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Strong remote points

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Abstract. Remote points constructed so far are actually strong remote. But we construct remote points of another type.

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1. Introduction

Let $X^* = \beta X \setminus X$ be the remainder in the Čech-Stone compactification βX of a completely regular space X. A point $p \in X^*$ is called a *remote point* of X if it is not in the closure of any nowhere dense subset of X. In 1962, Fine and Gillman [5] introduced remote points and proved that, under CH, the real line has remote points. Van Douwen [2], and, independently, Chae and Smith [1], showed that if X is a nonpseudocompact space with countable π-weight, then X has remote points. Alan Dow [4] showed that a nonpseudocompact space X with π-weight ω_1 has remote points if either X satisfies the ccc-condition, or ω^ω has some additional set-theoretical properties. Some counterexamples appeared [3], [4].

An inspection of the relevant results reveals that the remote points constructed so far satisfy the following

Definition 1.1 ([7]). A point $p \in X^*$ is called a *strong remote point* of X iff p is a remote point of X and

($*$) there is a p-chain σ such that for any family of open sets $\mathcal{W} \subset 2^X$ the following holds: if $\mathcal{W} < \sigma$ and $p \in \operatorname{Ex}\bigcup\mathcal{W}$, then there is a subfamily $\mathcal{W}' \subset \mathcal{W}$ such that $\mathcal{W}' <_{\text{fin}} \sigma$ and $p \in \operatorname{Ex}\bigcup\mathcal{W}'$.

A countable discrete family σ of open sets is called a p-chain if $p \in \operatorname{Ex}\bigcup\sigma$. In [7] one can see that ($*$) is, apparently, more useful in research. In the present paper we show that not every remote point has property ($*$).

Theorem 1.2. *Every zero-dimensional, nowhere locally compact, separable and metrizable space has a remote point that is not strongly remote.*

2. Proofs

Let 2^X be set of all subsets of X. Then $U \in 2^X$ is clopen if it is closed and open simultaneously, we write $\operatorname{Ex} U = \beta X \setminus \operatorname{Cl}_{\beta X}(X \setminus U)$. A subset π of 2^X is called clopen if it consists of clopen sets; cellular if its members are mutually disjoint and locally finite, if for every $x \in X$ there is a neighborhood $Ox \subset X$ meeting at most finitely many members of π. For any $\sigma \subset 2^X$ we say that π refines σ, denoted $\pi < \sigma$, if for every $U \in \pi$ there is a $V \in \sigma$ such that $U \subseteq V$. If, in addition, $\{U \in \pi : U \subset V\}$ is finite or empty for every $V \in \sigma$, then π finitarily refines σ, $\pi <_{\text{fin}} \sigma$. Let $\mathcal{S} = \{\mu : \mu = (i_0, \dots, i_m) \in \omega^{m+1}, m \in \omega\}$ be all finite sequences of numbers $i \in \omega$ and we let $\mathcal{F}$ denote the set of functions from $\mathcal{S}$ to the family of finite subsets of ω.

From now on the conditions of Theorem 1.2 hold. There is, obviously, a family $\{\mathcal{P}_m\}_{m\in\omega}$ of cellular clopen covers $\mathcal{P}_m = \{U_\mu : \mu \in \omega^{m+1}\}$, where $U_{\mu k} \subsetneq U_\mu$ for each $k \in \omega$, such that $\mathcal{B} = \bigcup_{m\in\omega} \mathcal{P}_m$ is a base in X. Let $\mathcal{B}^* = \{\pi \subset \mathcal{B} : \pi$ is a cellular cover of $X\}$ and $\pi(f) = \{U_{\mu k} : U_\mu \in \pi$ and $k \in \omega \setminus f(\mu)\}$ for every $\pi \in \mathcal{B}^*$ and $f \in \mathcal{F}$.

To begin we recall the remarkable construction by van Douwen [2, 4.1]: For any $U_\mu \in \mathcal{B}$ we index $\mathcal{B}(U_\mu) = \{V \in \mathcal{B} : V \subset U_\mu\}$ as $\mathcal{B}(U_\mu) = \{V_\alpha\}_{\alpha\in\omega}$. For a nowhere dense set $F \subset X$ put $\alpha_0 = \min\{\alpha \in \omega : V_\alpha \cap F = \emptyset\}$ and $\mathcal{D}(F, U_\mu, 0) = \{V_{\alpha_0}\}$. Let for some $j \in \omega$, $\alpha_j \in \omega$ and $\mathcal{D}(F, U_\mu, j) \subset \mathcal{B}(U_\mu)$ have been constructed. Then for every $\alpha \le \alpha_j$, $\alpha^* = \min\{\beta \in \omega : V_\beta \subset V_\alpha \setminus F\}$, $\mathcal{D}(F, U_\mu, j+1) = \mathcal{D}(F, U_\mu, j) \cup \{V_{\alpha^*} : \alpha \le \alpha_j\}$ and $\alpha_{j+1} = \max\{\alpha \in \omega : V_\alpha \in \mathcal{D}(F, U_\mu, j+1)\}$. Finally, the family $\{\bigcup \mathcal{D}(F, U_\mu, n) : F$ is a nowhere dense subset of $X\}$ is n-centered for each $n \in \omega$ [2, 4.1].

Now for any $\mu, \nu \in \mathcal{S}$, $U_\mu \subseteq U_\nu$ iff $\nu = (i_0, \dots, i_t)$ is an initial segment of $\mu = (i_0, \dots, i_t, \dots, i_m)$. We set

$$\mathcal{D}_0(F, U_\mu, n) = \bigcup \{\mathcal{D}(F, U_{\nu k}, n) : U_\mu \subseteq U_\nu \in \mathcal{B} \text{ and } k \in \omega\}.$$

If $\mathcal{D}_j(F, U_\mu, n)$ has been defined for some $j \in \omega$, then

$$\mathcal{D}_{j+1}(F, U_\mu, n) = \mathcal{D}_j(F, U_\mu, n) \cup \bigcup \{\mathcal{D}_0(F, V, n) : V \in \mathcal{D}_j(F, U_\mu, n)\}.$$

And, finally,

$$\mathcal{D}(F) = \bigcup_{U_n \in \mathcal{P}_0} \mathcal{D}_{n+1}(F, U_n, n).$$

Claim 1. *Let $U_\mu \in \mathcal{B}$. Then $\mathcal{D}_m(F, U_\mu, n)$ is locally finite in X for any $m, n \in \omega$ and nowhere dense $F \subset X$.*

Proof: As $\{U_{\nu k}\}_{k\in\omega} \subset \mathcal{P}_{m+1}$ for any $\nu \in \omega^{m+1}$ and $\mathcal{P}_{m+1}$ is a cellular clopen cover of X, the family $\bigcup_{k\in\omega} \mathcal{D}(F, U_{\nu k}, n)$ is locally finite in X. It follows by its definition that $\mathcal{D}_0(F, U_\mu, n)$ is locally finite.

Let $\mathcal{D}_j(F, U_\mu, n)$ be locally finite for some $j \in \omega$. Then for any $x \in X$, $x \in U_\nu$ for some $U_\nu \in \mathcal{B}$ meeting at most finitely many sets $V \in \mathcal{D}_j(F, U_\mu, n)$. For each of them, $\mathcal{D}_0(F, V, n)$ is locally finite as above. For any other $V \in \mathcal{D}_j(F, U_\mu, n)$, $U_\nu \cap V = \emptyset$ implies by the definition of the base $\mathcal{B}$ that

$$\{W \in \mathcal{D}_0(F, V, n) : U_\nu \cap W \neq \emptyset\} \subset \bigcup\{\mathcal{D}(F, U_\eta, n) : U_\nu \subseteq U_\eta \in \mathcal{B}\}.$$

As the last set is finite, the proof is done. □

With insignificant modifications Claim 2 has been proved in [6].

Claim 2. *Let $U_n \in \mathcal{P}_0$. Then*

$$\bigcap_{i=0}^{n}(\bigcup \pi_i(f_i)) \cap \bigcap_{j=0}^{n}(\bigcup \mathcal{D}_{n+1}(F_j, U_n, n)) \neq \emptyset$$

for every $\pi_i \in \mathcal{B}^*$, $f_i \in \mathcal{F}$ and nowhere dense sets $F_j \subset X$.

It follows that the point p in Claim 3 does really exist.

Claim 3. *Any point $p \in X^*$ such that*

$$p \in \bigcap\{\mathrm{Cl}_{\beta X} \bigcup \pi(f) : \pi \in \mathcal{B}^* \text{ and } f \in \mathcal{F}\} \cap$$
$$\bigcap\{\mathrm{Cl}_{\beta X} \bigcup \mathcal{D}(F) : F \text{ is a nowhere dense subset of } X\}$$

satisfies the conditions of the theorem.

Proof: Being in the intersection of the second family, p is a remote point. For any p-chain σ we just have to show that σ does not satisfy $(*)$. Indeed, as X is strongly zero-dimensional, $Op \subset \mathrm{Ex} \bigcup \sigma$ for a clopen neighborhood $Op \subset \beta X$. For any $x \in X$ define $U(x) \in \mathcal{B}$ to be the maximal neighborhood with the following properties: either $U(x) \subset Op \cap V$ for some $V \in \sigma$, or $U(x) \cap Op = \emptyset$. As the sets from the cover $\{U(x) : x \in X\}$ are pairwise either disjoint or equal, there is a cellular subcover π. Let $\mathcal{W} = \{U_{\mu k} : U_\mu \in \pi, U_\mu \subset Op \text{ and } k \in \omega\}$. Then $\mathcal{W} < \sigma$, $p \in \mathrm{Ex} \bigcup \mathcal{W}$ and for any $\mathcal{W}' \subset \mathcal{W}$, $\mathcal{W}' <_{\text{fin}} \sigma$ implies $\mathcal{W}' <_{\text{fin}} \pi$. Define $f \in \mathcal{F}$ for any $\mu \in \mathcal{S}$ as follows: $f(\mu) = \{k : U_{\mu k} \in \mathcal{W}'\}$. Then $\bigcup \pi(f) \cap (\bigcup \mathcal{W}') = \emptyset$ and, so, $p \notin \mathrm{Ex} \bigcup \mathcal{W}'$. □

Our proof is complete.

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