Annihilators and deductive systems in commutative Hilbert algebras

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Abstract. The properties of deductive systems in Hilbert algebras are treated. If a Hilbert algebra H considered as an ordered set is an upper semilattice then prime deductive systems coincide with meet-irreducible elements of the lattice $\operatorname{Ded} H$ of all deductive systems on H and every maximal deductive system is prime. Complements and relative complements of $\operatorname{Ded} H$ are characterized as the so called annihilators in H.

Keywords: (commutative) Hilbert algebra, deductive system (generated by a set), annihilator

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1. Introduction

Following the introduction of Hilbert algebras by L. Henkin in early 50-ties and A. Diego [9], the algebra and related concepts were developed by D. Busneag [2], [3], [4]. Y.B. Jun gave characterizations of deductive systems in Hilbert algebras (see [12], [13]), introduced the notion of commutative Hilbert algebras and gave some characterizations of a commutative Hilbert algebra (see [13]). In [11] S.M. Hong and Y.B. Jun showed that every bounded Hilbert algebra with at least two elements contains at least one maximal deductive system. In this paper, we consider properties of deductive systems in Hilbert algebras which are upper semilattices as posets. We show that every maximal deductive system is prime. We give a condition for a deductive system to be prime. We show that the annihilator of any non-empty subset of a Hilbert algebra is a deductive system which is an annihilator of the induced upper semilattice.

2. Preliminaries

We include some elementary aspects of Hilbert algebras that are necessary for this paper, and for more details we refer to [2], [3], [4] and [9].

A *Hilbert algebra* is a triple $(H, \cdot, 1)$, where H is a non-empty set, $\cdot$ is a binary operation on H, $1 \in H$ is an element such that the following three axioms are satisfied for every $x, y, z \in H$:

(H1) $x \cdot (y \cdot x) = 1$,

(H2) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1$,
(H3) if $x \cdot y = y \cdot x = 1$ then $x = y$.

In the sequel, the binary operation $\cdot$ will be denoted by juxtaposition. In a Hilbert algebra H, the following properties hold:

(P1) $x1 = 1$, $1x = x$ and $xx = 1$,
(P2) $x(yz) = (xy)(xz)$,
(P3) $x(yz) = y(xz)$.

If H is a Hilbert algebra, then the relation

$$x \leq y \quad \text{iff} \quad xy = 1$$

is a partial order on H, called the *natural ordering* on H. With respect to this ordering 1 is the greatest element of H and the following property is satisfied, see e.g. [9]:

(P4) $x \leq y$ implies $zx \leq zy$ and $yz \leq xz$.

For any x and y in a Hilbert algebra H, define $x \vee y$ as $(yx)x$. A Hilbert algebra H is said to be *commutative* (see [13, Definition 2.1]) if for all $x, y \in H$,

$$(yx)x = (xy)y, \text{ i.e., } x \vee y = y \vee x.$$

Note that $x \vee y$ is the least upper bound of x and y, hence each commutative Hilbert algebra H is a semilattice with respect to $\vee$ (see [13]) and hence $\vee$ is commutative and associative.

Examples. (1) Let $A = \{a, b, c, d, 1\}$ be an ordered set as shown on Figure 1.

Figure 1

Define the binary operation on A as follows:

$$xy = 1 \quad \text{if} \quad x \leq y \quad \text{and}$$

$$xy = y \quad \text{otherwise}.$$

One can check that $\mathcal{A} = (A, \cdot, 1)$ is a Hilbert algebra which is not commutative: $(ca)a = aa = 1 \neq c = 1c = (ac)c$.
In fact, a Hilbert algebra can be introduced on any ordered set with a greatest element 1, see e.g. [9].
(2) Let $A = \{a, b, c, d, 1\}$ be an ordered set as shown on Figure 2,

Figure 2

where the binary operation is defined by the table:

$\cdot$	a	b	c	d	1
a	1	1	1	d	1
b	c	1	c	d	1
c	b	b	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

One can show that $\mathcal{A} = (A, \cdot, 1)$ is an example of a commutative Hilbert algebra.

A subset D of a Hilbert algebra H is called a *deductive system* of H if:

(D1) $1 \in D$,
(D2) $x \in D$ and $xy \in D$ imply $y \in D$.

It is almost evident that if D is a deductive system of a Hilbert algebra H, then $x \leq y$ and $x \in D$ imply $y \in D$.

Denote by $\operatorname{Ded} H$ the set of all deductive systems of H. If $X \subseteq H$, then the set

$$\langle X \rangle := \bigcap \{D \in \operatorname{Ded} H \mid X \subseteq D\}$$

is called the *deductive system generated* by X. The mapping $X \longrightarrow \langle X \rangle$ is obviously a closure operator and hence $\operatorname{Ded} H$ is a complete lattice with respect to set inclusion.

The deductive system generated by a singleton $a \in H$ will be denoted by $\langle a\rangle$ and it is easy to verify that $\langle a\rangle = \{x \in A \mid a \leq x\}$, the so called *principal deductive system* (see [9]).

Proposition 2.1 (Busneag [2, Lemma 2.3]). *If H is a Hilbert algebra and $X \subseteq H$, then*

$$\langle X\rangle = \{x \in H \mid x_1(x_2(\dots(x_n x)\dots)) = 1 \ \text{ for some } \ x_1, x_2, \dots, x_n \in X\}.$$

Lemma 2.2. *Let H be a Hilbert algebra and $x, y \in H$. Then $\langle x\rangle \cap \langle y\rangle \supseteq \langle x \vee y\rangle$ with equality in commutative Hilbert algebras.*

Proof: The inclusion $\langle x \vee y\rangle \subseteq \langle x\rangle \cap \langle y\rangle$ is trivial.
Conversely, let H be commutative and $a \in \langle x\rangle \cap \langle y\rangle$. As mentioned above, $\langle x\rangle = \{b \in H \mid x \leq b\}$ and $\langle y\rangle = \{b \in H \mid y \leq b\}$ whence $x \vee y \leq a$ giving $a \in \langle x \vee y\rangle$. □

Let H be a Hilbert algebra and Θ a congruence on H. Denote by $[1]_\Theta$ the set $\{x \in H; \langle x, 1\rangle \in \Theta\}$, the so called *kernel of* Θ. As it was shown in [5], the correspondence $[1]_\Theta \longrightarrow \Theta$ is 1-1, i.e. in a Hilbert algebra every congruence is uniquely determined by its kernel (this property is called *weak regularity*).

Moreover, it was proved in [5], [6] that congruence kernels and deductive systems in Hilbert algebras coincide.

3. Maximal and prime deductive systems

In this section, unless otherwise specified, H will denote a commutative Hilbert algebra. For any $x, y \in H$ and natural number n we define $x^n y$ recursively as follows: $x^1 y = xy$ and $x^{n+1} y = x(x^n y)$.

Proposition 3.1. *Let $x, y, a \in H$. If $x^m a = 1$ and $y^n a = 1$ for some natural numbers n and m, then there exists a natural number $p \leq \frac{s(s+1)}{2}$, where $s = \max(m, n)$, such that $(x \vee y)^p a = 1$.*

Proof: Suppose $x^m a = 1$ and $y^n a = 1$ for some natural numbers m and n. It follows from (P1) that $x^m a = 1$ implies $x^k a = 1$ for $k \geq m$. Thus, without loss of generality, we may assume that $m = n$. Now it is sufficient to show that

$$\text{(3A)} \qquad \text{if } \ x^n a = 1 \ \text{ and } \ y^n a = 1 \ \text{ then } \ (x \vee y)^p a = 1$$

for some natural number $p = p_n \leq \frac{s(s+1)}{2}$. We will proceed by induction on n. For $n = 1$, we have $xa = 1$ and $ya = 1$, i.e., $x \leq a$ and $y \leq a$. Since H is an upper semilattice, we obtain $x \vee y \leq a$ or equivalently $(x \vee y)a = 1$. Hence (3A) is true for $n = 1$ and $p_n = 1$. Assume that (3A) holds for n, and we will prove that (3A) holds for $n + 1$. By (P1) and (P3) we have

$$\text{(3B)} \qquad 1 = x^{n+1} a = y^n(x^{n+1} a) = x(y^n(x^n a))$$

and

$$1 = y^{n+1}a = x^n(y^{n+1}a) = y(y^n(x^n a)). \tag{3C}$$

Since $x, y \leq x \vee y$, it follows from (P3) and (P4) that

$$1 = (x \vee y)(y^n(x^n a)) = y^n(x^n((x \vee y)a)). \tag{3D}$$

Note from $y^{n+1}a = 1$ and (P3) that

$$y^{n+1}((x \vee y)^k a) = 1 \tag{3E}$$

for any natural number k. By (P3) and (3E) we have

$$y(x^{n-1}(y^n((x \vee y)a))) = x^{n-1}(y^{n+1}((x \vee y)a)) = x^{n-1}1 = 1$$

and, by (P3) and (3D),

$$x(x^{n-1}(y^n((x \vee y)a))) = x^n(y^n((x \vee y)a)) = y^n(x^n((x \vee y)a)) = y^n 1 = 1. \tag{3F}$$

Since H is an upper semilattice, by (P3) and (P4) we obtain

$$x^{n-1}(y^n((x \vee y)^2 a)) = 1. \tag{3G}$$

In the same way, using (3E) and (3G), we get $x^{n-2}(y^n((x \vee y)^3 a)) = 1$. Repeating the above argument we conclude that

$$y^n((x \vee y)^{n+1}a) = 1. \tag{3H}$$

Similarly we have

$$x^n((x \vee y)^{n+1}a) = 1. \tag{3I}$$

Applying induction hypothesis to (3H) and (3I), we obtain

$$1 = (x \vee y)^{p_n}((x \vee y)^{n+1}a) = (x \vee y)^{p_n+n+1}a \tag{3J}$$

for some natural number p_n. By (3J), one can see $p_{n+1} \leq p_n + (n+1)$ and hence $p_n \leq \frac{s(s+1)}{2}$ for $s = \max(m, n)$. $\square$

Theorem 3.2. *Let $D \in \operatorname{Ded} H$. If $x \vee y \in D$, then*

$$\langle D \cup \{x\}\rangle \cap \langle D \cup \{y\}\rangle = D$$

for all $x, y \in H$.

Proof: Let $z \in \langle D \cup \{x\}\rangle \cap \langle D \cup \{y\}\rangle$. Then there exist $x_1, x_2, \dots, x_n \in D \cup \{x\}$ and $y_1, y_2, \dots, y_m \in D \cup \{y\}$ such that $x_1(x_2(\dots(x_n z)\dots)) = 1$ and $y_1(y_2(\dots(y_m z)\dots)) = 1$, respectively. Using (P3) we can write the above equalities in the following form:

$$x^p(a_1(\dots(a_s z)\dots)) = 1 \quad \text{where} \quad \{a_1, \dots, a_s\} = \{x_1, \dots, x_n\} \cap D,$$
$$y^q(b_1(\dots(b_t z)\dots)) = 1 \quad \text{where} \quad \{b_1, \dots, b_t\} = \{y_1, \dots, y_m\} \cap D.$$

It follows from (P1) and (P3) that

$$x^p(b_1(\dots(b_t(a_1(\dots(a_s z)\dots)))\dots)) = 1$$

and

$$y^q(b_1(\dots(b_t(a_1(\dots(a_s z)\dots)))\dots)) = 1.$$

Using Proposition 3.1 we have

$$(x \vee y)^k(b_1(\dots(b_t(a_1(\dots(a_s z)\dots)))\dots)) = 1$$

for some natural number k. Since $D \in \operatorname{Ded} H$ and $a_i, b_j, x \vee y \in D$, it follows that $z \in D$. This proves that

$$\langle D \cup \{x\}\rangle \cap \langle D \cup \{y\}\rangle \subseteq D.$$

The reverse inclusion is obvious, ending the proof. □

Theorem 3.3. *The set $\operatorname{Ded} H$ of all deductive systems of H is a complete and distributive lattice with respect to set inclusion. The least element of $\operatorname{Ded} H$ is $\{1\}$ and the greatest one is H. The operation meet coincides with set intersection and for $D_1, D_2 \in \operatorname{Ded} H$ we have $D_1 \vee D_2 = \langle D_1 \cup D_2\rangle$.*

Proof: Of course, $\langle\emptyset\rangle = \{1\}$ and $\langle H\rangle = H$ whence $\{1\}$ is the least and H the greatest element of $\operatorname{Ded} H$. We need only to prove distributivity of $\operatorname{Ded} H$. We use the fact that for every $D \in \operatorname{Ded} H$ there is a unique $\Theta_D \in \operatorname{Con} H$ such that $D = [1]_{\Theta_D}$. Moreover, it is clear that for $\Theta, \Phi \in \operatorname{Con} H$ we have $[1]_\Theta \cap [1]_\Phi = [1]_{\Theta\cap\Phi}$. We show also $[1]_\Theta \vee [1]_\Phi = [1]_{\Theta\vee\Phi}$. We know that the deductive system $[1]_\Theta \vee [1]_\Phi$ is a kernel of some $\Psi \in \operatorname{Con} H$, i.e., $[1]_\Theta \vee [1]_\Phi = [1]_\Psi$. Thus $[1]_\Theta \subseteq [1]_\Psi$, $[1]_\Phi \subseteq [1]_\Psi$ and, with respect to weak regularity, also $\Theta \subseteq \Psi$ and $\Phi \subseteq \Psi$ giving $\Theta \vee \Phi \subseteq \Psi$, i.e.,

$$[1]_\Theta \vee [1]_\Phi = [1]_\Psi \supseteq [1]_{\Theta\vee\Phi}.$$

Conversely, $[1]_\Theta \subseteq [1]_{\Theta\vee\Phi}$ and $[1]_\Phi \subseteq [1]_{\Theta\vee\Phi}$ whence $[1]_\Theta \vee [1]_\Phi \subseteq [1]_{\Theta\vee\Phi}$ proving the assertion. Now, we are ready to prove the distributivity of $\operatorname{Ded} H$. Let $D_1, D_2, D_3 \in \operatorname{Ded} H$. Then there exist $\Phi = \Theta_{D_1}$, $\Theta = \Theta_{D_2}$, $\Psi = \Theta_{D_3}$ such that $D_1 = [1]_\Phi$, $D_2 = [1]_\Theta$, $D_3 = [1]_\Psi$. We need only to show the inclusion

$$D_1 \wedge (D_2 \vee D_3) \subseteq (D_1 \wedge D_2) \vee (D_1 \wedge D_3).$$

For this, suppose $a \in D_1 \wedge (D_2 \vee D_3)$. Then

$$a \in [1]_\Phi \cap ([1]_\Theta \vee [1]_\Psi) = [1]_{\Phi\cap(\Theta\vee\Psi)}.$$

Hence $\langle 1, a\rangle \in \Phi \cap (\Theta \vee \Psi)$, i.e., $\langle 1, a\rangle \in \Phi$ and there exist $c_0, c_1, \dots, c_k \in H$ such that $c_0 = 1, c_k = a$ and

$$\langle c_j, c_{j+1}\rangle \in \Theta \;\text{ for } j \text{ even;} \quad \langle c_j, c_{j+1}\rangle \in \Psi \;\text{ for } j \text{ odd.}$$

Applying (P1), we have

$$\langle c_j a, 1\rangle = \langle c_j a, c_j 1\rangle \in \Phi, \quad \langle c_{j+1} a, 1\rangle = \langle c_{j+1} a, c_{j+1} 1\rangle \in \Phi$$

and, applying transitivity of Φ, also

$$\langle c_j a, c_{j+1} a\rangle \in \Phi \;\text{ for }\; j = 0, 1, \dots, k-1.$$

Then

$$a = 1a = c_0 a(\Phi\cap\Theta)c_1 a(\Phi\cap\Psi)c_2 a(\Phi\cap\Theta)\dots c_k a = aa = 1$$

whence $\langle a, 1\rangle \in (\Phi\cap\Theta) \vee (\Phi\cap\Psi)$, i.e.,

$$a \in [1]_{(\Phi\cap\Theta)\vee(\Phi\cap\Psi)} = ([1]_\Phi \cap [1]_\Psi) = (D_1 \wedge D_2) \vee (D_1 \wedge D_3)$$

which has to be shown. □

Definition 3.4. A deductive system D of H is said to be *prime* if for each $a, b \in H$, $a \vee b \in D$ implies $a \in D$ or $b \in D$. D is called *maximal* if $D \neq H$ and $D \subseteq D_1 \subseteq H$ implies $D = D_1$ or $D_1 = H$ for $D_1 \in \operatorname{Ded} H$.

Theorem 3.5. *Let S be a non-empty $\vee$-closed subset of H, i.e., $x \vee y \in S$ whenever $x, y \in S$. If $1 \notin S$, then $\mathcal{D} := \{D \in \operatorname{Ded} H \mid D \cap S = \emptyset\}$ has a maximal element M. Moreover M is a prime deductive system.*

Proof: Using Zorn's Lemma, we know that there exists a maximal deductive system M in $\mathcal{D}$ such that $M \cap S = \emptyset$. We prove that M is prime. If not, then there exist $x, y \in H$ such that $x \vee y \in M$, $x \notin M$ and $y \notin M$. Then M is properly contained in both $\langle M \cup \{x\}\rangle = M_1$ and $\langle M \cup \{y\}\rangle = M_2$. Since M is maximal, it follows that $M_1 \cap S \neq \emptyset$ and $M_2 \cap S \neq \emptyset$. Clearly $z_1, z_2 \leq z_1 \vee z_2$. In the following computations we used the fact that $x \vee y \in M$ and $\operatorname{Ded} H$ is distributive. Hence, by Lemma 2.2, $z_1 \vee z_2 \in M_1 \cap M_2 = \langle M \cup \{x\}\rangle \cap \langle M \cup \{y\}\rangle = (M \vee \langle x\rangle) \cap (M \vee \langle y\rangle) = M \vee (\langle x\rangle \cap \langle y\rangle) = M \vee \langle x \vee y\rangle = M$. Thus $z_1 \vee z_2 \in M \cap S$ contradicting $M \cap S = \emptyset$. □

Moreover, we can prove the following.

Theorem 3.6. *Let H be a commutative Hilbert algebra. A deductive system $\{1\} \neq M \in \operatorname{Ded} H$ is prime if and only if M is a $\wedge$-irreducible element of* $\operatorname{Ded} H$.

Proof: If M is a $\wedge$-irreducible element of $\operatorname{Ded} H$ and $x \vee y \in M$, then due to Lemma 2.2 and Theorem 3.2, we have

$$M = M \vee (\langle x \vee y \rangle) = M \vee (\langle x \rangle \cap \langle y \rangle) = (M \vee \langle x \rangle) \cap (M \vee \langle y \rangle).$$

Since M is $\wedge$-irreducible, either $M = M \vee \langle x \rangle$ or $M = M \vee \langle y \rangle$ giving either $x \in M$ or $y \in M$.

Conversely, let M be a prime deductive system and $M = M_1 \cap M_2$ for $M_1, M_2 \in \operatorname{Ded} H$. Suppose $M \subset M_1$, $M \subset M_2$ and $M_1 \neq M \neq M_2$. Then there are $x \in M_1 \setminus M$ and $y \in M_2 \setminus M$ with $x \vee y \in M_1 \cap M_2 = M$. Due to prime property, $x \in M$ or $y \in M$, a contradiction. □

Corollary 3.7. *Every proper (i.e., $\neq \{1\}$) deductive system D of a commutative Hilbert algebra H is the intersection of all prime deductive systems of H containing D.*

Theorem 3.8. *Every maximal deductive system of H is prime.*

Proof: Let M be a maximal deductive system of H. In view of Theorem 3.5, it is sufficient to show that $H \setminus M$ is $\vee$-closed. If $H \setminus M$ is not $\vee$-closed, then there exist $x, y \in A \setminus M$ such that $x \vee y \in M$. By the maximality of M, we have $\langle M \cup \{x\} \rangle = \langle M \cup \{y\} \rangle = H$. Thus M is properly contained in both $\langle M \cup \{x\} \rangle$ and $\langle M \cup \{y\} \rangle$, and so $M \subseteq \langle M \cup \{x\} \rangle \cap \langle M \cup \{y\} \rangle$. This contradicts Theorem 3.2, ending the proof. □

4. Annihilators in Hilbert algebras

The concept of an annihilator in Hilbert algebra was introduced by the first and second authors in [7]. Later on we will show how annihilators can be used for a description of all subdirectly irreducible finite commutative Hilbert algebras.

Definition 4.1 ([7])**.** Let C be a subset of a Hilbert algebra H. The set

$$A_H(C) := \{x \in H \mid xa = a \ \text{ for each } \ a \in C\}$$

is called the *annihilator* of C. If $C = \{a\}$ is a singleton, the annihilator $A_H(\{a\})$ will be denoted simply by $A_H(a)$ and called the *annihilator* of an element a.

It was proved in [7, Theorem 2] that for every $D \in \operatorname{Ded} H$ its annihilator $A_H(D)$ is also a deductive system of H and it is a pseudocomplement of D in the lattice $\operatorname{Ded} H$. Moreover, $A_H(a) \in \operatorname{Ded} H$ for every $a \in H$ and $A_H(D) = \bigcap\{A_H(d) \mid d \in D\}$ whence $A_H(C) \in \operatorname{Ded} H$ for any subset C of H.

In the case of commutative Hilbert algebras, we can state stronger results.

Lemma 4.2. *Let H be a commutative Hilbert algebra and $\emptyset \neq C \subseteq H$. Then*

$$A_H(C) = \{x \in H \mid x \vee a = 1 \ \textit{ for all } \ a \in C\}.$$

Proof: Denote by

$$M = \{x \in H \mid x \vee a = 1 \ \text{ for all } \ a \in C\}.$$

Suppose $x \in M$. Then $x \vee a = (xa)a = 1$ and thus $xa \leq a$. However, $a \leq xa$ thus $xa = a$, i.e., $x \in A_H(C)$. Conversely, if $x \in A_H(C)$ then $xa = a$ for each $a \in C$, whence $x \vee a = (xa)a = aa = 1$ giving $x \in M$. □

Theorem 4.3. *Let H be a commutative Hilbert algebra. For every subset M of H, we have $A_H(M) = A_H(\langle M \rangle)$.*

Proof: Since $M \subseteq \langle M \rangle$, we obtain $A_H(M) \supseteq A_H(\langle M \rangle)$. Now let $x \in A_H(M)$ and $a \in \langle M \rangle$. By Lemma 4.2, we have to show that $x \vee a = 1$. By Proposition 2.1, we get

$$m_1(m_2(\dots(m_n a)\dots)) = 1$$

for some $m_1, m_2, \dots, m_n \in M$. From $x \in A_H(M)$ it follows that $x \vee m_i = 1$ for $i = 1, 2, \dots, n$. By Lemma 4.2 this means that $m_i x = x$, $i = 1, 2, \dots, n$, and due to commutativity of H, also $x m_i = m_i$, $i = 1, 2, \dots, n$. Hence

$$\begin{aligned}
x = 1x &= (m_1(m_2(\dots(m_n a)\dots)))x \\
&= (m_1(m_2(\dots(m_n a)\dots)))(m_1 x) \\
&= m_1\Big(m_2(\dots(m_n a)\dots)x\Big) \\
&= m_1\Big(m_2(\dots(m_n a)\dots)(m_2 x)\Big) \\
&= m_1\Big(m_2(\dots(m_n a)\dots)x\Big) \\
&= \dots\dots\dots \\
&= m_1\Big(m_2(\dots(m_n(ax))\dots)\Big) \\
&= a\Big(m_1(m_2(\dots(m_n x)\dots))\Big).
\end{aligned}$$

This yields

$$\begin{aligned}
ax &= a\Big(a(m_1(m_2(\dots(m_n x)\dots)))\Big) \\
&= a\Big(m_1(m_2(\dots(m_n x)\dots))\Big) = x
\end{aligned}$$

and, by Lemma 4.2 again, $x \vee a = 1$. □

Theorem 4.4. *A finite commutative Hilbert algebra H is subdirectly irreducible if and only if H has a unique coatom with respect to the induced order.*

Proof: As it was stated in Section 2, the congruence lattice $\operatorname{Con} H$ is isomorphic to the lattice of congruence kernels, which is isomorphic to the lattice $\operatorname{Ded} H$ of deductive systems of H. Thus H is subdirectly irreducible iff there is a unique coatom in $\operatorname{Ded} H$, i.e. a unique maximal deductive system of H. By [7], $\operatorname{Ded} H$ is a pseudocomplemented lattice. Hence, H is subdirectly irreducible iff $A_H(a) = \{1\}$ for each $a \neq 1$. By Lemma 4.2, $A_H(a) = \{x \in H;\ x \vee a = 1\}$, thus in our case, $x \vee a = 1$ iff $x = 1$. Due to finiteness of H this yields that there is a unique coatom $b = \vee\{a \in H; a \neq 1\}$. □

Let L be a lattice and $a, b \in L$. By the *relative pseudocomplement of a with respect to b* we understand the greatest element (if it exists) $c \in L$ with the property $a \wedge c \leq b$.

Theorem 4.5. *Let H be a commutative Hilbert algebra. For any $B, C \in \operatorname{Ded} H$, the set*

$$M = \{x \in H \mid x \vee a \in B \ \text{ for all } \ a \in C\}$$

is the relative pseudocomplement of C with respect to B in the lattice $\operatorname{Ded} H$.

Proof: Let $x \in M$ and $xy \in M$. Then $x \vee a \in B$ and $(xy) \vee a \in B$, i.e., $((xy)a)a \in B$ for each $a \in C$. However, C is a deductive system of H and $xa \in C$, and thus $((xy)(xa))(xa) \in B$. Hence

$$u = (ya)(xa) = x((ya)a) = (x(ya))(xa) = ((xy)(xa))(xa) \in B$$

for all $a \in C$. Denote by $v = (ya)a$. We have

$$\begin{aligned}((xa)a)((ya)a) &= (ya)\Big(((xa)a)a\Big) = \Big((ya)((xa)a)\Big)\Big((ya)a\Big) \\ &= \Big(((ya)(xa))((ya)a)\Big)\Big((ya)a\Big) = (uv)v.\end{aligned}$$

Since $u \in B$ and B is a deductive system of H, it follows that

$$(uv)v = (1(uv))v \in B.$$

Together we have $((xa)a)((ya)a) \in B$ and since $(xa)a \in B$, also $y \vee a = (ya)a \in B$. Thus $y \in M$. It is almost clear that $1 \in M$, i.e., M is a deductive system of H. Now we show that $M \cap C \subseteq B$. Suppose $m \in M \cap C$. Then $m \vee a \in B$ for all $a \in C$. Setting $a = m$ we obtain $m \in B$. It is evident that M is the greatest set which is a deductive system of H and satisfying $M \cap C \subseteq B$, i.e., it is the relative pseudocomplement of C with respect to B in the lattice $\operatorname{Ded} H$. □

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