following question.

Problem 28. Fix $\xi \in \beta\omega$, and take the subspace $\omega \cup \{\xi\}$ of $\beta\omega$. Consider the space $C_p^b(\omega \cup \{\xi\})$ of bounded continuous real-valued functions on the space $\omega \cup \{\xi\}$, in the topology of pointwise convergence. Can the space $C_p^b(\omega \cup \{\xi\})$ be condensed onto a compact space?

Note, that the space $C_p^b(\omega \cup \{\xi\})$ in Problem 28 is not a Borel subset of the space $\mathbb{R}^{\omega\cup\{\xi\}}$, this was shown in [7]. Thus, to solve affirmatively Problem 28 we cannot just apply Pytkeev's results from [12]. Note also, that $C_p^b(\omega \cup \{\xi\})$ can be condensed onto a σ-compact space. It would be interesting to find out which of the results in this article on the non-existence of a condensation of certain spaces onto compacta can be strengthened to a conclusion that, for the same spaces, there is no condensation onto a σ-compact space. For example, we have the next question:

Problem 29. Can the space $C_p(A_\tau)$, where A_τ is the Alexandroff one-point compactification of an uncountable discrete space of cardinality τ, be condensed onto a σ-compact space? Can the space $C_p(A_{\omega_1})$ be condensed onto a σ-compact space?

Problem 30. Suppose that X is a non-metrizable Corson (Eberlein) compactum. Is then true that $C_p(X)$ cannot be condensed onto a σ-compact space?

See, in connection with Problem 30, Corollaries 22 and 21.

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