Gradient estimates for elliptic systems in Carnot-Carathéodory spaces

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Abstract. Let $X = (X_1, X_2, \dots, X_q)$ be a system of vector fields satisfying the Hörmander condition. We prove $L^{2,\lambda}_X$ local regularity for the gradient Xu of a solution of the following strongly elliptic system

$$-X^*_\alpha(a^{\alpha\beta}_{ij}(x)X_\beta u^j) = g_i - X^*_\alpha f^\alpha_i(x) \quad \forall i = 1, 2, \dots, N,$$

where $a^{\alpha\beta}_{ij}(x)$ are bounded functions and belong to Vanishing Mean Oscillation space.

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1. Introduction

In the last decades a considerable interest has been paid to the problem of local gradient estimates for the solutions of elliptic equations and systems. Namely, let us consider the uniformly elliptic system

$$-D_\alpha(a^{\alpha\beta}_{ij}(x)D_\beta u^j) = g_i - D_\alpha f^\alpha_i(x), \quad i = 1, 2, \dots, N, \tag{1}$$

where $i, j = 1, 2, \dots, N$ and $\alpha, \beta = 1, 2, \dots, n$. An interesting problem is to show that there exists $c \geq 0$ such that if $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ then

$$\|Du\|_{L^{2,\lambda}(\Omega')} \leq c\left(\|Du\|_{L^2(\Omega'')} + \|g\|_{L^{2Q/(Q+2),\lambda Q/(Q+2)}(\Omega)} + \|f\|_{L^{2,\lambda}(\Omega)}\right).$$

The first remarkable contribution is due to Agmon, Douglis and Nirenberg, (see [2] and [3]), at least in the case of elliptic systems with uniformly continuous coefficients.

Later, Miranda ([39]) generalized the estimates in the case of a single equation assuming $a_{ij} \in W^{1,n}$. At the same time (see [11]) Cordes obtained the same result without any smoothness assumption. However he supposed a geometric condition on the eigenvalues of the matrix of the coefficients a_{ij} to hold true.

The beginning of '90 saw another approach to the problem. In [13], a new hypothesis on the coefficients was introduced. In [13] it was assumed that the coefficients of the principal part belong to a class that may contain discontinuous functions. This class was introduced in [41] for other reasons and it is known as the class of the functions that have vanishing mean oscillation, i.e. VMO.

In [13] an equation in nondivergence form was studied but later the same technique has been adapted to cover the case of divergence form equation. The technique, introduced in [13], was based on an implicit representation formula for the derivatives of highest order. In that representation formula the highest order derivatives were expressed by particular integral operators. Using real analysis results, Chiarenza, Frasca, Longo got the desired estimates.

Later, these assumptions and techniques were generalized to a more abstract setting (e.g. [12], [14], [1], [20], [7], [17], [18], [19], [21], [32], [4], [6], [5]).

Huang in [33] was able to get similar estimates for uniformly elliptic systems of the kind (1) applying Campanato's technique ([9], [10], [29] and [30]).

We stress that all the previous results refer to uniformly elliptic systems. The case of degenerate equations and systems is much more delicate (see [4], [6], [5], [15], [16], [25], [42]). An approach that can overcome the difficulties in this context is to introduce some new metrics in $\mathbb{R}^n$ such that the system is no longer degenerate with respect to these metrics. One of these is the Carnot-Carathéodory metric, that is generated by the sub-unit curves with respect to a given system of vector fields $X = (X_1, X_2, \dots, X_q)$.

The aim of this work is to show that the gradient estimates still hold true in the very general setting of Carnot-Carathéodory spaces. We suppose that the system X of vector fields satisfies the Hörmander condition in $\mathbb{R}^n$; this means that the vector fields and their commutators up to some order generate $\mathbb{R}^n$ as vector space (e.g. [15], [16], [25], [31], [35], [42]).

More precisely we study the system

$$-X_\alpha^*(a_{ij}^{\alpha\beta}(x)X_\beta u^j) = g_i - X_\alpha^* f_i^\alpha(x), \quad i = 1, 2, \dots, N, \tag{2}$$

where $i, j = 1, 2, \dots, N$, $\alpha, \beta = 1, 2, \dots, q$ (in the sequel repeated indices denote summation) and coefficients $a_{ij}^{\alpha\beta}(x) \in L^\infty(\Omega) \cap \mathrm{VMO}_X(\Omega)$ (see Section 2 and [8] for definition), and

$$g_i \in L_X^{2Q/(Q+2), \lambda Q/(Q+2)}(\Omega), \quad f_i^\alpha \in L_X^{2,\lambda}(\Omega), \quad Q - n < \lambda < Q,$$
$$i = 1, 2, \dots, N, \quad \alpha = 1, 2, \dots, q,$$

where the space $L_X^{2,\lambda}$ is the intrinsic Morrey space with respect to the Carnot-Carathéodory metric (for the definition see Section 2).

We also assume the following strong ellipticity condition:
there exists $\nu > 0$ such that,

$$a_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \geq \nu \|\xi\|^2 \;\text{ a.e. } \; x \in \Omega \;\text{ and }\; \forall \xi \in \mathbb{R}^{qN}.$$

We mean that a function $u \in S^1_X(\Omega, \mathbb{R}^N)$ (see Section 2 for the definition of $S^1_X(\Omega, \mathbb{R}^N)$) is a solution of (2) if

$$\int_\Omega a^{\alpha\beta}_{ij}(x) X_\beta u^j X_\alpha \varphi^i \, dx = \int_\Omega (g_i \varphi^i + f^\alpha_i X_\alpha \varphi^i) \, dx \quad \forall \varphi \in S^1_{X,0}(\Omega, \mathbb{R}^N).$$

The goal of this paper is expressed in the following theorems:

Theorem 1.1. *Let $u \in S^1_X(\Omega, \mathbb{R}^N)$ be a solution of (2). Then*

$$Xu \in L^{2,\lambda}_{X,\mathrm{loc}}(\Omega, \mathbb{R}^{qN}),$$

and there exists $c \geq 0$ such that if $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ then

$$\|Xu\|_{L^{2,\lambda}_X(\Omega')} \leq c \left(\|Xu\|_{L^2(\Omega'')} + \|g\|_{L^{2Q/(Q+2),\lambda Q/(Q+2)}_X(\Omega)} + \|f\|_{L^{2,\lambda}_X(\Omega)} \right).$$

As an application of the results in Theorem 1.1 we get "global" Hölder continuity for solutions of the system (2).

Theorem 1.2. *Let $u \in S^1_X(\Omega, \mathbb{R}^N)$ be a solution of (2). If $Q - n < \lambda < 2$, then $u \in C^{0,\alpha}_X(\Omega, \mathbb{R}^N)$ with $\alpha = 1 - \frac{\lambda}{2}$.*

For definition of $C^{0,\alpha}_X(\Omega, \mathbb{R}^N)$ see Section 2.

2. Some preliminaries

Let us consider a system $X = (X_1, \dots, X_q)$, $q \leq n$, of vector fields in $\mathbb{R}^n$. For every multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_d)$ with $1 \leq \beta_i \leq q$, and $|\beta| = d$, set the commutator of length d as

$$X_\beta = [X_{\beta_d}, [X_{\beta_{d-1}}, \dots [X_{\beta_2}, X_{\beta_1}]..]].$$

Definition 2.1. *The system $X = (X_1, \dots, X_q)$ satisfies the Hörmander's condition of step s at some point x_0 of $\mathbb{R}^n$ if $\{X_\beta(x_0)\}_{|\beta| \leq s}$ spans $\mathbb{R}^n$ as vector space.*

Let $X = (X_1, \dots, X_q)$ satisfy the Hörmander condition in $\mathbb{R}^n$, let us assume X of the following kind:

$$X_j = \sum_{k=1}^n b_{jk} \frac{\partial}{\partial x_k}, \quad j = 1, \dots, q,$$

where b_{jk} are locally Lipschitz continuous functions. From now on we shall denote by $X^*_j = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{jk})$ the formal adjoint of X_j.

A piecewise C^1 curve $\gamma : [0,T] \to \mathbb{R}^n$ is called sub-unit, with respect to the system X, if whenever $\gamma'(t)$ exists one has

$$\langle \gamma'(t), \xi\rangle^2 \leq \sum_{j=1}^{q} \langle X_j(\gamma(t)), \xi\rangle^2, \quad \forall \xi \in \mathbb{R}^n.$$

We set $l_S(\gamma) = T$ the sub-unit length of γ. Given $x, y \in \mathbb{R}^n$, we denote by $\Phi(x,y)$ the collection of all sub-unit curves connecting x to y. It results $\Phi(x,y) \neq \emptyset$ $\forall x, y \in \mathbb{R}^n$. Then

$$d(x,y) = \inf \{l_S(\gamma) : \gamma \in \Phi(x,y)\}$$

defines a distance, usually called the Carnot-Carathéodory distance generated by X. We shall denote $B(x,R) = \{y \in \mathbb{R}^n \ : \ d(x,y) < R\}$ the metric ball centered at x of radius R and whenever x is not relevant we shall write B_R. We shall denote by $d_e(x,y)$ the usual Euclidean distance in $\mathbb{R}^n$.

Now we introduce the relevant quantitative assumptions.

(H1) $i : (\mathbb{R}^n, d_e) \to (\mathbb{R}^n, d)$ is continuous.

(H2) (Doubling condition) For every open bounded set $\Omega \subset \mathbb{R}^n$ there exist constants C_D, $R_D > 0$ such that for $x_0 \in \Omega$ and $0 < 2R < R_D$ one has $|B(x_0, 2R)| \leq C_D |B(x_0, R)|$.

(H3) (Weak-L^1 Poincarè type inequality) Given Ω as in (H2), there exist positive constants C_P and $\alpha \geq 1$ such that for any $x_0 \in \Omega$, $0 < R < R_D$ and $u \in C^1(B(x_0, \alpha R), \mathbb{R}^N)$, one has

$$\sup_{\lambda > 0} [\lambda |\{x \in B(x_0,R) : \|u(x) - u_{B(x_0,R)}\| > \lambda\}|] \leq C_P R \int_{B(x_0, \alpha R)} \|Xu\| \, dx,$$

where $u_{B(x_0,R)}$ denotes the integral average $|B(x_0,R)|^{-1} \int_{B(x_0,R)} \|u(y)\| \, dy$.

Finally we put $Q = \log_2 C_D$. It results $Q \geq n$, and Q will be the homogeneous dimension of Ω with respect to X.

We remark that, by doubling condition (H2), we have

$$|B_{tR}| \geq C_D t^Q |B_R| \quad \forall R \leq R_D \ \text{ and } \ \forall t \in (0,1). \tag{3}$$

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 3$, and $u : \Omega \to \mathbb{R}^N$, $N \geq 1$.

Definition 2.2. *Let $X = (X_1, X_2, \ldots, X_q)$ be a system of Lipschitz vector fields in $\mathbb{R}^n$, $1 \leq p \leq +\infty$, k a positive integer. We say that $u \in L^p(\Omega, \mathbb{R}^N)$ belongs to the Sobolev space $S_X^{k,p}(\Omega, \mathbb{R}^N)$ if*

$$\|u\|_{S_X^{k,p}(\Omega,\mathbb{R}^N)} \equiv \|u\|_{L^p(\Omega,\mathbb{R}^N)} + \sum_{h=1}^{k} \sum_{j_h=1}^{q} \|X_{j_1} X_{j_2} \ldots X_{j_h} u\|_{L^p(\Omega,\mathbb{R}^N)} < +\infty. \tag{4}$$

We also denote by $S^{k,p}_{X,0}(\Omega,\mathbb{R}^N)$ the closure of $C^\infty_{X,0}(\Omega,\mathbb{R}^N)$ in $S^{k,p}_X(\Omega,\mathbb{R}^N)$ with respect to the norm (4), and by $S^k_X(\Omega,\mathbb{R}^N)$ and $S^k_{X,0}$ the Sobolev spaces $S^{k,2}_X(\Omega,\mathbb{R}^N)$ and $S^{k,2}_{X,0}$ respectively.

In the sequel we shall use the following Sobolev embedding theorem.

Theorem 2.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary.*

If $1 \le p < \frac{Q}{k}$ then

$$S^{k,p}_X(\Omega,\mathbb{R}^N) \subset L^{p^*}(\Omega,\mathbb{R}^N), \quad \textit{where} \ \ \frac{1}{p^*} = \frac{1}{p} - \frac{k}{Q}\,,$$

and

$$\|u\|_{L^{p^*}(\Omega,\mathbb{R}^N)} \le c\|u\|_{S^{k,p}_X(\Omega,\mathbb{R}^N)}.$$

There are a lot of proofs of this theorem in literature. For what concerns Sobolev embedding theorems in metric spaces the reader can refer to [34], [33], [22], [23] and [43].

Now we define the Morrey spaces, the Campanato spaces, $C^{0,\alpha}_X$, BMO_X and VMO_X spaces with respect to the Carnot-Carathéodory metric ([37], [38]).

Definition 2.3. *Let $p \ge 1$. We say that $u \in L^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ belongs to $L^{p,\lambda}_X(\Omega,\mathbb{R}^N)$, for some $\lambda > 0$, if*

$$\|u\|_{L^{p,\lambda}_X(\Omega,\mathbb{R}^N)} = \sup_{x_0\in\Omega, 0<R<d_0} \left(\frac{R^\lambda}{|\Omega \cap B(x_0,R)|} \int_{\Omega\cap B(x_0,R)} \|u\|^p \,dx \right)^{\frac{1}{p}} < +\infty,$$

where $d_0 = \min(\mathrm{diam}(\Omega), R_D)$.

Definition 2.4. *Let $p \ge 1$. We say that $u \in L^p_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ belongs to $\mathcal{L}^{p,\lambda}_X(\Omega,\mathbb{R}^N)$, for $\lambda > -p$, if*

$$[u]_{\mathcal{L}^{p,\lambda}_X(\Omega,\mathbb{R}^N)} = \sup_{x_0\in\Omega, 0<R<d_0} \left(\frac{R^\lambda}{|\Omega \cap B(x_0,R)|} \int_{\Omega\cap B(x_0,R)} \|u - u_R\|^p \,dx \right)^{\frac{1}{p}} < +\infty,$$

where $d_0 = \min(\mathrm{diam}(\Omega), R_D)$.

$L^{p,\lambda}_X(\Omega,\mathbb{R}^N)$ and $\mathcal{L}^{p,\lambda}_X(\Omega,\mathbb{R}^N)$ are called Morrey space and Campanato space respectively.

Definition 2.5. *Let $\alpha \in (0,1[$. $C_X^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ is the Banach space of the functions $u : \overline{\Omega} \to \mathbb{R}^N$ α-Hölder continuous with the norm*

$$\|u\|_{C_X^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)} = \sup_{\Omega} \|u\| + \sup_{\overline{\Omega}} \frac{\|u(x) - u(y)\|}{[d(x,y)]^{\alpha}} .$$

We say that $u \in C_X^{0,\alpha}(\Omega, \mathbb{R}^N)$ if $u \in C_X^{0,\alpha}(K, \mathbb{R}^N)$ for every K compact subset of Ω.

Definition 2.6. *We say that $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ belongs to* $\mathrm{BMO}_X(\Omega, \mathbb{R}^N)$ *if*

$$\|u\|_{\mathrm{BMO}_X(\Omega,\mathbb{R}^N)} = \sup_{x_0\in\Omega, 0<R<d_0} \frac{1}{|\Omega \cap B(x_0,R)|} \int_{\Omega\cap B(x_0,R)} \|u - u_R\| \, dx < +\infty.$$

If $u \in \mathrm{BMO}_X(\Omega, \mathbb{R}^N)$ we say that u belongs to $\mathrm{VMO}_X(\Omega, \mathbb{R}^N)$ when

$$\eta(R) = \sup_{x_0\in\Omega, 0<\rho<R} \frac{1}{|\Omega \cap B(x_0,\rho)|} \int_{\Omega\cap B(x_0,\rho)} \|u - u_\rho\| \, dx \to 0$$

as $R \to 0$.

We observe that the spaces $L_X^{p,\lambda}(\Omega, \mathbb{R}^N)$ and $\mathcal{L}_X^{p,\lambda}(\Omega, \mathbb{R}^N)$, for $\lambda > Q$, are essentially the spaces $L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Moreover, the following theorem holds (see [37] and [27]).

Theorem 2.2. *If $\lambda > 0$, the Campanato space $\mathcal{L}_X^{p,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the Morrey space $L_X^{p,\lambda}(\Omega, \mathbb{R}^N)$. If $-p < \lambda < 0$, the Campanato space $\mathcal{L}_X^{p,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to $C_X^{0,\alpha}(\Omega, \mathbb{R}^N)$ with $\alpha = -\frac{\lambda}{p}$.*

3. Gradient estimates

Let us start by studying the following homogeneous system:

$$-X^*_\alpha(a_{ij}^{\alpha\beta}(x) X_\beta u^j) = 0 \quad i = 1, 2, \ldots, N, \tag{5}$$

with variable coefficients $a_{ij}^{\alpha\beta} \in L^\infty(\Omega) \cap \mathrm{VMO}_X(\Omega)$ satisfying the strong ellipticity condition:
there exists $\nu > 0$ such that,

$$a_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \geq \nu \|\xi\|^2 \ \text{ a.e. } \ x \in \Omega \ \text{ and } \ \forall \xi \in \mathbb{R}^{qN}.$$

We shall use the following energy estimate known as Caccioppoli inequality (see [42]).

Theorem 3.1. *Let $u \in S^1_X(\Omega, \mathbb{R}^N)$ be a solution of (5). Then there exists c such that $\forall \rho < R < d_0$*

$$\int_{B_\rho} \|Xu\|^2 \, dx \leq \frac{c}{(R-\rho)^2} \int_{B_R} \|u\|^2 \, dx.$$

Theorem 3.2. *Let $u \in S^1_X(\Omega, \mathbb{R}^N)$ be a solution of the system*

$$-X^*_\alpha(a^{\alpha\beta}_{ij} X_\beta u^j) = 0 \quad i = 1, 2, \ldots, N, \tag{6}$$

where $a^{\alpha\beta}_{ij} \in \mathbb{R}$ and satisfy the strong ellipticity condition. Then there exist two positive constants C and R_0 such that for every x_0 in Ω and $B_\rho = B(x_0, \rho)$, with $0 < \rho < R_0$, we have

$$\int_{B_\rho} \|Xu\|^2 \, dx \leq c \left(\frac{\rho}{R_0}\right)^Q \int_{B_{R_0}} \|Xu\|^2 \, dx. \tag{7}$$

For the proof of the last theorem see [44, Theorem 3.2].

In order to study the system (5) we recall one more definition ([28] and [30]).

Definition 3.1. *Given a functional $F : S^1_X(\Omega, \mathbb{R}^N) \to \mathbb{R}$ we call u a spherical quasi-minimum for F iff*

$$F(u; B_R) \leq cF(u + \varphi; B_R) \quad \forall \varphi \in S^1_{X,0}(B_R, \mathbb{R}^N) \ \text{ and } \ \forall B_R \subset \Omega.$$

If $u \in S^1_X(\Omega, \mathbb{R}^N)$ is a solution of the system (5) then u is a spherical quasi-minimum for the functional $F(u; \Omega) = \int_\Omega \|Xu\|^2 \, dx$. In fact, let $B_R \subset \Omega$ and $\varphi \in S^1_{X,0}(B_R, \mathbb{R}^N)$. Setting $v = u + \varphi$, using Definition 3.1 and Cauchy-Schwarz inequality, we have,

$$\begin{aligned}
\int_{B_R} \|Xu\|^2 \, dx &\leq \frac{1}{\nu} \int_{B_R} a^{\alpha\beta}_{ij}(x) X_\alpha u^i X_\beta u^j \, dx \\
&\leq \frac{1}{\nu} \int_{B_R} a^{\alpha\beta}_{ij}(x) X_\alpha u^i X_\beta v^j \, dx \\
&\leq c\left(\int_{B_R} \|Xu\|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{B_R} \|Xv\|^2 \, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Then

$$\int_{B_R} \|Xu\|^2 \, dx \leq c \int_{B_R} \|Xv\|^2 \, dx,$$

and the result follows.

Theorem 3.3. *Let $u \in S^1_X(\Omega, \mathbb{R}^N)$ be a solution of the system* (5). *Then there exists $p > 2$ such that*

$$u \in S^{1,p}_{X,\mathrm{loc}}(\Omega, \mathbb{R}^N).$$

Moreover $\forall B_R \subset\subset \Omega$

$$\left(\fint_{B_{\frac{R}{2}}} \|Xu\|^p \, dx \right)^{\frac{1}{p}} \le c \left(\fint_{B_R} \|Xu\|^2 \, dx \right)^{\frac{1}{2}},$$

where c does not depend on R.

Proof: For fixed $B_R \subset\subset \Omega$ and $\rho < R$, let η be a radial cutoff function, i.e. $\eta \in C_0^\infty(B_R)$, $0 \le \eta \le 1$, $\eta = 1$ in B_ρ, and $\|X\eta\| \le \frac{c}{R-\rho}$ (for the existence of this function see [5]). Since u is a spherical quasi-minimum for $\int_\Omega \|Xu\|^2 \, dx$, taking $\varphi = -\eta(u - u_R)$ we have

$$\begin{aligned} \int_{B_\rho} \|Xu\|^2 \, dx \le \int_{B_R} \|Xu\|^2 \, dx &\le c \int_{B_R} \|X(u - \eta(u - u_R))\|^2 \, dx \\ &\le c \int_{B_R} (1-\eta)^2 \|Xu\|^2 \, dx + c \int_{B_R} \|X\eta\|^2 \|u - u_R\|^2 \, dx \\ &\le c \int_{B_R \setminus B_\rho} \|Xu\|^2 \, dx + \frac{c}{(R-\rho)^2} \int_{B_R} \|u - u_R\|^2 \, dx, \end{aligned}$$

from which

$$\int_{B_\rho} \|Xu\|^2 \, dx \le \frac{c}{c+1} \int_{B_R} \|Xu\|^2 \, dx + \frac{c}{(R-\rho)^2} \int_{B_R} \|u - u_R\|^2 \, dx. \tag{8}$$

Applying Lemma 5.1 in [30] we obtain

$$\int_{B_\rho} \|Xu\|^2 \, dx \le \frac{c}{(R-\rho)^2} \int_{B_R} \|u - u_R\|^2 \, dx.$$

Now we choose $\rho = \frac{R}{2}$ and apply Poincaré inequality (see [42]) to get

$$\begin{aligned} \int_{B_{\frac{R}{2}}} \|Xu\|^2 \, dx &\le \frac{c}{R^2} \int_{B_R} \|u - u_R\|^2 \, dx \\ &\le \frac{c}{R^2} \left(\int_{B_R} \|Xu\|^{2_*} \, dx \right)^{\frac{2}{2_*}}, \quad \text{where } \frac{1}{2_*} = \frac{1}{2} + \frac{1}{Q}. \end{aligned} \tag{9}$$

Making use of (9), the doubling condition (H2) and Lemma 7 in [40], it follows

$$\left(\fint_{B_{\frac{R}{2}}} \|Xu\|^2\,dx\right)^{\frac{1}{2}} \leq \frac{c}{R|B_{\frac{R}{2}}|^{\frac{1}{2}}}\left(\int_{B_R} \|Xu\|^{2_*}\,dx\right)^{\frac{1}{2_*}}$$
$$\leq c\frac{|B_{\frac{R}{2}}|^{\frac{1}{Q}}}{R}\left(\fint_{B_R} \|Xu\|^{2_*}\,dx\right)^{\frac{1}{2_*}} \leq c\left(\fint_{B_R} \|Xu\|^{2_*}\,dx\right)^{\frac{1}{2_*}},$$

where c does not depend on R.

To get the conclusion we make use of Lemma 3 in [24] taking $f = \|Xu\|^{2_*}$, $s = 2/2_* > 1$. $\square$

Lemma 3.1. *Let $u \in S^1_X(\Omega, \mathbb{R}^N)$ be a solution of system (5), suppose $a^{\alpha\beta}_{ij} \in L^\infty(\Omega) \cap \mathrm{VMO}_X(\Omega)$ and the strong ellipticity condition holds true. Then there exists $0 < R_0 \leq d_0$ such that $\forall\ 0 < \mu < Q$*

$$\int_{B(x_0,\rho)} \|Xu\|^2\,dx \leq c\left(\frac{\rho}{R}\right)^{\mu} \int_{B(x_0,R)} \|Xu\|^2\,dx,$$

for any $\rho \leq R \leq \min(R_0, \mathrm{dist}(x_0, \partial\Omega))/2$.

Proof: Let $B(x_0, R) = B_R \subset\subset \Omega$ be a ball and let v, w be solutions of the following problems:

$$\begin{cases} -X^*_\alpha((a^{\alpha\beta}_{ij})_R X_\beta v^j) = 0 \text{ in } \ B_R, \\ v - u \in S^1_{X,0}(B_R, \mathbb{R}^N), \end{cases}$$

$$\begin{cases} -X^*_\alpha((a^{\alpha\beta}_{ij})_R X_\beta w^j) = -X^*_\alpha[((a^{\alpha\beta}_{ij})_R - a^{\alpha\beta}_{ij}(x))X_\beta u^j] \text{ in } \ B_R, \\ w \in S^1_{X,0}(B_R, \mathbb{R}^N). \end{cases}$$

Trivially we have $u = v + w$. Concerning the function v, it is solution of a system with constant coefficients, then, for any $0 < \rho < R$, we have

$$\int_{B_\rho} \|Xv\|^2\,dx \leq c\left(\frac{\rho}{R}\right)^{Q} \int_{B_R} \|Xv\|^2\,dx. \tag{10}$$

On the other hand, the function w satisfies the following estimate

$$\int_{B_R} \|Xw\|^2\,dx \leq c\int_{B_R} |(a^{\alpha\beta}_{ij})_R - a^{\alpha\beta}_{ij}(x)|^2 \|Xu\|^2\,dx. \tag{11}$$

Merging now (10) and (11), by Hölder inequality and Theorem 3.3 it follows, for any $0 < \rho < R$,

$$\begin{aligned}\int_{B_\rho} \|Xu\|^2\,dx &\le 2\int_{B_\rho}\|Xv\|^2\,dx + 2\int_{B_R}\|Xw\|^2\,dx \\ &\le c\left(\frac{\rho}{R}\right)^Q \int_{B_R}\|Xu\|^2\,dx \\ +c|B_R|&\left(\fint_{B_R} |(a_{ij}^{\alpha\beta})_R - a_{ij}^{\alpha\beta}(x)|^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}}\left(\fint_{B_R}\|Xu\|^p\right)^{2/p} \\ \le c\Bigg[\left(\frac{\rho}{R}\right)^Q &+ \left(\fint_{B_R} |(a_{ij}^{\alpha\beta})_R - a_{ij}^{\alpha\beta}(x)|^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}}\Bigg]\int_{B(2R)}\|Xu\|^2\,dx.\end{aligned}$$

Now the VMO_X assumption on the coefficients plays a role. Namely since $a_{ij}^{\alpha\beta} \in \mathrm{VMO}_X(\Omega)$ making use of Lemma 1.III, Chapter I in [10], we obtain that there exists $R_0 \le d_0$ such that $\forall \rho < R \le \min(R_0, \mathrm{dist}(x_0, \partial\Omega))/2$ and $\forall 0 < \mu < Q$

$$\int_{B_\rho}\|Xu\|^2\,dx \le c\left(\frac{\rho}{R}\right)^\mu \int_{B_R}\|Xu\|^2\,dx.$$

□

Now we can study the variable coefficients system

$$-X_\alpha^*(a_{ij}^{\alpha\beta}(x)X_\beta u^j) = g_i(x) - X_\alpha^* f_i^\alpha(x), \quad i = 1, 2, \ldots, N, \tag{12}$$

where

$$\begin{gathered} a_{ij}^{\alpha\beta} \in L^\infty(\Omega) \cap \mathrm{VMO}_X(\Omega), \\ g_i \in L_X^{2Q/(Q+2),\lambda Q/(Q+2)}(\Omega), \quad f_i^\alpha \in L_X^{2,\lambda}(\Omega), \quad \text{with } \ Q-n < \lambda < Q \end{gathered}$$

and the strong ellipticity condition holds true.

Theorem 3.4. *Let $u \in S_X^1(\Omega, \mathbb{R}^N)$ be a solution of* (12). *Then*

$$Xu \in L_{X,\mathrm{loc}}^{2,\lambda}(\Omega, \mathbb{R}^{qN}).$$

More precisely there exists $R_0 \le d_0$ such that $\forall \rho < R \le R_0$ and $B(x_0, R) \subset \Omega$, we have

$$\begin{aligned}\int_{B(x_0,\rho)} \|Xu\|^2\,dx &\le c\frac{|B(x_0,\rho)|}{\rho^\lambda}\,\cdot \\ &\cdot\left[\frac{R^\lambda}{|B(x_0,R)|}\int_{B(x_0,R))}\|Xu\|^2\,dx + \|g\|^2_{L_X^{2Q/(Q+2),\lambda Q/(Q+2)}(\Omega)} + \|f\|^2_{L_X^{2,\lambda}(\Omega)}\right],\end{aligned}$$

where c does not depend on ρ.

PROOF: For $x_0 \in \Omega$ fixed, we stress that by Lemma 7 in [40] there exists $R \leq \min(R_0, \mathrm{dist}(x_0, \partial\Omega))/2$ (R_0 appears in Lemma 3.1) such that $|B_R| \leq 1$.

Let v and w be the solutions of the following systems

$$\begin{cases} -X_\alpha^*(a_{ij}^{\alpha\beta}(x) X_\beta v^j) = 0 \ \text{ in } \ B_R, \\ v - u \in S^1_{X,0}(B_R, \mathbb{R}^N), \end{cases}$$

$$\begin{cases} -X_\alpha^*(a_{ij}^{\alpha\beta}(x) X_\beta w^j) = g_i - X_\alpha^* f_i^\alpha \ \text{ in } \ B_R, \\ w \in S^1_{X,0}(B_R, \mathbb{R}^N). \end{cases}$$

Let $Q - n < \mu < Q$. From Lemma 3.1 we get

$$\begin{aligned} &\int_{B_\rho} \|Xu\|^2\, dx \leq c \int_{B_\rho} \|Xv\|^2\, dx + c \int_{B_\rho} \|Xw\|^2\, dx \\ (13) \qquad &\leq c \Big(\frac{\rho}{R}\Big)^\mu \int_{B_R} \|Xv\|^2\, dx + c \int_{B_R} \|Xw\|^2\, dx \; lec \Big(\frac{\rho}{R}\Big)^\mu \int_{B_R} \|Xu\|^2\, dx \\ &\quad + c \int_{B_R} \|Xw\|^2\, dx. \end{aligned}$$

Now we estimate the last integral in (13). By definition of solution, Hölder and Sobolev inequalities, one has

$$\begin{aligned} \int_{B_R} \|Xw\|^2\, dx &\leq c \int_{B_R} \|w\| \|g\|\, dx + c \int_{B_R} \|Xw\| \|f\|\, dx \\ &\leq c \Big(\int_{B_R} \|w\|^{2^*} \Big)^{1/2^*} \Big(\int_{B_R} \|g\|^{2Q/(Q+2)}\, dx \Big)^{\frac{Q+2}{2Q}} \\ &\quad + c \Big(\int_{B_R} \|Xw\|^2\, dx \Big)^{1/2} \Big(\int_{B_R} \|f\|^2\, dx \Big)^{1/2} \\ &\leq c \Big(\int_{B_R} \|Xw\|^2\, dx \Big)^{1/2} \Big[\Big(\int_{B_R} \|g\|^{2Q/(Q+2)} \Big)^{\frac{Q+2}{2Q}} + \Big(\int_{B_R} \|f\|^2\, dx \Big)^{1/2} \Big], \end{aligned}$$

from which

$$\int_{B_R} \|Xw\|^2\, dx \leq c \Big(\int_{B_R} \|g\|^{2Q/(Q+2)} \Big)^{\frac{Q+2}{Q}} + c \int_{B_R} \|f\|^2\, dx.$$

Then we obtain

$$\begin{aligned}\int_{B_\rho} \|Xu\|^2\,dx &\le c\left(\frac{\rho}{R}\right)^\mu \int_{B_R} \|Xu\|^2\,dx \\ &\qquad + c\left(\int_{B_R} \|g\|^{2Q/(Q+2)}\right)^{\frac{Q+2}{Q}} + c\int_{B_R} \|f\|^2\,dx \\ &\le c\left(\frac{\rho}{R}\right)^\mu \int_{B_R} \|Xu\|^2\,dx + \frac{|B_R|^{\frac{Q+2}{Q}}}{R^\lambda}\|g\|^2_{L_X^{2Q/(Q+2),\lambda Q/(Q+2)}(\Omega)} + \frac{|B_R|}{R^\lambda}\|f\|^2_{L_X^{2,\lambda}(\Omega)} \\ &\le c\left(\frac{\rho}{R}\right)^\mu \int_{B_R} \|Xu\|^2\,dx + \frac{|B_R|}{R^\lambda}[\|g\|^2_{L_X^{2Q/(Q+2),\lambda Q/(Q+2)}(\Omega)} + \|f\|^2_{L_X^{2,\lambda}(\Omega)}].\end{aligned}$$

Since $Q-n<\mu<Q$, we can use Proposition 2.1 in [36] with $\beta=\mu$, $F(\rho)=\frac{|B_\rho|}{\rho^\lambda}$ and $Q-\lambda<\gamma<\mu$. We observe that $\frac{\rho^\gamma}{F(\rho)}$ is almost increasing: in fact from (3), since $\gamma>Q-\lambda$, it follows that $\forall t\in(0,1)$

$$\frac{t^{\gamma+\lambda}}{|B_{t\rho}|} \le C_D\frac{t^{\gamma+\lambda-Q}}{|B_\rho|} \le \frac{C_D}{|B_\rho|}\,.$$

Finally, we obtain $\forall\rho<R$, by Proposition 2.1 in [36]

$$\begin{aligned}&\int_{B_\rho} \|Xu\|^2\,dx \\ &\qquad \le c\frac{|B_\rho|}{\rho^\lambda}\left[\frac{R^\lambda}{|B_R|}\int_{B_R} \|Xu\|^2\,dx + \|g\|^2_{L_X^{2Q/(Q+2),\lambda Q/(Q+2)}(\Omega)} + \|f\|^2_{L_X^{2,\lambda}(\Omega)}\right].\end{aligned}$$

The last inequality ensures us that Xu belongs to the space $L^{2,\lambda}_{X,\mathrm{loc}}(\Omega,\mathbb{R}^{qN})$. $\square$

From the last theorem, Poincaré inequality (see [42]) and Theorem 2.2 we can obtain the following Hölder continuity result for the solution of the system (12).

Theorem 3.5. *Let $u\in S^1_X(\Omega,\mathbb{R}^N)$ be a solution of* (2)*. If $Q-n<\lambda<2$, then*

$$u\in C^{0,\alpha}_X(\Omega,\mathbb{R}^N) \quad \textit{with} \quad \alpha = 1-\frac{\lambda}{2}\,.$$

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