Spaces in which compact subsets are closed and the lattice of T_1-topologies on a set

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Abstract. We obtain some new properties of the class of KC-spaces, that is, those topological spaces in which compact sets are closed. The results are used to generalize theorems of Anderson [1] and Steiner and Steiner [12] concerning complementation in the lattice of T_1-topologies on a set X.

Keywords: KC-space, T_1-complementary topology, T_1-independent, sequential space

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The lattice $\mathcal{L}_1(X)$ of T_1-topologies on a set X has a least element 0 (the cofinite topology) and a greatest element 1 (the discrete topology) but it is known that there is T_1-topology τ (even a T_2-topology) with no T_1-complement, that is there is no topology μ such that $\mu \vee \tau = 1$ and $\mu \wedge \tau = 0$ (see [11] and [13]). These negative results notwithstanding, many T_1-spaces with "nice" properties have T_1-complements which do not share these properties. For example, it is known that the T_1-complements of many Hausdorff spaces are not Hausdorff (see [12] and [1]) and it is the purpose of this article to extend results of this kind. We study T_1-complementarity using two weaker properties:

Say that two T_1-topologies τ and τ' on a set X are T_1-*independent* (respectively, *transversal*) if $\tau \cap \tau'$ is the cofinite topology (respectively, $\tau \vee \tau'$ is the discrete topology). As we mentioned in the previous paragraph, if τ and τ' are both T_1-independent and transversal, they are said to be T_1-*complementary*.

Central to our results will be the following property: A topological space (X, τ) is said to be a KC-*space* if every compact subspace is closed. The topology will then be termed a KC-topology. Note that KC-spaces are T_1 and T_2-spaces are KC (but not vice versa necessarily) and that a sequence in a KC-space can converge to at most one point. The KC-spaces (which sometimes have been called T_B-spaces) have been studied by a number of authors (see for example [4] and [14]). We will obtain some new properties of this class of spaces with the aim of applying the results to problems concerning the lattice $\mathcal{L}_1(X)$.

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Following [7], we say that a space X has the *finite derived set property* (which we abbreviate as the FDS-*property*) if whenever A is infinite, there is an infinite subset $B \subseteq A$ such that B has only a finite number of accumulation points in X, that is to say, its derived set B^d is finite. It is not hard to show that each weakly Whyburn T_1-space (introduced and called a WAP-space in [8], but see [6] for the reasons for the change of name) and each sequential KC-space has the FDS-property.

To complete our list of definitions, we recall that if $\mathcal{P}$ is a topological property, then a space (X,τ) is said to be *minimal* $\mathcal{P}$ (respectively *maximal* $\mathcal{P}$) if (X,τ) has property $\mathcal{P}$ but no topology on X which is strictly smaller (respectively, strictly larger) than τ has $\mathcal{P}$. A space (X,τ) is said to be *Katětov* $\mathcal{P}$ if there is a topology $\sigma \subseteq \tau$ such that (X,σ) is minimal $\mathcal{P}$. Specifically, we are here interested in minimal KC-spaces, Katětov KC-spaces and maximal compact spaces. All other terms are standard and can be found in [3].

In 1967, Steiner and Steiner proved that no Hausdorff topology on a countably infinite set has a Hausdorff complement. In fact, although they did not explicitly say so, they proved that no Hausdorff topology on a countably infinite set has a complementary KC-topology. In the same article they showed that any complement of a first countable topology on an infinite set X must be countably compact on cofinite subspaces and Anderson [1] showed that such a complement cannot be both first countable and Hausdorff. In this paper, we generalize results of [7] to non-Hausdorff spaces and in the process, we generalize the above results of [1] and [12].

The following result is a slight generalization of Theorem 3.1 of [7].

Theorem 1. *Suppose (X,τ) is a T_1-space with the FDS-property and τ' is an independent topology for τ; if (X,τ') is a KC-space, it is countably compact and has no non-trivial convergent sequences.*

Proof: Suppose first that (X,τ') is not countably compact, then it contains some countably infinite closed discrete subspace D, whose complement we can also assume to be infinite. Since τ and τ' are complements, D is not closed in (X,τ) and since this latter space has the FDS-property, there is some $B \subseteq D$ such that B has only a finite number of accumulation points $\{x_1,\ldots,x_n\}$ in X. Since $B \cup \{x_1,\ldots,x_n\}$ is an infinite proper τ'-closed subset of X, we have constructed an infinite subset which is closed in both topologies, a contradiction.

Now suppose that S is a non-trivial convergent sequence in (X,τ') (convergent to x say) such that $X \setminus S$ is infinite. Since (X,τ') is a KC-space, $S \cup \{x\}$, being compact, is an infinite τ'-closed set and hence is not τ-closed. However, since (X,τ) has the FDS-property, there is some infinite $B \subseteq S$ with only a finite number of accumulation points which we again denote by $\{x_1,\ldots,x_n\}$. It is then clear that $B \cup \{x_1,\ldots,x_n,x\}$ is an infinite set which is closed in both topologies, a contradiction. $\square$

Note that in the first part of the above proof we need only to require that (X, τ') be a T_1-space.

Every countably infinite, compact KC-space has a non-trivial convergent sequence. Suppose X is such a space and let $p \in X$ be non-isolated. Then $X \setminus \{p\}$ is not compact, hence not countably compact and so there is an infinite closed discrete subspace $A \subseteq X \setminus \{p\}$. Enumerating $A = \{x_n : n \in \omega\}$, it is clear that $\{x_n\}$ converges to p in X. However, with a little care we can prove much more.

Theorem 2. *If X is a countable, compact T_1-space and $A \subseteq X$ then either A is compact or there is a sequence in A converging to a point of $X \setminus A$.*

Proof: Suppose $A \subset X$ is not compact. Let D be an infinite discrete subset of A which is closed in A. Since X is compact, $D^d \neq \emptyset$ and $D^d \subseteq X \setminus A$. We enumerate $\mathrm{cl}(D) \setminus D$ as $\{x_n : n \in \omega\}$ and we will show that for some n, x_n is the limit of a sequence in D, showing that A is not sequentially closed.

If each neighborhood U of $x_0 = z_0$ is such that $D \setminus U$ is finite, then any enumeration of D converges to z_0. If not, then pick an open set U_0 such that $D \setminus U_0$ is infinite and $z_0 \in U_0$; note that since X is compact and D is discrete, $(\mathrm{cl}(D) \setminus D) \setminus U_0 \neq \emptyset$. Now let $z_1 = x_{m_1}$, where $m_1 = \inf\{n \in \omega : x_n \notin U_0\}$. If each neighborhood U of z_1 is such that $(D \setminus U_0) \setminus U$ is finite, then any enumeration of $D \setminus U_0$ will converge to z_1. Having chosen points $z_0, \dots, z_{k-1}$ and open sets containing them $U_0, \dots, U_{k-1}$ in such a way that $D \setminus \bigcup\{U_j : 0 \leq j \leq k-1\}$ is infinite, it is clear as before that $(\mathrm{cl}(D) \setminus D) \setminus \bigcup\{U_j : 0 \leq j \leq k-1\}$ is non-empty and we let $z_k = x_{m_k}$ where $m_k = \inf\{n \in \omega : x_n \notin \bigcup\{U_j : 0 \leq j \leq k-1\}$. As before, either every neighborhood U of z_k is such that $(D \setminus U) \setminus \bigcup\{U_j : 0 \leq j \leq k-1\}$ is finite (in which case we obtain a sequence convergent to z_k) or there is some $U = U_k$ for which this set is infinite.

However, since D is locally compact, $\mathrm{cl}(D) \setminus D$ is compact and hence for some $n \in \omega$, $(\mathrm{cl}(D) \setminus D) \setminus \bigcup\{U_j : 0 \leq j \leq n\} = \emptyset$, but $(\mathrm{cl}(D) \setminus D) \setminus \bigcup\{U_j : 0 \leq j \leq n-1\} \neq \emptyset$. It is then the case that any enumeration of $D \setminus \bigcup\{U_j : 0 \leq j \leq n-1\}$ will converge to z_n. □

Corollary 3. *A compact, countable KC-space is sequential.*

Proof: If A is not closed, then it is not compact. The result now follows from the previous theorem. □

However, a compact countable KC-space does not have to be first countable as the one-point compactification (see [3, 3.5.11]) of a sequential, non-first countable space (for example, the space of [3, 1.6.19]), illustrates. Nor does such a space have to be scattered — the one-point compactification of the rationals is the relevant example here.

Corollary 4. *A countable KC-space has no non-trivial convergent sequences if and only if every compact subspace is finite.*

The next result generalizes Proposition 3.2 of [7].

Corollary 5. *No countably infinite sequential T_2-space has an independent topology which is KC.*

Proof: Let (X, τ) be a Hausdorff sequential space. (X, τ) can be condensed onto a second countable Hausdorff space (X, τ') and hence by Theorem 1, any independent T_1-topology μ must be compact and have no non-trivial convergent sequences. It follows from Corollary 3 that (X, μ) is not KC. □

A similar result was first proved by Steiner and Steiner who showed:

Theorem 6 ([12, Corollary to Theorem 2]). *If (X, τ) is a countable Hausdorff space and τ' is T_1-complementary, then every cofinite subset of (X, τ') is compact.*

The next corollary improves (for countable spaces) a result of Wilansky [14, Theorem 5], who showed that the 1-point compactification of a KC-space is KC if and only if X is a k-space (see [3, 3.3.18]). We note in passing that a countable Hausdorff k-space is clearly sequential, but we are not aware of a direct proof that a countable KC-space which is a k-space is sequential.

Corollary 7. *The 1-point compactification of a countable KC-space X is KC if and only if X is sequential.*

Proof: The sufficiency is clear since an open subspace of a sequential space is sequential.

For the necessity, suppose that C is a compact subspace of the 1-point compactification $Y = X \cup \{\infty\}$ of X. If $\infty \notin C$ then C is a compact subspace of X, hence closed and so $Y \setminus C$ is open in Y. If on the other hand $\infty \in C$, then if C is not closed in Y, $C \cap X$ is not closed in X and hence there is a sequence $\{x_n\}$ in $C \cap X$ converging to some $p \notin C$. Since X is KC, the compact set $S = \{p\} \cup \{x_n : n \in \omega\}$ is closed in X and so $Y \setminus S$ is a neighborhood of ∞ and so ∞ is not an accumulation point of $\{x_n\}$, implying that C is not compact, a contradiction. □

A problem attributed to R. Larson by Fleissner in [4] is whether a space is maximal compact if and only if it is minimal KC. It was shown in [9] that a maximal compact space is KC, and hence is minimal KC, since any topology weaker than a compact KC topology cannot be KC. However, the converse problem of whether every minimal KC topology is compact appears to be still open. We now show that Larson's question has a positive answer in the case of countable spaces, but for clarity, we split the proof into two parts. First we show that a countable KC-space has the FDS-property.

Lemma 8. *If X is a countable KC-space, then every infinite $D \subseteq X$ contains an infinite subset with only a finite number of accumulation points (in X).*

Proof: Enumerate X as $\{x_n : n \in \omega\}$ and suppose that $D \subseteq X$ is infinite and every infinite subset of D has infinitely many accumulation points. Let $n_0 \in \omega$ be the smallest integer such that x_{n_0} is an accumulation point of D. If each neighborhood V of x_{n_0} has the property that $D \setminus V$ is finite, then any enumeration of D converges to x_{n_0} and hence D has only one accumulation point, a contradiction. Thus we may choose an open neighborhood V_0 of x_{n_0} such that $D_1 = D \setminus V_0$ is infinite. Having chosen points $x_{n_0}, x_{n_1}, \dots, x_{n_{j-1}}$ and open sets $V_0, V_1, \dots, V_{j-1}$ such that $x_{n_k} \in V_k$ for each $1 \le k \le j-1$ and $D_j = D \setminus (\bigcup\{V_k : 1 \le k \le j-1\})$ is infinite, we let n_j be the least integer such that x_{n_j} is an accumulation point of D_j and we choose a neighborhood V_j of x_{n_j} such that $D_j \setminus V_j = D_{j+1}$ is infinite. Such a choice is again possible for if every neighborhood V of x_{n_j} is such that $D_j \setminus V$ is finite, then any enumeration of D_j is a sequence which converges to x_{n_j} and hence D_j has only one accumulation point.

Now for each $j \in \omega$, we choose $y_j \in D_j \setminus \{y_0, y_1, \dots, y_{j-1}\}$ and we denote the set $\{y_n : n \in \omega\}$ by S. It is clear that S is infinite and all but finitely many points of S are contained in D_j for each $j \in \omega$ and so an accumulation point of S is an accumulation point of $S \cap D_j$ for each $j \in \omega$. Thus S can have no accumulation point, since if p were such a point, then for some $k \in \omega$, $p = x_k$ and from the construction, we would have that $k \ge n_j$ for each $j \in \omega$, which is absurd. □

Lemma 9. *If (X, τ) is a countable non-compact KC-space with the FDS-property, then X can be condensed onto a weaker KC-space.*

Proof: Since X is not countably compact, there is some countably infinite closed discrete subspace $D = \{d_n : n \in \omega\} \subseteq X$. Fix $p \in X$ and $\mathcal{F} \in \beta\omega \setminus \omega$ and define a new topology σ on X as follows:

(i) if $p \notin U$, then $U \in \sigma$ if and only if $U \in \tau$,

and

(ii) if $p \in U$, then $U \in \sigma$ if and only if $U \in \tau$ and $\{n \in \omega : d_n \in U\} \in \mathcal{F}$.

Clearly (X, σ) is a T_1-space, $\sigma \subset \tau$ and for each $B \subseteq X$, $\mathrm{cl}_\sigma(B) \subseteq \mathrm{cl}_\tau(B) \cup \{p\}$. We show that (X, σ) is a KC-space. To this end, suppose to the contrary that A is a non-closed compact subset of (X, σ). Obviously $p \in \mathrm{cl}_\sigma(A)$ and there are two cases to consider:

(a) If $p \notin A$, then $\sigma|A = \tau|A$ and so A is compact and hence closed in (X, τ). Thus there is some $U \in \tau$ such that $p \in U$ and $U \cap A = \emptyset$. If $\{n \in \omega : d_n \in A\} \notin \mathcal{F}$, then $\{n \in \omega : d_n \in D \setminus A\} \in \mathcal{F}$ and for each $t \in D \setminus A$ we can choose $U_t \in \tau$ such that $t \in U_t$ and $U_t \cap A = \emptyset$. Then $p \in U \cup \bigcup\{U_t : t \in D \setminus A\} \in \sigma$ contradicting the fact that $p \in \mathrm{cl}_\sigma(A)$. Thus $\{n \in \omega : d_n \in A\} \in \mathcal{F}$ and then there is some infinite set $S \subset A \cap D$ such that $S \notin \mathcal{F}$ and S is then an infinite closed discrete subset of A in (X, σ), implying that $(A, \sigma|A)$ is not compact.

(b) If $p \in A$, then $\mathrm{cl}_\sigma(A) = \mathrm{cl}_\tau(A)$. If A is not closed in (X, τ), then A is not compact (thus not countably compact) in (X, τ), and so there is an infinite discrete subset $C \subseteq A$ which is closed in $(A, \tau|A)$. However, C is not closed in $(A, \sigma|A)$ and so $\mathrm{cl}_\sigma(C) \cap A = C \cup \{p\}$. This implies that $\{n : d_n \in \mathrm{cl}_\tau(C)\} \in \mathcal{F}$.

Since (X,τ) has the FDS-property, there is some infinite subset $B \subseteq C$ with only a finite number of accumulation points in X. Thus $\{n : d_n \in \mathrm{cl}_\tau(B)\} \notin \mathcal{F}$ which implies that B is closed and discrete in $(A, \sigma|A)$, implying in its turn that A is not compact in (X,σ). □

The following result, an immediate consequence of the previous two lemmas, is then a partial positive answer to the above-mentioned question of Larson.

Theorem 10. *Every minimal KC-topology on a countable set is compact.*

These results should be contrasted with the case of minimal Hausdorff spaces. An example of a countable minimal Hausdorff space which is not countably compact is given in [10, Example 100].

We also note that in [4], Fleissner constructed a countably compact KC-topology t on ω_1 which is not Katětov KC, that is to say, if $\tau \subseteq t$ is KC then there is a $\tau' \subset \tau$ which is also KC. It is easy to see that (ω_1, t) has the FDS-property and furthermore, if $\tau \subseteq t$ has the FDS-property then so does τ'. Thus (ω_1, t) cannot be condensed onto a space which is minimal with respect to being both KC and having the FDS-property.

We turn now to the problem of whether a second countable KC-topology can have a KC-topology which is complementary. Recall that if κ is a cardinal, then a space is *κ-discrete* if it is the union of (at most) κ discrete subspaces, (however, if $\kappa = \omega$, then we use the standard terminology, σ-discrete). First we need some preliminary results, the first of which is a slight generalization of Theorem 2.3 of [13] and we omit the similar proof. For a definition of network and network weight we refer the reader to [3, 3.1.17].

Lemma 11. *Let (X,τ) be a space of network weight κ; if μ is a transversal for τ, then μ is κ-discrete.*

Our aim now is to show that each infinite, countably compact σ-discrete T_1-space has a non-trivial convergent sequence, but for convenience, we first prove a preliminary lemma and separate the cases of countable and uncountable X.

Lemma 12. *If X is an infinite countably compact T_1-space which is the union of two discrete subspaces, then X has a non-trivial convergent sequence.*

Proof: Suppose X is the union of two discrete subsets, E and F. Without loss of generality, we assume that the points of E are isolated in X and those of F are the accumulation points of X; since X is countably compact, F is finite, say $F = \{x_j : 1 \le j \le n\}$. If $X \setminus \{x_1\}$ is not countably compact, then there is a discrete subspace $G_1 \subseteq X \setminus \{x_1\}$ whose unique accumulation point is x_1. $G_1 \cup \{x_1\}$ is then a countably compact (and hence compact) Hausdorff space with only one non-isolated point and thus must contain a non-trivial convergent sequence. If $X \setminus \{x_1\}$ is countably compact, then we replace X by $X \setminus \{x_1\}$ and consider the subspace $X \setminus \{x_1, x_2\}$. Since $X \setminus F$ is not countably compact, there is some m $(1 \leq m \leq n)$ for which $X \setminus \{x_1, \dots, x_{m-1}\}$ is countably compact but $X \setminus \{x_1, \dots, x_m\}$ is not. There is then a sequence converging to x_m. □

Lemma 13. *A countably infinite, compact T_1-space has a non-trivial convergent sequence.*

Proof: Let (X, τ) be such a space. We consider two cases, either (a) all discrete subspaces of X are finite, or (b) X has an infinite discrete subspace.

(a) Let $X = X_0$; if all discrete subspaces are finite, then either X has the cofinite topology and hence has a non-trivial convergent sequence or there is an infinite proper closed subset $X_1 \subset X_0$. In this case, $X_0 \setminus X_1 \neq \emptyset$ and we can choose $x_0 \in X_0 \setminus X_1$. Having chosen closed sets X_k and points x_k for each $k < n$ with the property that X_k is an infinite proper closed subset of X_{k-1} for each $k \in \{1, \dots, n-1\}$, and $x_k \in X_{k-1} \setminus X_k$ for each $k \in \{1, \dots, n-1\}$, there are two possibilities:

> Either X_{n-1} has the cofinite topology, in which case it has a non-trivial convergent sequence and the recursive process ends, or there is some infinite closed proper subspace $C \subset X_{n-1}$, in which case we define $X_n = C$ and choose $x_n \in X_{n-1} \setminus X_n$.

If it were the case that for all $n \in \omega$, X_n contains a proper closed infinite subset, then for each n, we would have that $U_n = X \setminus (X_n \cup \{x_k : 1 \leq k \leq n-1\})$ is an open set with the property that $x_k \in U_n$ if and only if $k = n$. Thus $\{x_k : k \in \omega\}$ would be an infinite relatively discrete set, contradicting the hypothesis. Hence there is some $m \in \omega$ for which the infinite set X_m contains no proper closed infinite subset, implying that X_m has the cofinite topology and thus contains a non-trivial convergent sequence.

(b) Suppose now that X contains an infinite discrete subspace D_0; denote by F_0 the closed subspace $\mathrm{cl}(D_0) \setminus D_0 \subseteq X$. Having defined closed subspaces F_γ for each $\gamma < \alpha$ $(\alpha < \omega_1)$, we define F_α as follows:

> If α is a limit ordinal, then $F_\alpha = \bigcap\{F_\gamma : \gamma < \alpha\}$. If $\alpha = \beta + 1$ and F_β contains an infinite discrete subset D_β, then let $F_\alpha = \mathrm{cl}(D_\beta) \setminus D_\beta$; otherwise define $F_\alpha = F_\beta$.

Note that if F_α contains an infinite discrete subspace, then $F_{\alpha+1}$ is a proper subset of F_α. The family $\{F_\alpha : \alpha \in \omega_1\}$ is a nested family of closed sets in the compact T_1-space X, and hence has non-empty intersection. Furthermore, since X is countable, there is some minimal $\lambda < \omega_1$ such that $F_\alpha = F_\lambda$ for all $\alpha > \lambda$; thus F_λ can contain no infinite discrete subspace. There are now three cases to consider:

(i) If F_λ is infinite, then we apply (a) above to obtain a non-trivial convergent sequence in F_λ.

(ii) If F_λ is finite and λ is a non-limit ordinal, say $\lambda = \gamma + 1$ then since $\gamma < \lambda$, $F_\lambda = \mathrm{cl}(D_\gamma) \setminus D_\gamma$ where D_γ is an infinite discrete subspace of F_γ. Since F_λ

is finite, it is discrete and so F_γ is the union of two discrete subspaces, namely D_γ and F_λ. The existence of a non-trivial convergent sequence now follows from Lemma 12.

(iii) If F_λ is finite and λ is a limit ordinal, say $\lambda = \sup\{\lambda_n : n \in \omega\}$ where $\lambda_k < \lambda_{k+1} \in \omega + 1$, then since $\lambda_n < \lambda$ for each $n \in \omega$, it follows that $F_{\lambda_n} \setminus F_\lambda$ is infinite for each n. Thus we can choose $p_n \in F_{\lambda_n} \setminus (F_\lambda \cup \{p_k : 1 \leq k \leq n-1\})$. Using an argument similar to that in (a) above, for each $m \in \omega$, $U_m = X \setminus (F_{\lambda_{m+1}} \cup \{p_k : 1 \leq k \leq m-1\})$ is an open set meeting $\{p_n : n \in \omega\}$ in $\{p_m\}$; thus $\{p_n : n \in \omega\}$ is discrete and hence $C = F_\lambda \cup \{p_n : n \in \omega\}$ is the union of two discrete subspaces. Furthermore, since $p_n \in F_{\lambda_n}$ and $F_\lambda = \bigcap\{F_{\lambda_n} : n \in \omega\}$, it follows that all the accumulation points of $\{p_n : n \in \omega\}$ lie in F_λ and so $C = \{p_n : n \in \omega\} \cup F_\lambda$ is compact. The existence of a non-trivial convergent sequence in C again follows from Lemma 12. □

Theorem 14. *An infinite, countably compact, σ-discrete T_1-space X has a non-trivial convergent sequence.*

Proof: If X is countable, the result follows from the previous lemma. If X is uncountable, then suppose $X = \bigcup\{D_n : n \in \omega\}$, where D_n is discrete for each $n \in \omega$. At least one of the sets D_n is necessarily infinite and we denote by n_0, the smallest integer for which this occurs. We define $X_0 = \mathrm{cl}(D_{n_0}) \setminus D_{n_0}$ and note that $X_0 \subset X$ is closed. There are three alternatives:

i) X_0 is finite and hence discrete, in which case $\mathrm{cl}(D_{n_0})$ is an infinite countably compact T_1-space which is the union of two discrete subspaces; the existence of a non-trivial convergent sequence in X_0 now follows from Lemma 12. Or,

ii) X_0 is countably infinite, in which case X_0 is a countably infinite compact T_1-space and the existence of a non-trivial convergent sequence in X_0 now follows from Lemma 13. Or,

iii) $X_0 \subseteq \bigcup\{D_n : n \in \omega \setminus \{n_0\}\}$ is uncountable, and hence for some $n \in \omega$, $X_0 \cap D_n$ is uncountable, in which case we denote by n_1 the smallest integer for which this occurs and let $X_1 = \mathrm{cl}(D_{n_1} \cap X_0) \setminus (D_{n_1} \cap X_0)$. The above process can now be repeated with X_1 in place of X_0.

Proceeding in this way, either:

(a) for some $j \in \omega$, the closed subspace X_j constructed at the jth step of the recursion is countable, in which case the arguments of i) or ii) above apply and we obtain a non-trivial convergent sequence in $X_j \subseteq X$, or
(b) condition iii) holds for each $j \in \omega$ and we obtain a nested (infinite) sequence of uncountable, countably compact closed subspaces $\{X_j : j \in \omega\}$, in which case we let $Y = \bigcap\{X_j : j \in \omega\}$.

Clearly Y is a non-empty, closed subset of X which meets each of the discrete sets D_n in a finite set and hence Y is countable. If Y is infinite, the existence of a non-trivial convergent sequence in Y follows from Lemma 13. If, on the other hand, Y is finite then it is discrete, and for each $k \in \omega$ we can choose $p_k \in X_k \setminus (Y \cup \{p_0, \dots, p_{k-1}\})$. As in the proof of Lemma 13, $\{p_n : n \in \omega\}$ is discrete and the subspace $Z = Y \cup \{p_n : n \in \omega\}$ is compact. Thus Z is the union of two discrete subsets, $\{p_n : n \in \omega\}$ and Y. The existence of a non-trivial sequence again follows from Lemma 12. □

Theorem 15. *No infinite KC-space with a countable network and the FDS-property* (*in particular, no infinite second countable KC-space*) *has a T_1-complementary topology which is KC.*

PROOF: Suppose (X, τ) is an infinite KC-space with a countable network and the FDS-property and μ is a complement for τ. By Lemma 11, (X, μ) is σ-discrete and by Theorem 1, μ is countably compact and has no non-trivial convergent sequences. This contradicts Theorem 14. □

For countable spaces we can do better, applying Lemma 8, we have the following strengthening of Theorem 3 of [12]:

Corollary 16. *No KC-topology on a countably infinite set has a complementary KC-topology.*

Steiner and Steiner [12, Theorem 2] have shown that any T_1-complement of an infinite first countable Hausdorff space must have non-closed countably compact subspaces, while Anderson and Stewart [2, Theorem 2] have shown that such a T_1-complement cannot be both Hausdorff and first countable. Furthermore, Anderson [1, Corollary 1] showed that every Hausdorff Fréchet-Urysohn space has (at least) one T_1-complement which is not KC. These results should be compared with the following theorem which is an immediate consequence of Theorems 1, 11, 15 and the fact that a sequential KC-space has the FDS-property:

Theorem 17. *A T_1-complement of an infinite sequential KC-space with a countable network is countably compact, σ-discrete, has no non-trivial convergent sequences and is not KC.*

A number of questions still remain open; some may have been posed before, but still seem interesting.

Question A. *Can every KC-space which is not countably compact be condensed onto a strictly weaker KC-topology?*

Theorem 10 gives a positive answer for countable spaces and in the general case a positive answer obviously implies that minimal KC-spaces are countably compact. Note that a KC-space cannot necessarily be condensed onto a KC-space with a convergent sequence — any compact Hausdorff space with no non-trivial convergent sequences is an example.

Question B. *When can a KC-space of network weight κ be condensed onto a KC-space of weight κ (or even onto a KC-space with the FDS-property)?*

In the case $\kappa = \aleph_0$, the answer is negative as the example of the 1-point compactification of the space of [3, 1.6.19] shows. This space is countable, compact KC (and hence minimal KC) with uncountable weight. Hence we are led to ask:

Question C. *Is every countable KC-space Katětov KC?*

It turns out that Question C has a somewhat simpler formulation; we need the following result:

Theorem 18. *A countable KC-space (X, τ) is Katětov KC if and only if there is a weaker sequential KC-topology $\sigma \subseteq \tau$.*

Proof: If (X, τ) is a countable Katětov KC space, then by Corollary 12, there is a weaker compact KC-topology σ on X. However, by Corollary 3, (X, σ) is sequential and the necessity follows.

For the sufficiency, suppose that (X, μ) is a countable KC-space and that $\tau \subseteq \mu$ is a sequential KC-topology. If (X, τ) is compact, then it is minimal KC and hence (X, μ) is Katětov KC. So we assume that (X, τ) is not compact. It follows from Lemma 7 that the one-point compactification $(\omega X, \omega)$ is sequential. Following [3, 3.5.11], we identify X with $\omega(X) \subseteq \omega X$ and denote the singleton $\omega X \setminus \omega(X) = \omega X \setminus X$ by $\{\Omega\}$. The topology of ωX will be denoted by τ_ω. Let y be any point of X and define a partition $\mathcal{P}$ of ωX by $\mathcal{P} = \{\{x\} : x \in X \text{ and } x \neq y\} \cup \{\{y, \Omega\}\}$ and denote by σ the quotient topology on $\mathcal{P}$. To further simplify the notation, we identify $x \in X$ $(x \neq y)$ with $\{x\} \in \mathcal{P}$ and $y \in X$ with $\{y, \Omega\} \in \mathcal{P}$ and in future we refer to (X, σ) rather than $(\mathcal{P}, \sigma)$. The quotient map from $(\omega X, \tau_\omega)$, to (X, σ) will be denoted by q and so:

† Since X is identified with a subset of ωX, if $x \in (X, \sigma)$ and $x \neq y$, then $q^{-1}[x] = x$ and $q^{-1}[y] = \{y, \Omega\}$.

Note that if $x \neq y$, then U is a τ-neighborhood of $x \in X$ if and only if it is a σ-neighborhood of x and $W \in \sigma$ is a σ-neighborhood of y if and only if $q^{-1}[W]$ is a τ_ω-open set containing $\{y, \Omega\}$. Thus $\sigma \subset \tau \subseteq \mu$ and clearly (X, σ) is a compact T_1-space. We will show that the space (X, σ) is KC and hence is minimal KC.

To this end, we note first that it follows from [5, Proposition 1.2] that any quotient of a sequential space is sequential and hence (X, σ) is sequential. To show that (X, σ) is a KC-space, suppose to the contrary that C is compact but not closed in (X, σ). Since (X, σ) is sequential:

‡ There is a sequence of distinct points $\{s_n\} \subseteq C$ convergent to $s \notin C$ in (X, σ) and since this space is T_1, we can assume without loss of generality that for each n, $s_n \neq y$.

However, C is compact and so $\{s_n\}$ must have an accumulation point $z \in C$, showing that the compact subspace $A = \{s_n : n \in \omega\} \cup \{s\}$ is not closed. Thus to prove that (X, σ) is a KC-space it suffices to show that the convergent sequence $\{s_n\}$ together with its limit s, is closed in (X, σ). However, if A is not closed, then since (X, σ) is sequential, there is a sequence in A converging to $t \notin A$. This sequence is a subsequence of the original sequence $\{x_n\}$ and hence must also converge to s. Thus in (X, σ) there is a sequence with two distinct limits s and t. We show that this leads to a contradiction.

Now if $y \notin \{s, t\}$, then by † and ‡, $\{s_n\}$ is sequence with two distinct limits s and t in ωX, contradicting the fact that ωX is a KC-space. Alternatively, if $y \in \{s, t\}$, say $y = s$, then since $t \neq y$, it again follows from † and ‡, that $\{s_n\}$ converges to t in the space ωX.

Now, since $S = \{s_n : n \in \omega\} \cup \{t\}$ is compact in the KC-space ωX, it is closed and hence y is not an accumulation point of the sequence $\{s_n\}$ in ωX. Thus by ‡, there is $U \in \tau_\omega$ with $y \in U$ such that $s_n \notin U$ for all n (and we can assume that $\Omega \notin U$ so that with the identifications we are making, $q[U] = U$). Furthermore, since S is compact, $V = \omega X \setminus S$ is an open neighborhood of Ω in ωX and $s_n \notin q[V]$ for all n. Now, since $q^{-1}[q[U \cup V]] = U \cup V$, it follows that $q[U \cup V]$ is a σ-neighborhood of y with the property that $s_n \notin q[U \cup V]$ for all n, contradicting the fact that the sequence $\{s_n\}$ converges to y in (X, σ). Clearly, the case $y = t$ is identical and we are done. □

Thus Question C is equivalent to the following:

Question C′. *Can every countable KC-space be condensed onto a KC-space which is sequential?*

Since each infinite compact Hausdorff space of size less than the continuum is scattered and has a non-trivial convergent sequence, we are led to ask:

Question D. *Does every countably compact KC-space of size less then $2^{\aleph_0}$ have the FDS-property?*

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