Holomorphic subordinated semigroups

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Abstract. If $(e^{-tA})_{t>0}$ is a strongly continuous and contractive semigroup on a complex Banach space B, then $-(-A)^\alpha$, $0<\alpha<1$, generates a holomorphic semigroup on B. This was proved by K. Yosida in [7]. Using similar techniques, we present a class H of Bernstein functions such that for all $f \in H$, the operator $-f(-A)$ generates a holomorphic semigroup.

Keywords: holomorphic semigroup, Bernstein function

Classification: 35K65, 35B65, 35B40

Introduction

According to K. Yosida [7], if A is a generator of a bounded semigroup on a complex Banach space, we can define $-(-A)^\alpha$, $0<\alpha<1$, as a generator of a holomorphic semigroup. It is also known through G. Lumer's and L. Paquet's works [5], that some solutions of evolutionary equations for the Cauchy problem constitute some holomorphic semigroups. In 1989, L. Paquet has proved in [6] that the distributions T with support in $\mathbb{R}_+$ such that $LT = -(\int (\cdot)^{-\alpha}\, d\nu(\alpha))^{-1}$, where ν is a positive measure on $[0,1]$ and $\nu([0,1])>0$, are generators of pseudoholomorphic semigroups of measures on $\mathbb{R}_+$ in the sense of [6]. We will study in this work the holomorphy of the subordinated semigroups. More precisely, we will present a class of Bernstein functions such that for any strongly continuous and contractive semigroup $(T_t)_{t>0}$ on a complex Banach space the subordinated semigroup to $(T_t)_{t>0}$ is holomorphic.

Being inspired by the holomorphy of fractionary semigroups [7], we consider the set of Bernstein functions f verifying $\operatorname{Re} f(z) \geq c|\operatorname{Im} z|^\alpha$ in a sector of the complex plane and for $|\operatorname{Im} z| > \rho > 0$. Such functions have a semigroup of subprobability measures $(\rho_t)_{t>0}$ which is absolutely continuous with respect to Lebesgue measure on $[0,+\infty[$.

In Theorem 1, we give an integral representation of the density $f_t(s)$ by means of the function f. Moreover, we show that for all $s>0$ the density $f_t(s)$ is differentiable with respect to t, on $[0,+\infty[$.

We are also interested in the holomorphy of the semigroup $(\rho_t)_{t>0}$. Using the homogeneity of the function s^α, $0<\alpha<1$, K. Yosida has shown that the associated semigroup is holomorphic. In the general case many difficulties arise in the study of this last property. For this reason we add the hypothesis of regularity on the Bernstein function f, $f(s) \le c' s^{\alpha}$, $s > 1$, that allows us to deduce the holomorphy of the subordinated semigroup $(T_t^f)_{t>0}$ to a strongly continuous and contractive semigroup $(T_t)_{t>0}$. We may note that we can find again the result in the case of the fractional powers.

1. Bernstein functions and associated convolution semigroups

Definition 1. A positive function f on $[0,+\infty[$ is called a Bernstein function if f is C^{∞} on $[0,+\infty[$ and for all $n \in \mathbb{N}^*$, $(-1)^n f^{(n)} \le 0$.

In the following f denotes a Bernstein function.

According to [2, Theorem 9.8, p. 64] we have the following property:

Every Bernstein function f possesses the following representation

$$f(s) = a + bs + \int_0^{+\infty} (1 - e^{-rs})\mu(dr) \tag{1}$$

where a, b are two positive reals and μ is a positive measure on $[0,+\infty[$ such that $\int_0^{+\infty} \frac{r}{1+r}\mu(dr) < +\infty$.

If the measure in (1) is absolutely continuous with respect to Lebesgue measure on $[0,+\infty[$ and the density is completely monotone, then f is said to be a complete Bernstein function and by applying Bernstein theorem ([2, Theorem 9.3, p. 62]) to the density and Fubini's theorem, representation (1) becomes

$$f(s) = a + bs + \int_0^{+\infty} \frac{s}{s+r}\rho(dr) \tag{2}$$

where ρ is a positive measure on $[0,+\infty[$ verifying $\int_0^{+\infty} \frac{1}{1+r}\rho(dr) < +\infty$.

For every $t > 0$ and for every Bernstein function f, the function defined on $[0,+\infty[$ by $s \to e^{-tf(s)}$ is completely monotone. From [2, Proposition 9.2 and Theorem 9.3] there exists one positive measure ρ_t on $[0,+\infty[$ such that

$$\int_0^{+\infty} e^{-rs}\rho_t(dr) = e^{-tf(s)} \quad \forall\, s > 0,$$

and by [2, Theorem 9.18] the family of measures $(\rho_t)_{t>0}$ forms a convolution semigroup on $\mathbb{R}_+$.

For every complex number z such that $\operatorname{Re} z \ge 0$ and for every $r \ge 0$, we have

$$|1 - e^{-rz}| \le r|z| \quad \text{and} \quad |1 - e^{-rz}| \le 2 \tag{2'}$$

which shows according to (1) that every Bernstein function is extendable to a continuous function on $\mathbb{C}_+ := \{z \in \mathbb{C}, \operatorname{Re} z \geq 0\}$ and to a holomorphic function on $\mathbb{C}_+^* := \{z \in \mathbb{C}, \operatorname{Re} z > 0\}$. This extension verifies

$$\overline{f(z)} = f(\overline{z}) \quad \text{and} \quad \operatorname{Re} f(z) \geq 0 \quad \forall\, z \in \mathbb{C}_+. \tag{3}$$

Moreover it follows easily from (1) and (2′) that every Bernstein function is in modulus dominated by an affine function of $|z|$ on $\mathbb{C}_+^*$.

It is known that if the function φ_t defined on $\mathbb{C}_+$ by $\varphi_t(z) := e^{-tf(z)}$ $(t > 0)$ is integrable on the line $D = \{\sigma + iy, y \in \mathbb{R}\}$ for some $\sigma \geq 0$, then the semigroup $(\rho_t)_{t>0}$ is absolutely continuous with respect to the Lebesgue measure and the density f_t is given by

$$f_t(s) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{sz - tf(z)} dz. \tag{4}$$

Moreover for all $s, t > 0$, we have

$$\int_0^{+\infty} f_t(r)\, dr = e^{-tf(0)} \quad \text{and} \quad f_t * f_s = f_{t+s}. \tag{5}$$

For $\theta \in]0, \pi[$, denote $\Lambda(\theta) = \{z \in \mathbb{C}^*, |\operatorname{Arg} z| < \theta\}$.
In the following assume that

(H_1): There exists $\varphi \in]\frac{\pi}{2}, \pi[$ such that the Bernstein function f has a holomorphic extension on $\Lambda(\varphi)$, and for all $z \in \Lambda(\varphi) \cap \{z \in \mathbb{C}, |\operatorname{Im} z| > \rho\}$, we have $\operatorname{Re} f(z) \geq c|\operatorname{Im} z|^\alpha$, for some $\rho > 0$, $c > 0$ and $\alpha > 0$.

Remark 1. Every complete Bernstein function has a holomorphic extension on $\mathbb{C} \setminus \mathbb{R}_-$.

Examples of Bernstein functions verifying (H_1):

Bernstein function $f(s)$	$\rho(dr)$
s^α, $0 < \alpha < 1$	$\frac{\sin \alpha\pi}{\pi}\, r^{\alpha-1}\, dr$
$s^{1/2}(1 - \exp(-4s^{1/2}))$	$\frac{2}{\pi} r^{-1/2} (\sin 2r^{1/2})^2 dr$
$s^{1/2} \log(1 + \coth s^{1/2})$	$\frac{1}{2\pi} r^{1/2} \log(1 + \operatorname{cotg} r^{1/2})^2 dr$
$s\frac{(s^{1/4}-1)}{s-1}$, $f(1) = \frac{1}{4}$	$\frac{\sqrt{2}}{2\pi} \frac{r^{1/4}}{1+r}\, dr$
$s^{1/2} \log(s^{1/2} + 1)$	$\frac{2}{2\pi} r^{-1/2} \log(1 + r)\, dr$
$1 - e^{-\beta s} + s^\alpha$, $0 < \alpha < 1$, $\beta > 0$	$\nexists$

Theorem 1. *Let f be a Bernstein function satisfying assumption* (H_1)*. For all* $\frac{\pi}{2} \leq \theta < \varphi$*,* $s > 0$ *and* $t > 0$ *we have:*

$$f_t(s) = \frac{1}{\pi} \int_0^{+\infty} \exp(rs\cos\theta - t \operatorname{Re} f(re^{i\theta})) \sin(sr\sin\theta - t \operatorname{Im} f(re^{i\theta}) + \theta) dr.$$

PROOF: Choose the closed contour (Γ_θ) below.

Since for all $t > 0$ the function $z \to e^{-tf(z)}$ is holomorphic on a neighborhood of (Γ_θ), we have by Cauchy theorem

$$\begin{aligned}
0 &= \frac{1}{2i\pi} \int_{\Gamma_\theta} e^{sz - tf(z)}\, dz = \frac{1}{2\pi} \int_{-\beta}^{\beta} e^{s(1+iy) - tf(1+iy)}\, dy \\
&+ \frac{1}{2i\pi} \int_1^0 e^{s(r+i\beta) - tf(r+i\beta)}\, dr \\
&+ \frac{1}{2\pi} \int_{\pi/2}^{\theta} e^{s\beta e^{i\psi} - tf(\beta e^{i\psi})} \beta e^{i\psi}\, d\psi + \frac{1}{2i\pi} \int_{\beta}^{\varepsilon} e^{sre^{i\theta} - tf(re^{i\theta})} e^{i\theta}\, dr \\
&+ \frac{1}{2\pi} \int_{\theta}^{-\theta} e^{s\varepsilon e^{i\psi} - tf(\varepsilon e^{i\psi})} \varepsilon e^{i\psi}\, d\psi + \frac{1}{2i\pi} \int_{\varepsilon}^{\beta} e^{sre^{-i\theta} - tf(re^{-i\theta})} e^{-i\theta}\, dr \\
&+ \frac{1}{2\pi} \int_{-\theta}^{-\pi/2} e^{s\beta e^{i\psi} - tf(\beta e^{i\psi})} \beta e^{i\psi}\, d\psi + \frac{1}{2i\pi} \int_0^1 e^{s(r-i\beta) - tf(r-i\beta)}\, dr \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8,
\end{aligned}$$

where

$$\begin{aligned}
I_2 + I_8 &= \frac{1}{2i\pi} \int_0^1 \left(e^{s(r-i\beta) - tf(r-i\beta)} - e^{s(r+i\beta) - tf(r+i\beta)} \right) dr \\
&= \frac{1}{2i\pi} \int_0^1 e^{sr - t\operatorname{Re} f(r+i\beta)} \left(e^{i(-sr + t\operatorname{Im} f(r+i\beta))} - e^{i(sr - t\operatorname{Im} f(r+i\beta))} \right) dr \\
&= \frac{1}{\pi} \exp(sr - t\operatorname{Re} f(r+i\beta)) \sin(-sr + t\operatorname{Im} f(r+i\beta))\, dr.
\end{aligned}$$

Assumption (H_1) implies that for sufficiently large β, we have

$$|I_2 + I_8| \le \frac{1}{\pi}\int_0^1 e^{sr - ct\beta^\alpha}\, dr$$

which tends to zero when β reaches to the infinity.

$$I_3 + I_7 = \frac{1}{2\pi}\int_{\pi/2}^{\theta}\left[e^{s\beta e^{i\psi} - tf(\beta e^{i\psi})}\beta e^{i\psi} - e^{s\beta e^{-i\psi} - tf(\beta e^{-i\psi})}\beta e^{-i\psi}\right] d\psi.$$

Thus

$$\begin{aligned}|I_3 + I_7| &\le \frac{1}{\pi}\int_{\pi/2}^{\theta} \beta e^{s\beta\cos\psi - ct|\beta\sin\psi|^\alpha}\left|\sin(s\beta\sin\psi - t\operatorname{Im} f(\beta e^{-i\psi}) + \psi)\right| d\psi \\ &\le \frac{1}{\pi}\int_{\pi/2}^{\theta} e^{-ct\beta^\alpha(\sin\theta)^\alpha}\, d\psi,\end{aligned}$$

which tends to zero when β reaches to the infinity.

$$I_5 = \frac{1}{2\pi}\int_{\theta}^{-\theta} e^{s\varepsilon e^{i\psi} - tf(\varepsilon e^{i\psi})}\varepsilon e^{i\psi}\, d\psi,$$

which reaches to zero when ε tends to zero. And

$$\begin{aligned}I_4 + I_6 &= \frac{1}{2i\pi}\int_{\beta}^{\varepsilon} e^{sre^{i\theta} - tf(re^{i\theta}) + i\theta}\, dr - \frac{1}{2i\pi}\int_{\beta}^{\varepsilon} e^{sre^{-i\theta} - tf(re^{-i\theta}) - i\theta}\, dr \\ &= \frac{1}{\pi}\int_{\beta}^{\varepsilon} \exp(sr\cos\theta - t\operatorname{Re} f(re^{i\theta}))\sin(sr\sin\theta - t\operatorname{Im} f(re^{i\theta}) + \theta)\, dr\end{aligned}$$

which proves that

$$f_t(s) = \frac{1}{\pi}\int_0^{+\infty} \exp(sr\cos\theta - t\operatorname{Re} f(re^{i\theta}))\sin(sr\sin\theta - t\operatorname{Im} f(re^{i\theta}) + \theta)\, dr.$$

□

Corollary. *For all $s > 0$ and for all Bernstein functions f satisfying assumption (H_1), we have:*
$t \to f_t(s)$ is differentiable on $[0, +\infty[$ and

(6)
$$\begin{aligned}&\frac{\partial}{\partial t} f_t(s) \\ &= -\frac{1}{\pi}\int_0^{+\infty}\left|f(re^{i\theta})\right| \exp(sr\cos\theta - t\operatorname{Re} f(re^{i\theta}))\sin(sr\sin\theta - t\operatorname{Im} f(re^{i\theta}) \\ &\qquad + \operatorname{Arg} f(re^{i\theta}) + \theta)\, dr,\end{aligned}$$

where $\frac{\pi}{2} \leq \theta < \varphi$.

PROOF: For all $z \in \Lambda(\varphi)$ and $s > 0$, the function g defined by $g(t) = e^{sz-tf(z)}$ is differentiable on $[0,+\infty[$ and the derivative g' is given by $g'(t) = -f(z)g(t)$.

Let $z = 1+iy$, $|y| > \rho > 0$ and $0 < a < t$, there exist two positive constants A and B such that for all $s > 0$ we have

$$|g'(t)| = |f(z)g(t)| \leq (A+By)e^{-ac|y|^\alpha+s},$$

which is integrable with respect to y on $\mathbb{R}$. Using the derivation theorem, the function $t \to f_t(s)$ is differentiable on $[0,+\infty[$ and we have

$$\frac{\partial}{\partial t} f_t(s) = -\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} f(z)e^{sz-tf(z)}\,dz, \quad \text{where} \quad s,\sigma > 0.$$

By integrating on the same contour (Γ_θ), we obtain

$$\begin{aligned}\frac{\partial}{\partial t} f_t(s) = -\frac{1}{\pi}\int_0^{+\infty} \Big[&\operatorname{Re} f(re^{i\theta}) \exp(sr\cos\theta - t\operatorname{Re} f(re^{i\theta}))\\ &\times \sin(sr\sin\theta - t\operatorname{Im} f(re^{i\theta}) + \theta)\\ &+ \operatorname{Im} f(re^{i\theta}) \exp(sr\cos\theta - t\operatorname{Re} f(re^{i\theta})) \cos(sr\sin\theta - t\operatorname{Im} f(re^{i\theta}) + \theta)\Big]\,dr.\end{aligned}$$

So

$$\begin{aligned}&\frac{\partial}{\partial t} f_t(s)\\ &= -\frac{1}{\pi}\int_0^{+\infty} \left|f(re^{i\theta})\right| \exp(sr\cos\theta - t\operatorname{Re} f(re^{i\theta})) \sin(sr\sin\theta - t\operatorname{Im} f(re^{i\theta})\\ &\qquad + \operatorname{Arg} f(re^{i\theta}) + \theta)\,dr. \qquad \square\end{aligned}$$

Proposition 1. *Let f be a Bernstein function verifying* (H_1). *Then* $g : t \to \int_0^\infty f_t(s)\,ds$ *is differentiable on* $[0,+\infty[$ *and if moreover we have*

(H_2): $\int_0^1 \frac{|f(re^{i\theta})|}{r}\,dr < +\infty$ *for some* $\frac{\pi}{2} < \theta < \varphi$,

then $\int_0^{+\infty} \frac{\partial}{\partial t} f_t(s)\,ds = 0$.

PROOF: The differentiability of the function g follows directly form (5). Now if f verifies $\int_0^1 \frac{|f(re^{i\theta})|}{r}\,dr < +\infty$, then this last assumption implies necessarily that $f(0) = 0$ and that the derivative of g verifies

$$g'(t) = \int_0^{+\infty} \frac{\partial}{\partial t} f_t(s)\,ds = \frac{\partial}{\partial t}\left(\int_0^{+\infty} f_t(s)\,ds\right) = \frac{\partial}{\partial t}\left(e^{-tf(0)}\right) = 0.$$

□

Remark 2. We note that the complete Bernstein function f defined by $f(s) = \int_0^{1/2} \frac{s}{(s+r)r(\log r)^2}\, dr$ verifies $\int_0^1 \frac{|f(re^{i\theta})|}{r}\, dr = +\infty$ for all $\frac{\pi}{2} < \theta < \pi$.

2. Holomorphic semigroups

In this part, we will consider a strongly continuous semigroup $(T_t)_{t>0}$ of contractive operators on a complex Banach space $(B, \|\cdot\|)$.

Definition 2. The semigroup $(T_t)_{t>0}$ is said to be θ-holomorphic, $0 < \theta \leq \frac{\pi}{2}$, if there exists a holomorphic extension $z \to T_z$ to $S_\theta = \{z \in \mathbb{C}^*; |\operatorname{Arg} z| < \theta\}$ such that

(i) $\forall\, z, z' \in s_\theta$, $T_{z+z'} = T_z \circ T_{z'}$;
(ii) $\forall\, \theta' \in]0, \theta[$, $\forall\, u \in B$, $\lim_{\substack{z \in s_\theta \\ z \to 0}} T_z u = u$.

For a Bernstein function f and the associated convolution semigroup $(\rho_t)_{t>0}$, we give the following definition.

Definition 3. The family of operators $(T_t^f)_{t>0}$, defined on B by

$$T_t^f u = \int_0^{+\infty} T_s u \rho_t(ds), \;\; u \in B$$

forms a semigroup on B, it is called the semigroup subordinated to $(T_t)_{t>0}$ with respect to f (or $(\rho_t)_{t>0}$).

Now assume that f is a Bernstein function verifying (H_1), (H_2) and that

(H_3): there exists a positive constant c' such that $f(r) < c' r^\alpha$ for $r > \rho' > 0$, α being the constant in (H_1).

For any function we deduce the central result of this work.

Theorem 2. *The subordinated semigroup $(T_t^f)_{t>0}$ is holomorphic.*

Proof: We will use the holomorphic semigroup characterization given in [7]. Let $t > 0$, $u \in B$ be as in Theorem 1. $T_t^f u$ is given by

$$T_t^f u = \frac{1}{\pi} \int_0^{+\infty} T_s u \int_0^{+\infty} \exp(sr\cos\theta - t \operatorname{Re} f(re^{i\theta})) \\ \times \sin(sr\sin\theta - t \operatorname{Im} f(re^{i\theta}) + \theta)\, dr\, ds$$

where $\frac{\pi}{2} < \theta < \varphi$, fixed by (H_2).

Assumption (H_2) implies that the function $t \to T_t^f u$ is differentiable on $[0,+\infty[$ and we have

$$(7) \quad \begin{aligned}(T_t^f)'u &= \frac{\partial}{\partial t} T_t^f u \\ &= -\frac{1}{\pi}\int_0^{+\infty} T_s u \int_0^{+\infty} \left| f(re^{i\theta})\right| \exp(sr\cos\theta - t\operatorname{Re} f(re^{i\theta})) \\ &\qquad \times \sin(sr\sin\theta - t\operatorname{Im} f(re^{i\theta}) + \operatorname{Arg} f(re^{i\theta}) + \theta)\, dr\, ds.\end{aligned}$$

Since $(T_t)_{t>0}$ is a contractive semigroup on B, then by Fubini theorem we will have

$$\|(T_t^f)'u\| \le \frac{\|u\|}{\|\cos\theta\|}\int_0^{+\infty} \frac{|f(re^{i\theta})|}{r} \exp(-t\operatorname{Re} f(re^{i\theta}))\, dr.$$

Let now $t \in]0,1[$ for sufficiently large $\beta > 0$. We have

$$t\|(T_t^f)'u\| \le \frac{\|u\|}{\|\cos\theta\|}\left(\int_0^{\beta} \frac{M|f(re^{i\theta})|}{r}\, dr + \int_\beta^{+\infty} \frac{c'tr^\alpha}{r}\exp(-tcr^\alpha \sin^\alpha\theta)\, dr\right),$$

where $M = \sup \exp\{-t\operatorname{Re} f(re^{i\theta}), (t,r) \in]0,1[\times[0,\beta]\}$. Since $\int_0^1 \frac{|f(re^{i\theta})|}{r}\, dr$ is finite, the first integral is finite as well.

For the second integral, by a change of variable $v = tcr^\alpha$, we obtain

$$\int_\beta^{+\infty} \frac{c'tr^\alpha}{r}\exp(-tcr^\alpha\sin^\alpha\theta)\, dr \le \frac{c'}{c\sin^\alpha\theta}\int_0^{+\infty}\frac{e^{-v}}{\alpha}\, dv = \frac{c'}{\alpha\sin^\alpha\theta}\,.$$

Then we can find a positive constant K such that

$$\forall\, t \in]0,1[,\ \|t(T_t^f)'\| \le K.$$

That implies, according to K. Yosida's theorem ([7, p. 254]) that the subordinated semigroup is holomorphic on the section Ω defined by $\Omega := \{z \in \mathbb{C}^*, |\operatorname{Arg} z| < tg^{-1}(\frac{1}{eK})\}$, and for all $z \in \Omega$, $u \in B$, $T_t^f u$ is given locally by

$$T_t^f u = \sum_{n\ge 0} \frac{(z-t)^n}{n!}(T_t^f)^{(n)}u.$$

□

Remark 3. (1) Let f be a Bernstein function. If the convolution semigroup associated with f is $(\rho_t)_{t>0}$, then for all positive constants λ, the convolution semigroup associated to $f+\lambda$ is $(\mu_t)_{t>0}$ where $\mu_t = e^{-\lambda t}\rho_t$. The semigroup

$(\rho_t)_{t>0}$ is holomorphic if and only if $(\mu_t)_{t>0}$ is holomorphic ([6]). In particular we can assume that $f(0)=0$.

(2) We note that Theorem 2 is also true for the Bernstein function $f(s)=\sqrt{s}\log(1+\sqrt{s})$, though, condition ($\mathrm{H}_3$) is not satisfied.

Below we shall present a direct proof.

For $\frac{\pi}{2}<\theta<\pi$, we have in this case

$$\begin{aligned}\operatorname{Re} f(re^{i\theta}) &\cong \cos\frac{\theta}{2}\sqrt{r}\log\sqrt{r},\ \ (r\to+\infty),\\ |f(re^{i\theta})| &\cong \sqrt{r}\log\sqrt{r},\ \ (r\to+\infty),\end{aligned}$$

and

$$|f(re^{i\theta})|\cong r,\ \ (r\to 0).$$

By using in (7) the change of variable, $sr=v$, we obtain

$$\begin{aligned}(T_t^f)'u = -\frac{1}{\pi}\int_0^{+\infty} T_{\frac{v}{r}}u\int_0^{+\infty}\frac{|f(re^{i\theta})|}{r}&\exp(v\cos\theta - t\operatorname{Re} f(re^{i\theta}))\\ &\times\sin(v\sin\theta - t\operatorname{Im} f(re^{i\theta})+\operatorname{Arg}(re^{i\theta})+\theta)\,dr\,dv.\end{aligned}$$

That gives for all $t>0$

$$\|(T_t^f)'u\|\le\frac{\|u\|}{\pi}\int_0^{+\infty}\frac{|f(re^{i\theta})|}{r}\exp(-t\operatorname{Re} f(re^{i\theta}))\,dr\cdot\int_0^{+\infty}\exp(v\cos\theta)\,dv.$$

Let $0<t\le 1$, and a smooth positive real β

$$\begin{aligned}\|(T_t^f)'u\| &\le \frac{\|u\|}{\pi|\cos\theta|}\int_0^{+\infty}\frac{|f(re^{i\theta})|}{r}\exp(-t\operatorname{Re} f(re^{i\theta}))\,dt\\ &\le\frac{k\|u\|}{\pi|\cos\theta|}\left[1+\int_\beta^{+\infty}\frac{t\log\sqrt{r}}{\sqrt{r}}\exp(-t\sqrt{r}\log\sqrt{r})\,dr\right]\end{aligned}$$

where k is a positive constant.

A second change of variable $\omega=t\sqrt{r}\log\sqrt{r}$ gives $\|t(T_tf)'u\|\le\frac{2k\|u\|}{\pi|\cos\theta|}$, $t\in\,]0,1[$ and $u\in B$. The proof is achieved according to K. Yosida [7]. $\square$

Now we start from a Bernstein function f, $(\rho_t)_{t>0}$ is the associated convolution semigroup and we suppose that the semigroup $(\rho_t *,\cdot)_{t>0}$ is holomorphic on $C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ vanishing at infinity, and $(\rho_t *,\cdot)_{t>0}$ acts on $C_0(\mathbb{R})$ by

$$(\rho_t * h)(x)=\int_0^{+\infty}h(x-s)\rho_t(ds).$$

A necessary condition is proved in [1] and we have the following characterization

Theorem 3. *If the semigroup $(\rho_t *, \cdot)_{t>0}$ is holomorphic on $C_0(\mathbb{R})$, then the Bernstein function f satisfies the condition*

$$|f(z)| \leq C|z|^\gamma, \ \operatorname{Re} z > 0, \ |z| \geq 1, \ 0 < \gamma < 1 \ \text{ and } \ C = \frac{3(1+f(1))}{1-e^{-1}}.$$

Remark 4. (1) This result shows that the introduced hypothesis (H_3) is natural.

(2) If f is a Bernstein function, then the function g defined by $g(s) = [f(\frac{1}{s})]^{-1}$ is also a Bernstein function (see [3, Lemma 5]). In particular if ν is a measure on $[0,1]$ of the form $\sum_n c_n \delta_{\alpha_n}$ such that $0 \leq \alpha_n \leq 1$ and $\sum_n c_n < +\infty$, then the Bernstein function $(\int (\cdot)^{-\alpha}\, d\nu(\alpha))^{-1}$ verifies (H_1), (H_2) and (H_3) if and only if $\sup_n \alpha_n < 1$.

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(*Received* May 7, 2001, *revised* November 26, 2001)