Topological games and product spaces

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Abstract. In this paper, we deal with the product of spaces which are either $\mathcal{G}$-spaces or $\mathcal{G}_p$-spaces, for some $p \in \omega^*$. These spaces are defined in terms of a two-person infinite game over a topological space. All countably compact spaces are $\mathcal{G}$-spaces, and every $\mathcal{G}_p$-space is a $\mathcal{G}$-space, for every $p \in \omega^*$. We prove that if $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $X = \prod_{\mu<\omega_1} X_\mu$ is a $\mathcal{G}$-space, then there is $A \in [\omega_1]^{\leq\omega}$ such that X_μ is countably compact for every $\mu \in \omega_1 \setminus A$. As a consequence, X^{ω_1} is a $\mathcal{G}$-space iff X^{ω_1} is countably compact, and if $X^{2^{\mathfrak{c}}}$ is a $\mathcal{G}$-space, then all powers of X are countably compact. It is easy to prove that the product of a countable family of $\mathcal{G}_p$ spaces is a $\mathcal{G}_p$-space, for every $p \in \omega^*$. For every $1 \leq n < \omega$, we construct a space X such that X^n is countably compact and X^{n+1} is not a $\mathcal{G}$-space. If $p, q \in \omega^*$ are RK-incomparable, then we construct a $\mathcal{G}_p$-space X and a $\mathcal{G}_q$-space Y such that $X \times Y$ is not a $\mathcal{G}$-space. We give an example of two free ultrafilters p and q on ω such that $p <_{RK} q$, p and q are RF-incomparable, $p \approx_C q$ ($\leq_C$ is the *Comfort* order on ω^*) and there are a $\mathcal{G}_p$-space X and a $\mathcal{G}_q$-space Y whose product $X \times Y$ is not a $\mathcal{G}$-space.

Keywords: RF-order, RK-order, *Comfort*-order, p-limit, p-compact, $\mathcal{G}$-space, $\mathcal{G}_p$-space, countably compact

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1. Introduction

All the spaces are assumed to be Tychonoff. The Stone-Čech compactification $\beta\omega$ of the countable discrete space ω is identified with the set of all ultrafilters on ω and its remainder $\omega^* = \beta\omega \setminus \omega$ is identified with the set of all free ultrafilters on ω.

Let us define the basic common rules of our games:

Let X be a space and $x \in X$. We have two players, I and II who are going to play around the point x. Player I makes the first move by choosing an open neighborhood $U_0 \in \mathcal{N}(x)$. Then, player II responds by choosing $x_0 \in U_0$. Player I then chooses another open neighborhood $U_1 \in \mathcal{N}(x)$, and then player II responds by choosing $x_1 \in U_1$ and so on. Both players repeat this procedure infinitely many times. At the end of the game we have a sequence $(x_n)_{n<\omega}$ of points in X, and a sequence $(U_n)_{n<\omega}$ of neighborhoods of x such that $x_n \in U_n$, for every $n < \omega$. The games differ from each other in the winning condition. Following

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A. Bouziad [Bo], we say that player I *wins* in the $\mathcal{G}(x, X)$-*game* if $\{x_n : n < \omega\}$ has an adherent point in X. Otherwise, player II is declared to be the winner in the $\mathcal{G}(x, X)$-game. To define the main infinite games of this paper, we shall recall the definition of the p-limit point of a sequence of points of a space, for an ultrafilter $p \in \omega^*$.

Definition 1.1 (R.A. Bernstein [Be]). *Let $p \in \omega^*$. A point x of a space X is said to be the p-limit point of a sequence $(x_n)_{n<\omega}$ in X, in symbols $x = p-\lim_{n\to\omega} x_n$, if for every neighborhood U of x, $\{n < \omega : x_n \in V\} \in p$.*

Bernstein's notion characterizes the points lying in the closure of countable subsets of a space: A point $x \in X$ is an adherent point of a countable subset A of X iff there are a sequence $(x_n)_{n<\omega}$ in A and $p \in \omega^*$ such that $x = p-\lim_{n\to\omega} x_n$.

We are ready to state the winning condition of our games. Fix $p \in \omega^*$. As in the paper [GG], we say that player I *wins* in the $\mathcal{G}_p(x, X)$-*game* if the sequence $(x_n)_{n<\omega}$ has a p-limit point in the space X. Otherwise, the second player wins the $\mathcal{G}_p(x, X)$-game. All these games are natural generalizations of the $W(x, X)$-game introduced by G. Gruenhage in [Gru].

Definition 1.2. *Let X be a space and $p \in \omega^*$. A* strategy *for player I is a sequence $\sigma = \{\sigma_n : n < \omega\}$ of functions, where $\sigma_n : X^{n+1} \to \mathcal{N}(x)$ for every $n < \omega$. Given a strategy σ we say that a sequence $(x_n)_{n<\omega}$ in X is a σ-sequence if $x_{n+1} \in \sigma_n((x_0, x_1, \dots, x_n))$, for each $n < \omega$. For $x \in X$, a strategy $\sigma = \{\sigma_n : n < \omega\}$ for player I in the $\mathcal{G}(x, X)$-game (respectively, $\mathcal{G}_p(x, X)$-game) is said to be a winning strategy, if each σ-sequence has an adherent point (respectively, a p-limit point) in X. A space X is called a $\mathcal{G}$-space (respectively, $\mathcal{G}_p$-space) if the first player I has a winning strategy in the $\mathcal{G}(x, X)$-game (respectively, $\mathcal{G}_p(x, X)$-game), for every $x \in X$.*

Every countably compact space is a $\mathcal{G}$-space, every $\mathcal{G}_p$-space is a $\mathcal{G}$-space and every p-compact space is a $\mathcal{G}_p$-space, for $p \in \omega^*$ (a space X is called p-*compact* provided that every sequence in X has a p-limit point in X).

In this paper, we mainly study the product of $\mathcal{G}$-spaces. In the second section, it is shown that if $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $\prod_{\mu<\omega_1} X_\mu$ is a $\mathcal{G}$-space, then the X_μ's are countably compact except for countably many. It follows that if X^{ω_1} is a $\mathcal{G}$-space, then X^{ω_1} is countably compact. At the end of the second section, we shall prove that the product of a countable family of $\mathcal{G}_p$-spaces is a $\mathcal{G}_p$-space, for every $p \in \omega^*$. In the last section, we study the finite products of some $\mathcal{G}$-spaces.

2. Infinite products

In the first theorem of this section, we will give a necessary condition for a product of ω_1-many $\mathcal{G}$-spaces to be a $\mathcal{G}$-space.

A subbasic open set of the product space $X = \prod_{i\in I} X_i$ is denoted by $[i, V] = \{x \in X : x(i) \in V\}$, where $i \in I$ and V is a nonempty open subset of X_i.

Theorem 2.1. *If $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $X = \prod_{\mu<\omega_1} X_\mu$ is a $\mathcal{G}$-space, then there is $A \in [\omega_1]^{\leq\omega}$ such that X_μ is countably compact, for every $\mu \in \omega_1 \setminus A$.*

PROOF: Suppose that there is a set $A \in [\omega_1]^{\omega_1}$ such that X_μ is not countably compact for any $\mu \in A$. Then, for every $\mu \in A$, there is a closed discrete countable subset $\{y_n^\mu : n < \omega\}$ of X_μ. Fix $x \in X$ and let us play the $\mathcal{G}(x, X)$-game. We are going to define a winning strategy for player II. Indeed, player I starts the game by picking $V_0 = \bigcap_{\mu\in F_0}[\mu, W_\mu^0]$, where $F_0 \in [\omega_1]^{<\omega}$ and $W_\mu^0 \in \mathcal{N}(x(\mu))$, for every $\mu \in F_0$. Then, player II chooses $x_0 \in X$ which is defined by

$$x_0(\mu) = \begin{cases} x(\mu) & \text{if } \mu \in (\omega_1 \setminus A) \cup F_0, \\ y_0^\mu & \text{if } \mu \in A \setminus F_0. \end{cases}$$

Now, player I choose $V_1 = \bigcap_{\mu\in F_1}[\mu, W_\mu^1]$, where $F_1 \in [\omega_1]^{<\omega}$ and $W_\mu^0 \in \mathcal{N}(x(\mu))$ for every $\mu \in F_1$. Then, player II responds by picking the point $x_1 \in X$ defined by

$$x_1(\mu) = \begin{cases} x(\mu) & \text{if } \mu \in (\omega_1 \setminus A) \cup F_1, \\ y_1^\mu & \text{if } \mu \in A \setminus F_1, \end{cases}$$

and so on. Clearly, $x_n \in V_n = \bigcap_{\mu\in F_n}[\mu, W_\mu^n]$, for every $n < \omega$. Since the set $\bigcup_{n<\omega} F_n$ is countable there is $\nu \in A \setminus (\bigcup_{n<\omega} F_n)$. Then, the set $\{x_n(\nu) : n < \omega\} = \{y_n^\nu : n < \omega\}$ does not have an accumulation point in X_ν; hence, the set $\{x_n : n < \omega\}$ cannot have an accumulation point in X, which contradicts the hypothesis. Therefore, A is countable. $\square$

Corollary 2.2. *Let X be a space. Then, X^{ω_1} is a $\mathcal{G}$-space iff X^{ω_1} is countably compact.*

A countably compact space X whose square is not countably compact (see for instance [Va, Example 4.8]) is an example of a $\mathcal{G}$-space such that X^{ω_1} cannot be a $\mathcal{G}$-space. The proof of the following theorem is analogous to the proof of Theorem 2.1.

Theorem 2.3. *Let $p \in \omega^*$. If $\{X_\mu : \mu < \omega_1\}$ is a set of spaces whose product $X = \prod_{\mu<\omega_1} X_\mu$ is a $\mathcal{G}_p$-space, then there is $A \in [\omega_1]^{\leq\omega}$ such that X_μ is p-compact for every $\mu \in \omega_1 \setminus A$.*

J. Ginsburg and V. Saks [GS] (see also [Va, Theorem 4.11]) proved that all powers of a space are countably compact the $2^{\mathfrak{c}}$ power of the space is countably compact iff it is p-compact, for some $p \in \omega^*$. This characterization and Theorem 2.3 imply the following two corollaries:

Corollary 2.4. *For $p \in \omega^*$, the following are equivalent.*

(1) *All powers of X are $\mathcal{G}_p$-spaces.*
(2) *X^{ω_1} is a $\mathcal{G}_p$-space.*
(3) *X is p-compact.*

Corollary 2.5. *For a space X, the following are equivalent.*

(1) *All powers of X are $\mathcal{G}$-spaces.*
(2) *$X^{2^{\mathfrak{c}}}$ is a $\mathcal{G}$-space.*
(3) *All powers of X are countably compact.*
(4) *X is p-compact for some $p \in \omega^*$.*

Proof: The equivalence $3 \Leftrightarrow 4$ is taken from [GS]. It suffices to show the implication $2 \Rightarrow 3$. Consider the space $Y = (X^{2^{\mathfrak{c}}})^{\omega_1}$ which is homeomorphic to $X^{2^{\mathfrak{c}}}$. By assumption, Y is a $\mathcal{G}$-space. So, by Corollary 2.2, $X^{2^{\mathfrak{c}}}$ is countably compact. By the characterization of J. Ginsburg and V. Saks [GS] quoted above, we conclude that all powers of X are countably compact. □

Theorem 2.6. *For $p \in \omega^*$, the product of countably many $\mathcal{G}_p$-spaces is a $\mathcal{G}_p$-space.*

Proof: Let $\{X_i : i < \omega\}$ be a family of $\mathcal{G}_p$-spaces. Put $X = \prod_{i<\omega} X_i$ and fix $x \in X$. For every $i < \omega$, let $\sigma^i = \{\sigma^i_n : X_i^{n+1} \to \mathcal{N}(x(i)) : n < \omega\}$ be a winning strategy for the $\mathcal{G}_p(x(i), X_i)$-game. For every $n < \omega$, we define $\sigma_n : X^{n+1} \to \mathcal{N}(x)$ by $\sigma_n(y_0, \dots, y_n) = \bigcap_{i \leq n}[i, \sigma^i_n(y_0(i), \dots, y_n(i))]$, for every $(y_0, \dots, y_n) \in X^{n+1}$. We claim that $\sigma = \{\sigma_n : n < \omega\}$ is a winning strategy for the $\mathcal{G}_p(x, X)$-game. In fact, let $(y_n)_{n<\omega}$ be a σ-sequence. By definition, we have that $y_{n+1} \in \sigma_n(y_0, \dots, y_n)$, for every $n < \omega$. Hence, $y_{n+1}(i) \in \sigma^i_n(y_0(i), \dots, y_n(i))$ for every $i, n < \omega$. That is, $(y_n(i))_{n<\omega}$ is a σ^i-sequence, for every $i < \omega$. By assumption, for every $i < \omega$, we have that $y(i) = p - \lim_{n\to\omega} y_n(i)$ exists. Thus, we obtain that $y = p - \lim_{n\to\omega} y_n$. This proves the claim and the theorem as well. □

For $p \in \omega^*$, $X_p = \beta_p(\omega) \setminus \{p\}$ is a $\mathcal{G}_p$-space that is not p-compact, where $\beta_p(\omega)$ is the p-compactification of ω (see [G]). It follows from Corollary 2.6 that X_p^ω is a $\mathcal{G}_p$-space and $X_p^{\omega_1}$ is not a $\mathcal{G}_p$-space, for every $p \in \omega^*$.

3. Finite products

We have shown in Theorem 2.6 that the product of countably many $\mathcal{G}_p$-spaces is again a $\mathcal{G}_p$-space, for each $p \in \omega^*$. However, for $\mathcal{G}$-spaces this is not true as we will see in the first example of this section. For this task, we will slightly modify Frolík's Example given in [Fro]. We shall present most of the details of Frolík's construction since his notation, in the original paper [Fro], is not standard:

If $p, q \in \omega^*$, then $p \approx q$ means that there is a bijection $f : \omega \to \omega$ such that $\hat{f}(p) = q$, where $\hat{f} : \beta(\omega) \to \beta(\omega)$ denotes the Stone-Čech extension of f. It is clear that $\approx$ is an equivalence relation on ω^* and the equivalence class of a point $p \in \omega^*$ is called the *type* of p and it is denoted by $T(p) = \{q \in \omega^* : q \approx p\}$. We know that $p \approx q$ iff there is a function $f : \omega \to \omega$ and $A \in p$ such that $\hat{f}(p) = q$ and $f|_A$ is one-to-one (for a proof see [CN, Theorem 9.2(b)]). The *RK-ordering* and the *RF-ordering* on ω^* are defined as follows:

For $p, q \in \omega^*$, we say that $p \leq_{RK} q$ if there is a function $f : \omega \to \omega$ such that $\hat{f}(q) = p$, and we say that $p \leq_{RF} q$ if there is an embedding $f : \omega \to \beta(\omega)$ such that $\hat{f}(p) = q$.
It is known that $\leq_{RF} \subset \leq_{RK}$, and $p \approx q$ iff $p \leq_{RK} q$ and $q \leq_{RK} p$, for $p, q \in \omega^*$. For $p, q \in \omega^*$, $p <_{RK} q$ will mean that $p \leq_{RK} q$ and $p \not\approx q$. For $p \in \omega^*$, we let $P_{RK}(p) = \{q \in \omega^* : q \leq_{RK} p\}$ and $S_{RF}(p) = \{q \in \omega^* : p <_{RF} q\}$. Z. Frolík [Fro] proved that $|S_{RF}(p)| = 2^{\mathfrak{c}}$, for all $p \in \omega^*$.

Lemma 3.1. *There is a family $\{X_\mu : \mu < \omega_1\}$ of subsets of ω^* and a set $\{p_\mu : \mu < \omega_1\}$ of points in ω^* such that:*

i. p_μ *and* p_ν *are RK-incomparable ultrafilters for distinct* $\mu, \nu < \omega_1$*;*

ii. $X_\mu = \{\hat{f}(p_\mu) : f : \omega \to \bigcup_{\nu<\mu} X_\nu$ is an embedding $\} \subseteq S_{RF}(p_\mu)$, *for every* $0 < \mu < \omega_1$*;*

iii. $|X_\mu| \leq \mathfrak{c}$, *for every* $\mu < \omega_1$*;*

iv. $X_\mu \cap X_\nu = \emptyset$, *whenever* $\mu < \nu < \omega_1$.

Proof: We know that there is a set W of size $2^{\mathfrak{c}}$ consisting of pairwise RK-incomparable weak P-points in ω^* (see [Ku] and [Si]). In virtue of Theorem 16.16 of [CN], we have that $S_{RF}(s) \cap S_{RF}(t) = \emptyset$ for distinct $s, r \in W$. Take $p_0 \in W$ and let $X_0 = T(p_0)$. Now, assume that p_μ and X_μ have been defined satisfying conditions *i*–*iv*, for each $\mu < \theta < \omega_1$. Put $X = \bigcup_{\mu<\theta} X_\mu$. It follows from *iii* that $|X| \leq \mathfrak{c}$. Then, choose $p_\theta \in W \setminus \{p_\mu : \mu < \theta\}$. Then, we have that $S_{RF}(p_\theta) \cap X = \emptyset$. Thus, we define $X_\theta = \{\hat{f}(p_\theta) : f : \omega \to \bigcup_{\mu<\theta} X_\mu$ is an embedding $\} \subseteq S_{RF}(p_\theta)$. □

For $A \subseteq \omega$, let $A^{\nearrow\omega} = \{f \in A^\omega : f$ is strictly increasing$\}$.

Example 3.2. *For every $1 \leq n < \omega$, there exists a space X such that X^n is countably compact and X^{n+1} is not a $\mathcal{G}$-space.*

Proof: Let $\{X_\mu : \mu < \omega_1\}$ and $\{p_\mu : \mu < \omega_1\}$ be subsets of ω^* satisfying all the properties given in Lemma 3.1. We remark, by Theorem 16.16 of [CN], that $S_{RF}(p_\mu) \cap S_{RF}(p_\nu) = \emptyset$ whenever $\mu < \nu < \mathfrak{c}$. For $\emptyset \neq I \subseteq \omega_1$, we define $X_I = \omega \cup (\bigcup_{\mu\in I} X_\mu)$. We need the following fact which is Theorem D of [Fro]:

($*$) Let $\{I_n : n < \omega\} \subseteq \mathcal{P}(\omega_1)$ be nonempty sets. If $\bigcap_{n<\omega} I_n$ is unbounded in ω_1, then $\prod_{n<\omega} X_{I_n}$ is countably compact. If $\bigcap_{n<\omega} I_n = \emptyset$, then $\prod_{n<\omega} X_{I_n}$ is not countably compact.

Fix $1 \leq n < \omega$. For each $k \leq n$, let $I_k = \{\mu < \omega_1 : \mu \not\equiv k \bmod (n+1)\}$. Let us consider the topological sum $X = \oplus_{k\leq n} X_{I_k}$. Since $\bigcap_{1\leq i\leq n} I_{k_i}$ is unbounded in

ω_1, by $(*)$, $X_{I_{k_1}} \times \cdots \times X_{I_{k_n}}$ is countably compact, for every $k_1, \ldots, k_n \in n+1$. It follows that $X^n = \bigcup_{k_1,\ldots,k_n \in n+1} X_{I_{k_1}} \times \cdots \times X_{I_{k_n}}$ is countably compact. To prove that X^{n+1} cannot be a $\mathcal{G}$-space it suffices to show that $X_{I_0} \times X_{I_1} \times \cdots \times X_{I_n}$ is not a $\mathcal{G}$-space. For every $k \leq n$, fix a non-isolated point $x_k \in X_{I_k}$. Now, we will prove that player II has a winning strategy in the $\mathcal{G}((x_0, \ldots, x_n), X_{I_0} \times \cdots \times X_{I_n})$-game. Indeed, let $\sigma = \{\sigma_m : (X_{I_0} \times \cdots \times X_{I_n})^{m+1} \to \mathcal{N}((x_0, \ldots, x_n)) : m < \omega\}$ be a strategy for player I. It is not hard to prove that, for every $k \leq n$, player II may choose $f_k \in \omega^{\nearrow\omega}$ so that

$$(f_1(m+1), \ldots, f_n(m+1)) \in \sigma_m((f_1(0), \ldots, f_n(0)), \ldots, (f_1(m), \ldots, f_n(m))),$$

for every $m < \omega$ and for every $k \leq n$. Suppose that there is $r \in \omega^*$ such that $\hat{f}_k(r) \in X_{I_k}$, for all $k \leq n$. Then, for each $k \leq n$, we have that $\hat{f}_k(r) \in X_{\mu_k}$, for some $\mu_k \in I_k$. By definition, we know that $p_{\mu_k} \leq_{RF} \hat{f}_k(r) \approx r$, for every $k \leq n$. Theorem 16.16 of [CN] implies that $p_{\mu_0} = p_{\mu_1} = \cdots = p_{\mu_n}$ and so $\mu_0 = \mu_1 = \cdots = \mu_n \in \bigcap_{k \leq n} I_k = \emptyset$, which is a contradiction. Thus, the set $\{(f_1(m), \ldots, f_n(m)) : m < \omega\}$ does not have any accumulation point in $X_{I_0} \times \cdots \times X_{I_n}$. Therefore, $X_{I_0} \times \cdots \times X_{I_n}$ is not a $\mathcal{G}$-space. □

Our next example shows that the product of a $\mathcal{G}_p$-space and a $\mathcal{G}$-space is not in general a $\mathcal{G}$-space. First, we prove some preliminary results.

Lemma 3.3. *Let $p, q \in \omega^*$ and let $f, g : \omega \to \omega$ be two one-to-one functions. If $p \not\approx q$, then (p, q) is not an accumulation point of $\{(f(n), g(n)) : n < \omega\}$ in $\beta(\omega) \times \beta(\omega)$.*

Proof: Suppose the contrary. Then, there is $r \in \omega^*$ such that $p = \hat{f}(r)$ and $q = \hat{g}(r)$. By a fact quoted above [CN, Theorem 9.2(b)], we must have that $p \approx r$ and $q \approx r$. So $p \approx q$, but this is a contradiction. □

Lemma 3.4. *Let $\omega \subseteq X, Y \subseteq \beta(\omega)$ be two non-discrete spaces. If $X \cap (\bigcup\{T(p) : p \in Y \cap \omega^*\}) = \emptyset = Y \cap (\bigcup\{T(q) : q \in X \cap \omega^*\})$, then $X \times Y$ cannot be a $\mathcal{G}$-space.*

Proof: Let $x \in X$ and $y \in Y$ be accumulation points of X and Y, respectively. We will prove that player II has a winning strategy in the $\mathcal{G}((x, y), X \times Y)$-game. Indeed, suppose that $(V_n \times W_n)_{n<\omega}$ is a sequence of basic neighborhoods of (x, y) in $X \times Y$. Then, we may find two strictly increasing functions $f, g : \omega \to \omega$ such that $f(n) \in V_n$ and $g(n) \in W_n$, for every $n < \omega$, and $f[\omega] \cap g[\omega] = \emptyset$. By Lemma 3.3, the countable set $\{(f(n), g(n)) : n < \omega\}$ does not have an accumulation point in $X \times Y$. This shows that player II has a winning strategy. Therefore, $X \times Y$ is not a $\mathcal{G}$-space. □

Theorem 3.5. *Let $M \in [\omega^*]^{\leq \mathfrak{c}}$. If $\omega \subseteq X \subseteq \beta(\omega)$ satisfies $|X| \leq \mathfrak{c}$, then there are $p \in \omega^*$ and a countably compact $\mathcal{G}_p$-space Y such that $X \times Y$ is not a $\mathcal{G}$-space and $r <_{RK} p$, for every $r \in M$.*

Proof: By Theorem 10.9 of [CN], we can find $q \in \omega^*$ so that $r <_{RK} q$, for every $r \in M \cup (X \cap \omega^*)$. Choose $p \in \omega^*$ so that $q <_{RK} p$ and let $\Gamma_q = \beta(\omega) \setminus P_{RK}(q)$. We know that Γ_q is countably compact. Theorem 2.1 from [GG] assures that Γ_q is a $\mathcal{G}_p$-space. Suppose that $X \times \Gamma_q$ is a $\mathcal{G}$-space. So, by Lemma 3.4, either $X \cap \{T(s) : s \in \Gamma_q \cap \omega^*\} \neq \emptyset$ or $\Gamma_q \cap \{T(t) : t \in X \cap \omega^*\} \neq \emptyset$, but this is impossible. Therefore, $X \times \Gamma_q$ cannot be a $\mathcal{G}$-space and $r <_{RK} p$, for all $r \in M$. □

Corollary 3.6. *For every $p \in \omega^*$, there are $q \in \omega^*$, a $\mathcal{G}_p$-space X and a countably compact $\mathcal{G}_q$-space Y such that $X \times Y$ is not a $\mathcal{G}$-space and $p <_{RK} q$.*

Proof: Let $p \in \omega^*$. We apply Theorem 3.5 to the p-compactification $\beta_p(\omega)$ of ω and $M = \{p\}$. □

Theorem 3.7. *Let $p, q \in \omega^*$. If $q \in R(p) = \{\hat{f}(p) :\ f \in \omega^\omega$ and $\exists A \in p(f|_A$ is strictly increasing$)\}$, then the product of a $\mathcal{G}_p$-space and a $\mathcal{G}_q$-space is a $\mathcal{G}_q$-space.*

Proof: This theorem is a direct consequence of Theorem 2.6 and Theorem 2.4 of [GG]. □

Example 3.8. *Let $p, q \in \omega^*$. If $q \in T(p) \setminus R(p)$, then there are a $\mathcal{G}_p$-space X and a $\mathcal{G}_q$-space Y such that $X \times Y$ is not a $\mathcal{G}_q$-space.*

Proof: Let $\Omega(p)$ the space defined in Theorem 2.3 from [GG]. We know that $\Omega(p)$ is a $\mathcal{G}_p$-space that is not a $\mathcal{G}_q$-space. Thus, $\Omega(p) \times \beta_q(\omega)$ is a $\mathcal{G}_p$-space that is not a $\mathcal{G}_q$-space and $\beta_q(\omega)$ is a $\mathcal{G}_q$-space. □

It was proved in [HST] that $p \in \omega^*$ is a Q-point iff $T(p) = R(p)$. Example 3.8 shows that the condition "$q \in R(p)$" given in Theorem 3.7 is essential. Next, we will give some relationships between the game and some of the orderings on ω^*.

Theorem 3.9. *Let $p, q \in \omega^*$ be RK-incomparable. Then there are a $\mathcal{G}_p$-space X and a $\mathcal{G}_q$-space Y such that $X \times Y$ is not a $\mathcal{G}$-space.*

Proof: Our spaces are $X = S_{RF}(p) \cup T(p) \cup \omega$ and $Y = S_{RF}(q) \cup T(q) \cup \omega$. It is not hard to see that X is a $\mathcal{G}_p$-space and Y is a $\mathcal{G}_q$-space. Fix $(x, y) \in X \times Y \setminus (\omega \times \omega)$. Let us see that player II has a winning strategy in the $\mathcal{G}((x, y), X \times Y)$-game. Indeed, suppose that $\sigma = \{\sigma_n : (X \times Y)^{n+1} \to \mathcal{N}((x, y)) : n < \omega\}$ is a strategy for player I. Then, player II can always choose two functions $f, g \in \omega^{\nearrow \omega}$ so that $((f(n), g(n)))_{n<\omega}$ is a σ-sequence with $f[\omega] \cap g[\omega] = \emptyset$. Suppose that $(s, t) \in X \times Y$ is an accumulation point for $\{(f(n), g(n)) : n < \omega\}$. By Lemma 3.3, we must have that $s \approx t$. On the other hand, by definition, there are two embeddings $e : \omega \to X$ and $h : \omega \to Y$ for which $\hat{e}(p) = s$ and $\hat{h}(q) = t$. Since $p \leq_{RF} s$, $q \leq_{RF} t$ and $s \approx t$, by Theorem 16.16 of [CN], either $p \leq_{RK} q$ or $q \leq_{RK} p$, but this is a contradiction. Therefore, $X \times Y$ is not a $\mathcal{G}$-space. □

W.W. Comfort introduced in [G] the following order on ω^*:
We say that $p \leq_C q$ if every q-compact space is p-compact.

It is known that $\leq_{RK} \subset \leq_C$ and they are different from each other. For $p, q \in \omega^*$, $p \approx_C q$ will mean that $q \leq_C p$ and $p \leq_C q$. For $p \in \omega^*$, $T_C(p) = \{q \in \omega^* : p \approx_C q\}$ is the *Comfort type* of p. It is not hard to see that $\Delta_p = \omega \cup T_C(p)$ is a $\mathcal{G}_p$-space, for each $p \in \omega^*$. The next theorem follows directly from Theorem 2.6 and Lemma 3.4.

Theorem 3.10. *Let $p, q \in \omega^*$. Then, $\Delta_p \times \Delta_q$ is a $\mathcal{G}$-space iff $p \approx_C q$.*

The following is a direct consequence of Theorems 3.9 and 3.10.

Corollary 3.11. *Let $p, q \in \omega^*$. If the product of a $\mathcal{G}_p$-space and a $\mathcal{G}_q$-space is a $\mathcal{G}$-space, then p and q are RK-comparable and $p \approx_C q$.*

We know that $S_{RF}(p)$ is a $\mathcal{G}_p$-space, for every $p \in \omega^*$.

Theorem 3.12. *Let $p, q \in \omega^*$. If $S_{RF}(p) \times S_{RF}(q)$ is a $\mathcal{G}$-space, then either*
i. p and q are RF-comparable; or
ii. there is $r \in \omega^*$ such that $r <_{RF} p$ and $r <_{RF} q$.

Proof: Fix $(x, y) \in X = S_{RF}(p) \times S_{RF}(q)$. Let $\sigma = \{\sigma_n : n < \omega\}$ be a winning strategy for player I in the $\mathcal{G}((x, y), X)$-game. Then, we may choose a σ-sequence $((f(n), g(n)))_{n<\omega}$ so that $f : \omega \to S_{RF}(p)$ and $g : \omega \to S_{RF}(q)$ are embeddings. By assumption, there is $r \in \omega^*$ such that $\hat{f}(r) \in S_{RF}(p)$ and $\hat{g}(r) \in S_{RF}(q)$. It then follows that $r <_{RF} \hat{f}(r)$, $r <_{RF} \hat{g}(r)$, $p <_{RF} \hat{f}(r)$ and $q <_{RF} \hat{g}(r)$. According to Theorem 16.16 from [CN], r and p are RF-comparable and also r and q are RF-comparable. The conclusion then follows from these relations. $\square$

Next, we shall prove that the first clause i of the conclusion of Theorem 3.14 suffices to get the converse of the same theorem.

Theorem 3.13. *Let $p, q \in \omega^*$. If $p \leq_{RF} q$, then $S_{RF}(p) \times S_{RF}(q)$ is a $\mathcal{G}_q$-space.*

Proof: Let $f : \omega \to S_{RF}(q)$ and $g : \omega \to S_{RF}(p)$ be two embeddings. By the transitivity of the RF-order, we get that $\hat{f}(q) \in S_{RF}(p)$ and $\hat{g}(q) \in S_{RF}(q)$. So, $(\hat{f}(q), \hat{g}(q))$ is an accumulation point of $\{(f(n), g(n)) : n < \omega\}$. This shows that $S_{RF}(p) \times S_{RF}(q)$ is a $\mathcal{G}_q$-space. $\square$

To give more examples we need the following notion:
The *tensor product* of two ultrafilters $p, q \in \omega^*$ is the ultrafilter

$$p \otimes q = \{A \subseteq \omega \times \omega : \{n < \omega : \{m < \omega : (n, m) \in A\} \in q\} \in p\}$$

on $\omega \times \omega$. For $p, q \in \omega^*$, $p \otimes q$ can be viewed as an ultrafilter on ω via a fixed bijection between $\omega \times \omega$ and ω. It is know that $p <_{RF} p \otimes q$ and $q <_{RK} p \otimes q$, for every $p, q \in \omega^*$. We list some relevant properties of the tensor product:

1. It was proved in [G] that if r and s are RK-incomparable free ultrafilters on ω and $r <_{RK} p$ and $s <_{RK} p$, then $p \otimes r \approx_C p \approx_C p \otimes s$ and $p \otimes r$ and $p \otimes s$ are RK-incomparable too.

2. Let s and t be two RF-minimal and RK-incomparable free ultrafilters on ω (see [Ku]), and let $r \in \omega^*$ be such that $s <_{RK} r$ and $t <_{RK} r$. Put $p = s \otimes r$ and $q = t \otimes (s \otimes r)$. Then, we have that $p <_{RK} q$, $p \approx_C q$ (for this fact see [G]), p and q are RF-incomparable and do not have a common RF-predecessor. From Theorem 3.12 we get that $S_{RF}(p) \times S_{RF}(q)$ is not a $\mathcal{G}$-space. This shows that the converse of Corollary 3.11 fails.

We will see that the second condition *ii* of Theorem 3.14 implies its converse under some additional conditions.

Theorem 3.14. *Let $p, q \in \omega^*$. Suppose that there are $r \in \omega^*$ and two embeddings $f, g : \omega \to \omega^*$ such that $\hat{f}(r) = p$, $\hat{g}(r) = q$ and $f(n) \leq_{RF} p$ and $g(n) \leq_{RF} q$, for every $n < \omega$. Then, $S_{RF}(p) \times S_{RF}(q)$ is a $\mathcal{G}_r$-space.*

Proof: Let $e : \omega \to S_{RF}(p)$ and $h : \omega \to S_{RF}(q)$ be two embeddings. Since $f(n) <_{RF} e(n)$ and $g(n) <_{RF} h(n)$, for every $n < \omega$, by Lemma 2.20 from [Boo], $p = \hat{f}(r) <_{RF} \hat{e}(r)$ and $q = \hat{g}(r) <_{RF} \hat{h}(r)$. Then, $(\hat{e}(r), \hat{h}(r)) \in S_{RF}(p) \times S_{RF}(q)$ is an accumulation point of the set $\{(e(n), h(n)) : n < \omega\}$. This shows that $S_{RF}(p) \times S_{RF}(q)$ is a $\mathcal{G}_r$-space. $\square$

As a consequence of Theorem 3.14, $S_{RF}(r \otimes p) \times S_{RF}(r \otimes q)$ is a $\mathcal{G}$-space, for every $p, q, r \in \omega^*$. But, the next example shows that the second clause *ii* of the conclusion of Theorem 3.14 does not imply its converse.

Example 3.15. *Let $r \in \omega^*$ and let $f, g : \omega \to \omega^*$ be two embeddings such that r, $f(n)$ and $g(n)$, for every $n < \omega$, are all pairwise RK-incomparable and RF-minimal. If $\hat{f}(r) = p$ and $\hat{g}(r) = q$, then $S_{RF}(p) \times S_{RF}(q)$ is not a $\mathcal{G}$-space.*

Proof: It is not hard to see, by Lemma 2.20 of [Boo], that p and q are RF-incomparable, and notice that $r <_{RF} p$ and $r <_{RF} q$. Let $e : \omega \to S_{RF}(p)$ and $h : \omega \to S_{RF}(q)$ be two embeddings. First, suppose that there is $s \in T(r)$ such that $\hat{e}(s) \in S_{RF}(p)$ and $\hat{h}(s) \in S_{RF}(q)$. Fix a bijection $\sigma : \omega \to \omega$ such that $\hat{\sigma}(r) = s$. Then, we have that $p = \hat{f}(r) <_{RF} \hat{e}(s) = \hat{e}(\hat{\sigma}(r))$ and $q = \hat{g}(r) <_{RF} \hat{h}(s) = \hat{h}(\hat{\sigma}(r))$. According to Lemma 2.20 from [Boo], $A = \{n < \omega : f(n) <_{RF} \hat{e}(\sigma(n))\} \cap \{n < \omega : g(n) <_{RF} \hat{h}(\sigma(n))\} \in r$. Take two distinct points $m, n \in A$. So, we have that $f(n) <_{RF} \hat{e}(\sigma(n))$, $f(m) <_{RF} \hat{e}(\sigma(m))$, $p <_{RF} \hat{e}(\sigma(n))$ and $p <_{RF} \hat{e}(\sigma(m))$. Then, by Theorem 16.16 from [CN] and our hypothesis, $f(n) <_{RF} p$ and $f(m) <_{RF} p$. Hence, by Theorem 16.16 of [CN], we conclude that $f(n)$ and $f(m)$ are RF-comparable, but this is a contradiction. A similar contradiction is obtained if we replace f by g. This proves that for every $s \in T(r)$ we have that $\hat{e}(s) \notin S_{RF}(p)$ and $\hat{h}(s) \notin S_{RF}(q)$. Now, let us assume that there is $t \in \omega^*$ such that $r <_{RF} t <_{RF} p$ and $r <_{RF} t <_{RF} q$. Choose an embedding $j : \omega \to \omega^*$ such that $\hat{j}(r) = t$. By Lemma 2.20 of [Boo], we have that $B = \{n < \omega : j(n) <_{RF} f(n)\} \cap \{n < \omega : j(n) <_{RF} g(n)\} \in r$, which is impossible since $f(n)$ and $g(n)$ are RF-minimal, for every $n < \omega$. Therefore, there is no $t \in \omega^*$ with $r <_{RF} t <_{RF} p$ and $r <_{RF} t <_{RF} q$. By Theorem 3.14, we obtain that $S_{RF}(p) \times S_{RF}(q)$ is not a $\mathcal{G}$-space. □

Now, we give a necessary condition for the product of two subspaces of ω^* to fail be a $\mathcal{G}$-space. For our purposes we need a lemma:

Lemma 3.16. *Let $p, q \in \omega^*$ and let $f, g : \omega \to \omega^*$ be two embeddings. If p and q do not have a common RF-predecessor, then (p, q) is not an accumulation point of $\{(f(n), g(n)) : n < \omega\}$.*

Proof: Suppose the contrary. Then, there is $r \in \omega^*$ such that $(p, q) = r - \lim_{n\to\omega}(f(n), g(n))$. Hence, $p = r - \lim_{n\to\omega} f(n)$ and $q = r - \lim_{n\to\omega} g(n)$. Since f and g are embeddings, $r <_{RF} p$ and $r <_{RF} q$, which is a contradiction. □

Theorem 3.17. *If $X, Y \subseteq \omega^*$ satisfy that $P_{RK}(p) \cap P_{RK}(q) = \emptyset$ for every $p \in X$ and for every $q \in Y$, then $X \times Y$ is not a $\mathcal{G}$-space.*

Proof: We apply an argument similar to the one used in the proof of Lemma 3.4 by using Lemma 3.16. □

We end by listing some open questions that the authors were unable to respond.

Question 3.18. *Are there spaces X and Y such that X is a $\mathcal{G}_p$-space, for all $p \in \omega^*$, and Y is a $\mathcal{G}$-space, but $X \times Y$ is not a $\mathcal{G}$-space?*

Question 3.19. *If $p, q \in \omega^*$ and $q \approx_C p <_{RF} q$, are there a $\mathcal{G}_p$-space and a $\mathcal{G}_q$-space whose product is not a $\mathcal{G}$-space?*

We point out that Lemma 3.12.10 of [En] implies that if X is a $\mathcal{G}$-k-space and Y is a $\mathcal{G}$-space, then $X \times Y$ is a $\mathcal{G}$-space.

Question 3.20. *For $n < \omega$, is there a topological group G such that G^n is a $\mathcal{G}$-space but G^{n+1} is not a $\mathcal{G}$-space?*

Question 3.21. *Is there a topological group G that is a $\mathcal{G}$-space and it is not a $\mathcal{G}_p$-space for any $p \in \omega^*$?*

Question 3.22. *Let $p, q \in \omega^*$ be RK-incomparable. Are there a $\mathcal{G}_p$-topological group G and a $\mathcal{G}_q$-topological group H such that $G \times H$ is not a $\mathcal{G}$-space?*

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