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# Integral and derivative operators of functional order on generalized Besov and Triebel-Lizorkin spaces in the setting of spaces of homogeneous type

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*Abstract.* In the setting of spaces of homogeneous-type, we define the Integral, $I_\phi$, and Derivative, $D_\phi$, operators of order $\phi$, where $\phi$ is a function of positive lower type and upper type less than 1, and show that $I_\phi$ and $D_\phi$ are bounded from Lipschitz spaces $\Lambda^\xi$ to $\Lambda^{\xi\phi}$ and $\Lambda^{\xi/\phi}$ respectively, with suitable restrictions on the quasi-increasing function $\xi$ in each case. We also prove that $I_\phi$ and $D_\phi$ are bounded from the generalized Besov $\dot{B}_p^{\psi,q}$, with $1 \leq p, q < \infty$, and Triebel-Lizorkin spaces $\dot{F}_p^{\psi,q}$, with $1 < p, q < \infty$, of order $\psi$ to those of order $\phi\psi$ and $\psi/\phi$ respectively, where $\psi$ is the quotient of two quasi-increasing functions of adequate upper types.

*Keywords:* integral and derivative operators of functional order, fractional integral operator, fractional derivative operator, spaces of homogeneous type, Besov spaces, Triebel-Lizorkin spaces

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## 1. Introduction

In the context of normal spaces of homogeneous-type $(X, \delta, \mu)$ of order $\theta \leq 1$ (see the definitions below), the fractional integral and derivative operators of order $\alpha$, with $0 < \alpha < \theta$, were defined by Gatto, Segovia and Vági [GSV] by linking them to quasi-distances constructed through the kernels $\{s_t(x,y)\}_{t>0}$ of a symmetric approximation to the identity. Namely, if $\delta_\alpha : X \times X \to [0, \infty)$ is defined by

$$\delta_\alpha(x,y) = \left(\int_0^\infty t^{\alpha-1} s_t(x,y)\, dt\right)^{1/(\alpha-1)} \tag{1.1}$$

for $x \neq y$ and $\delta_\alpha(x,y) = 0$ for $x = y$; and $\delta_{-\alpha} : X \times X \to [0,\infty)$ by

$$\delta_{-\alpha}(x,y) = \left(\int_0^\infty t^{-\alpha-1} s_t(x,y)\, dt\right)^{1/(-\alpha-1)} \tag{1.2}$$

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for $x \neq y$ and $\delta_{-\alpha}(x,y) = 0$ for $x = y$, then the authors proved that $\delta_\alpha$ and $\delta_{-\alpha}$ are quasi-metrics equivalent to $\delta$. The fractional integral $I_\alpha$ was thus defined by

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta_\alpha^{1-\alpha}(x,y)}\, d\mu(y),$$

for $f \in \Lambda^\beta \cap L^1$, and the fractional derivative $D_\alpha$ by

$$D_\alpha f(x) = \int_X \frac{f(y) - f(x)}{\delta_{-\alpha}^{1+\alpha}(x,y)}\, d\mu(y)$$

for $f \in \Lambda^\beta \cap L^\infty$ and $\alpha < \beta \leq \theta$. The definition of the quasi-metrics and the resulting operators allowed the authors to prove that the composition $T_\alpha = D_\alpha I_\alpha$ is a Calderón-Zygmund operator and that it is invertible in $L^2$ for small positive values of $\alpha$.

The purpose of this work is to show that these technics can also be used to define the integral, $I_\phi$, and derivative, $D_\phi$, operators whose kernels are equivalent to $\phi(\delta(x,y))/\delta(x,y)$ and $1/\phi(\delta(x,y))\delta(x,y)$ respectively and $\phi$ belongs to a class of quasi-increasing functions. This class includes the potentials $t^\alpha$, $0 < \alpha < 1$, but also functions as, for example, $\max(t^\alpha, t^\beta)$, $\min(t^\alpha, t^\beta)$, with $0 < \alpha < \beta < 1$, and $t^\beta(1 + \log^+ t)$, $0 \leq \beta < 1$.

We then prove that those operators are bounded on Lipschitz spaces $\Lambda^\xi$ defined by functions whose moduli of continuity are dominated by a function $\xi(t)$ in a class of growth functions.

We finally study boundedness of the integral and derivative operators on the Besov $\dot{B}_p^{\psi,q}$, $1 \leq p,q < \infty$, and Triebel-Lizorkin spaces $\dot{F}_p^{\psi,q}$, $1 < p,q < \infty$, of distributions of order $\psi$, where $\psi$ is the quotient of two quasi-increasing functions of adequate upper types. These spaces were defined in [HV] and they are a generalization of the spaces $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$, $-\theta < \alpha < \theta$, given by Han and Sawyer [HS], in the setting of spaces of homogeneous type.

The authors proved Calderón-type reproduction formulas which are essential for the definition of those spaces and to prove T1-theorems on them. In this work those formulas are used to prove the boundedness of the operators on the generalized Besov and Triebel-Lizorkin spaces.

This work is organized in the following way:

In Section 2 the class of functions involved in the 'order' of the integral and derivative operators and in local regularity of our function and distribution spaces is defined. Also the structure of normal spaces of homogeneous type, the test function spaces, the notions of discrete and continuous (in the time variable) approximations to the identity and the definitions of the generalized Besov and Triebel-Lizorkin spaces are set there. The integral and derivative operators are defined in Section 3 and the main theorems are stated in Section 4. Known results on the class of quasi-increasing functions and some consequences of them, the Calderón-type reproduction formula and properties of the generalized Besov and Triebel-Lizorkin spaces are given in Section 5. In Section 6 new representations of the kernels of the integral and derivative operators are obtained and size and smoothness properties are proved on them. Theorems of boundedness of the operators on Lipschitz spaces are proved in Section 7. Lemmas needed to prove boundedness theorems on Besov and Triebel-Lizorkin spaces are given in Section 8. Finally, the proofs of those theorems are in Section 9.

Along this paper the constant $C$ may change from one occurrence to another.

## 2. Preliminaries

In this paragraph we define the class of functions, moduli of continuity, controlling local regularity of the distribution spaces concerned and that are also related to the operators defined in this work. Let us consider nonnegative functions $\phi$ defined on the positive real numbers. The function $\phi(t)$ is *quasi-increasing* if there is a positive constant $C$ such that $\phi(t_1) \le C\phi(t_2)$ whenever $t_1 < t_2$.
Analogously, $\phi(t)$ is *quasi-decreasing* if there is a positive constant $C$ such that if $\phi(t_2) \le C\phi(t_1)$ for all $t_1 < t_2$.
Two functions $\psi(t)$ and $\phi(t)$ are *equivalent*, $\psi \simeq \phi$, if there are positive constants $C_1$ and $C_2$ such that $C_1 \le \phi/\psi \le C_2$.

The function $\phi(t)$, is of, or has, *lower type* $\alpha$, $0 \le \alpha < \infty$, if there is a constant $C_1 > 0$ such that

$$\phi(uv) \le C_1 u^\alpha \phi(v) \quad \text{for} \quad u < 1 \quad \text{and all} \quad v > 0. \tag{2.3}$$

Similarly, $\phi(t)$ is of *upper type* $\alpha$, $0 \le \alpha < \infty$ if there is a constant $C_2 > 0$ such that

$$\phi(uv) \le C_2 u^\alpha \phi(v) \quad \text{for} \quad u \ge 1 \quad \text{and all} \quad v > 0. \tag{2.4}$$

Nonnegative quasi-increasing functions $\phi$ such that $\lim_{t\to 0^+} \phi(t) = 0$ and having finite upper type will be called *growth functions*.
Let notice that if $\phi(t)$ is of both lower type $\alpha$ and upper type $\beta$ then $\alpha \le \beta$. Also, if $\phi(t)$ is of lower type $\alpha$ and $0 \le \beta < \alpha$ then $\phi$ is of lower type $\beta$. Moreover, since the condition $\phi(t)$ quasi-increasing implies, at least, lower-type 0 for $\phi$, a function $\phi(t)$ is quasi-increasing if, and only if, it is of lower type $\alpha$ for some $\alpha \ge 0$.
On the other hand, if $\phi(t)$ is of upper type $\alpha$ and $\beta > \alpha$ then $\phi$ is of upper type $\beta$, and thus, if $\phi$ is of finite upper type there is a right half line of upper types for $\phi$.
Let us notice that the condition of having finite upper type is equivalent to the Orlicz condition $\Delta_2$, $\phi(2t) \le A\phi(t)$ for some positive constant $A$.
For example, the potential $t^\alpha$, with $\alpha \ge 0$, is of lower and upper type $\alpha$. The functions $\max(t^\alpha, t^\beta)$ and $\min(t^\alpha, t^\beta)$, with $\alpha < \beta$, are both of lower type $\alpha$ and

upper type $\beta$. Also, $t^\beta(1+\log^+ t)$, with $\beta \geq 0$, is of lower type $\beta$ and of upper type $\beta+\epsilon$, for every $\epsilon > 0$. More specific properties of this class that will be used later in the main lemmas and theorems stated in Section 5.

Let us now define the structure of spaces of homogeneous type which is the underlying geometry for the test functions spaces defined in this work.
Given a set $X$, a real valued function $\delta(x,y)$ defined on $X \times X$ is a quasi-distance on $X$ if there exists a constant $A > 1$ such that for all $x, y, z \in X$ it verifies:

$$\begin{aligned}
&\delta(x,y) \geq 0 \quad \text{and} \quad \delta(x,y) = 0 \quad \text{if and only if} \quad x = y \\
&\delta(x,y) = \delta(y,x) \\
&\delta(x,y) \leq A[\delta(x,z) + \delta(z,y)].
\end{aligned}$$

In a set $X$ endowed with a quasi-distance $\delta(x,y)$, the balls $B_\delta(x,r) = \{y : \delta(x,y) < r\}$ form a basis of neighborhoods of $x$ for the topology induced by the uniform structure on $X$. Let $\mu$ be a positive measure on a $\sigma$- algebra of subsets of $X$ which contains the open set and the balls $B_\delta(x,r)$. The triple $X := (X, \delta, \mu)$ is a *space of homogeneous type* if there exists a finite constant $A' > 0$ such that $\mu(B_\delta(x,2r)) \leq A'\mu(B_\delta(x,r))$ for all $x \in X$ and $r > 0$. Macías and Segovia, [MS], showed that it is always possible to find a quasi-distance $d(x,y)$ equivalent to $\delta(x,y)$ and $0 < \theta \leq 1$, such that

$$|d(x,y) - d(x',y)| \leq Cr^{1-\theta} d(x,x')^\theta \tag{2.5}$$

holds whenever $d(x,y) < r$ and $d(x',y) < r$. If $\delta$ satisfies (2.5) then $X$ is said to be *of order* $\theta$. Furthermore, $X$ is a *normal space* if $A_1 r \leq \mu(B_\delta(x,r)) \leq A_2 r$ for every $x \in X$ and $r > 0$ and some positive constants $A_1$ and $A_2$.

*In this work $X := (X, \delta, \mu)$ means a normal space of homogeneous type of order $\theta$ and $A$ denotes the constant of the triangular inequality associated to $\delta$.*

Let us now introduce the test function spaces which concern us in this work.
Given a quasi-increasing function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0} \xi(t) = 0$ and $\lim_{t\to\infty} \xi(t) = \infty$, the Lipschitz space $\Lambda^\xi$ is the class of all functions $f : X \to \mathbb{C}$ such that

$$|f(x) - f(y)| \leq C\xi(\delta(x,y)) \quad \text{for every} \quad x, y \in X,$$

and the number $|f|_\xi$ denoting the infimum of the constants $C$ appearing above, defines a semi-norm on $\Lambda^\xi$, since $|f|_\xi = 0$ for all constant functions $f$.
Furthermore, given a ball $B$ in $X$, $\Lambda^\xi(B)$ denotes the set of functions $f \in \Lambda^\xi$ with support in $B$. Since, a function belonging to this space is bounded, the number $\|f\|_\xi = \|f\|_\infty + |f|_\xi$ defines a norm that gives a Banach structure to $\Lambda^\xi(B)$.
We say that a function $f$ belongs to $\Lambda_0^\xi$ if $f \in \Lambda^\xi(B)$ for some ball $B$. The space $\Lambda_0^\xi$ is the inductive limit of the Banach spaces $\Lambda^\xi(B)$.

Finally, $(\Lambda_0^\xi)'$ will mean the space of all continuous linear functionals on $\Lambda_0^\xi$. When $\xi(t)=t^\beta$, with $0<\beta\le\theta$, we have the classical Lipschitz spaces $\Lambda^\beta$ and $\Lambda_0^\beta$.

Another suitable class of test functions, the set $M^{(\beta,\gamma)}$, was defined in [HS]. Indeed, given $0<\beta\le 1$, $\gamma>0$ and $x_0\in X$ fixed, a function $f$ is called a *smooth molecule of type* $(\beta,\gamma)$ *of width* $d$ *centered in* $x_0$, if there exists a constant $C>0$ such that

$$|f(x)|\le C\frac{d^\gamma}{(d+\delta(x,x_0))^{1+\gamma}}, \tag{2.6}$$

$$|f(x)-f(x')|\le C\left(\frac{\delta(x,x')}{d}\right)^\beta\left(\frac{d^\gamma}{(d+\delta(x,x_0))^{1+\gamma}}+\frac{d^\gamma}{(d+\delta(x',x_0))^{1+\gamma}}\right), \tag{2.7}$$

$$\int f(x)\,d\mu(x)=0, \tag{2.8}$$

hold for every $x\in X$.
If the norm $\|f\|_{(\beta,\gamma)}$ is defined by the infimum of the constants appearing in (2.6) and (2.7), the set $M^{(\beta,\gamma)}(x_0,d)$ of all smooth molecules of type $(\beta,\gamma)$ of width $d$ centered in $x_0$ is a Banach space. Fixing $x_0\in X$ and $d=1$, that space will be named $M^{(\beta,\gamma)}$, and the set of all linear continuous functionals on $M^{(\beta,\gamma)}$ will be called $(M^{(\beta,\gamma)})'$. Along this work $\langle h,f\rangle$ denotes the natural application of $h\in(M^{(\beta,\gamma)})'$ to $f\in M^{(\beta,\gamma)}$.

In order to define the generalized Besov and Triebel-Lizorkin spaces of distributions the definition of an approximation to the identity as given in [HS] is needed.
A sequence $(S_k)_{k\in\mathbb{Z}}$ of integral operators is called an *approximation to the identity*, if the kernels $S_k(x,y)$ associated to $S_k$ are functions from $X\times X$ to $\mathbb{C}$ and there exist $0<\epsilon\le\theta$ and a finite constant $C$ such that for all $k\in\mathbb{Z}$ and $x,x',y,y'\in X$ they satisfy

$$S_k(x,y)=0 \ \text{ if } \ \delta(x,y)\ge(2A)^{-k} \ \text{ and } \ \|S_k\|_\infty\le C(2A)^k, \tag{2.9}$$

$$|S_k(x,y)-S_k(x',y)|\le C(2A)^{k(1+\epsilon)}\delta(x,x')^\epsilon, \tag{2.10}$$

$$\begin{aligned}&|S_k(x,y)-S_k(x,y')|\le C(2A)^{k(1+\epsilon)}\delta(y,y')^\epsilon,\\ &|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\\ &\qquad\qquad\le C(2A)^{k(1+2\epsilon)}\delta(x,x')^\epsilon\delta(y,y')^\epsilon,\end{aligned} \tag{2.11}$$

$$\int_X S_k(x,y)\,d\mu(y)=\int_X S_k(x,y)\,d\mu(x)=1. \tag{2.12}$$

*Throughout this paper, $\epsilon$ ($0 < \epsilon \le \theta$) will denote the constant associated to an approximation to the identity satisfying* (2.10), (2.11) *and* (2.12).

If $(S_k)_{k\in\mathbb{Z}}$ is an approximation to the identity then the family of operators $D_k = S_k - S_{k-1}$ satisfy $\sum_{k\in\mathbb{Z}} D_k = I$ in $L^2$, since $\lim_{k\to\infty} S_k f = f$ and $\lim_{k\to-\infty} S_k f = 0$ in $L^2$. Moreover, their associated kernels $D_k(x,y)$ satisfy properties (2.9) to (2.12) and

$$\int_X D_k(x,y)\,d\mu(y) = \int_X D_k(x,y)\,d\mu(x) = 0. \tag{2.13}$$

Let us now define the spaces of distributions.

*In the sequel we denote by $\psi$ the function $\psi = \phi_1/\phi_2$, where $\phi_1(t)$ and $\phi_2(t)$ are quasi-increasing functions of upper types $s_1 < \epsilon$ and $s_2 < \epsilon$, respectively.*

For $f \in (M^{(\beta,\gamma)})'$, with $0 < \beta, \gamma < \epsilon$, a norm is defined by

$$\|f\|_{\dot{B}_p^{\psi,q}} = \left(\sum_{k\in\mathbb{Z}} \Big(\frac{1}{\psi((2A)^{-k})}\|D_k f\|_p\Big)^q\right)^{\frac{1}{q}} \quad \text{if } 1 \le p \le \infty,\ 1 \le q \le \infty, \tag{2.14}$$

with the obvious change for the case $q = \infty$. By interchanging the order of the norms in $L^p$ and $l^q$, we also define

$$\|f\|_{\dot{F}_p^{\psi,q}} = \left\| \left(\sum_{k\in\mathbb{Z}} \Big(\frac{1}{\psi((2A)^{-k})}|D_k f|\Big)^q\right)^{\frac{1}{q}} \right\|_{L^p}, \quad \text{if } 1 < p, q < \infty. \tag{2.15}$$

The *Besov space* $\dot{B}_p^{\psi,q}$, $1 \le p, q \le \infty$, is the set of all $f \in \left(M^{(\beta,\gamma)}\right)'$ with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{B}_p^{\psi,q}} < \infty \quad \text{and} \quad |\langle f, h\rangle| \le C\|f\|_{\dot{B}_p^{\psi,q}}\|h\|_{(\beta,\gamma)},$$

for all $h \in M^{(\beta,\gamma)}$.

Analogously, The *Triebel-Lizorkin space* $\dot{F}_p^{\psi,q}$, with $1 < p, q < \infty$, is the set of all $f \in (M^{(\beta,\gamma)})'$, with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{F}_p^{\psi,q}} < \infty, \quad \text{and} \quad |\langle f, h\rangle| \le \|f\|_{\dot{F}_p^{\psi,q}}\|h\|_{(\beta,\gamma)},$$

for all $h \in M^{(\beta,\gamma)}$.

When $\psi(t) = t^\alpha$ and $-\epsilon < \alpha < \epsilon$, the definitions of the Besov spaces $\dot{B}_p^{\alpha,q}$ and the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ given in [HS] are recovered.

Finally, to build our operators we consider a symmetric approximation to the identity, $\{S_t\}_{t>0}$, as defined in [GSV]. This collection can be built in a similar

fashion to that of the family $\{S_k\}_{k\in\mathbb{Z}}$ and the kernel $s_t(x,y)$ associated to $S_t$ satisfies the following properties:
There are positive constants, $b_1, b_2, c_1, c_2$ and $c_3$, such that for all $x, y \in X$ and $t > 0$, $s_t(x,y)$ satisfies

$$s_t(x,y) = s_t(y,x), \tag{2.16}$$

$$0 \le s_t(x,y) \le c_1/t, \tag{2.17}$$

$$s_t(x,y) = 0 \text{ if } \delta(x,y) > b_1 t \text{ and } c_2/t < s_t(x,y) \text{ if } \delta(x,y) < b_2 t, \tag{2.18}$$

$$|s_t(x,y) - s_t(x',y)| < c_3 \delta^\theta(x,x')/t^{1+\theta} \text{ for all } x, x', y \in X, \tag{2.19}$$

$$\int s_t(x,y)\, d\mu(y) = 1, \text{ for all } x \in X, \tag{2.20}$$

$$s_t(x,y) \text{ is continuously differentiable in } t. \tag{2.21}$$

## 3. Integral and derivative operators of order $\phi$

Consider a symmetric approximation to the identity, $\{S_t\}_{t>0}$, whose kernels satisfy properties (2.16) to (2.21), and a quasi-increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0^+} \phi(t) = 0$.
We now define

$$K_\phi(x,y) = \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)\, dt \quad \text{for } x \neq y.$$

Clearly, $K_\phi(x,y) > 0$ and $K_\phi(x,y) = K_\phi(y,x)$ for every $(x,y)$.
If $\phi$ is of positive lower type and upper type $s_\phi < 1$ the *integral operator of order* $\phi$, $I_\phi$, and its extension $\tilde{I}_\phi$ are defined in the following way:
Let $\xi$ be a quasi-increasing function of upper type $\beta$.
If $\beta > 0$ and $f \in \Lambda^\xi \cap L^1$ then

$$I_\phi f(x) := \int_X K_\phi(x,y) f(y)\, d\mu(y), \tag{3.22}$$

If $\beta + s_\phi < \theta$ and $f \in \Lambda^\xi$ then

$$\tilde{I}_\phi f(x) := \int_X (K_\phi(x,y) - K_\phi(x_0,y)) f(y)\, d\mu(y), \tag{3.23}$$

for every $x \in X$ and an arbitrary fixed $x_0 \in X$.
On the other hand, for $\phi$ of finite upper-type we define

$$K_{1/\phi}(x,y) = \int_0^\infty \frac{1}{\phi(t)t} s_t(x,y)\, dt, \quad \text{for } x \neq y.$$

Clearly $K_{1/\phi}$ is positive and symmetric.
If $\phi$ is a function of lower type $i_\phi > 0$ and upper type $s_\phi$ the *derivative operator of order* $\phi$, $D_\phi$, and its extension, $\tilde{D}_\phi$ are defined as follows:
For any function $\xi$ of lower type $\alpha$ and of upper type $\beta$, such that $s_\phi < \alpha$,
if $f \in \Lambda^\xi \cap L^\infty$, then

$$D_\phi f(x) = \int_X K_{1/\phi}(x, y)(f(y) - f(x))\, d\mu(y) \tag{3.24}$$

and if $f \in \Lambda^\xi$, then

$$\tilde{D}_\phi f(x) = \int_X (K_{1/\phi}(x, y)(f(y) - f(x)) - K_{1/\phi}(x_0, y)(f(y) - f(x_0)))\, d\mu(y) \tag{3.25}$$

for each $x \in X$ and an arbitrary, but fixed, $x_0 \in X$.

## 4. Main theorems

**Theorem 4.1.** *Let $\phi$ be of lower type $i_\phi > 0$ and upper type $s_\phi < 1$ and $\xi$ a quasi-increasing function of upper type $\beta$.*

*If $f \in \Lambda^\xi \cap L^1$ and $\beta > 0$ then $I_\phi f(x)$ converges absolutely for all $x$ and if, moreover, $\beta + s_\phi < \theta$ then there is a constant $C > 0$, independent of $f$, such that*

$$|I_\phi f|_{\Lambda^{\xi\phi}} \le C |f|_{\Lambda^\xi}.$$

*If $f \in \Lambda^\xi$ and $\beta + s_\phi < \theta$ then $\tilde{I}_\phi f(x)$ converges absolutely for all $x$ and there is a constant $C > 0$, independent of $f$, such that*

$$|\tilde{I}_\phi f|_{\Lambda^{\xi\phi}} \le C |f|_{\Lambda^\xi}.$$

*Moreover, if $f \in \Lambda^\xi \cap L^1$, then $\tilde{I}_\phi f$ coincides with $I_\phi f$ as an element of $\Lambda^{\xi\phi}$ (since $\tilde{I}_\phi f(x) = I_\phi f(x) - I_\phi f(x_0)$).*

**Theorem 4.2.** *Let $\phi$ be a function of lower type $i_\phi > 0$ and upper type $s_\phi$. Let also $\xi$ be a quasi-increasing function of lower type $\alpha$ and upper type $\beta$. If $f \in \Lambda^\xi \cap L^\infty$ and $s_\phi < \alpha$ then $D_\phi f(x)$ is absolutely convergent for every $x \in X$ and if, also, $\beta < \theta + i_\phi$ then*

$$\|D_\phi f\|_{\xi/\phi} \le C \|f\|_\xi.$$

*If $f \in \Lambda^\xi$, $s_\phi < \alpha$ and $\beta < \theta + i_\phi$ then $\tilde{D}_\phi f(x)$ is absolutely convergent for every $x \in X$ and*

$$|\tilde{D}_\phi f|_{\xi/\phi} \le C |f|_\xi.$$

*Moreover, if $f \in \Lambda^\xi \cap L^\infty$, then $\tilde{D}_\phi f$ coincides with $D_\phi f$ as an element of $\Lambda^\xi$, (since $\tilde{D}_\phi f(x) = D_\phi f(x) - D_\phi f(x_0)$).*

In the following theorems we use the notation $\psi = \psi_1/\psi_2$, where $\psi_1$ and $\psi_2$ are quasi-increasing functions of upper types $s_1$ and $s_2$ respectively.

**Theorem 4.3.** *Let $\phi$ be a function of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$. If $s_1 + s_\phi < \epsilon$ and $s_2 + s_\phi - i_\phi < \epsilon$, then $I_\phi$ is a linear continuous operator from $\dot{F}_p^{\psi,q}$ to $\dot{F}_p^{\phi\psi,q}$, for $1 < p, q < \infty$.*

**Theorem 4.4.** *Let $\phi$ be a function of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$. If $s_1 + s_\phi < \epsilon$ and $s_2 + s_\phi - i_\phi < \epsilon$ then $I_\phi$ is a linear continuous operator from $\dot{B}_p^{\psi,q}$ to $\dot{B}_p^{\phi\psi,q}$, for $1 \leq p, q < \infty$.*

**Theorem 4.5.** *Let $\phi$ be a function of positive lower type and of upper type $s_\phi < \epsilon$. If $s_1 < \epsilon$ and $s_\phi + s_2 < \epsilon$ then, $D_\phi$ is a linear continuous operator from $\dot{F}_p^{\psi,q}$ to $\dot{F}_p^{\psi/\phi,q}$, for $1 < p, q < \infty$.*

**Theorem 4.6.** *Let $\phi$ be of positive lower type and of upper type $s_\phi < \epsilon$. If $s_1 < \epsilon$ and $s_\phi + s_2 < \epsilon$ then $D_\phi$ is a linear continuous operator from $\dot{B}_p^{\psi,q}$ to $\dot{B}_p^{\psi/\phi,q}$, for $1 \leq p, q < \infty$.*

## 5. Previous results

We begin this section giving some special properties of the class of functions acting as moduli of continuity but also as order of integration and derivation. These properties will be used later in the proof of the main lemmas and theorems. Unless special difficulty or interest of their proof, these are omitted for the sake of briefness.
The following two statements are alternative definitions of lower and upper type. If $\phi(t)$ is of upper type $s_\phi$ then there is a constant $C > 0$ such that

$$\phi(uv) \geq \frac{1}{C} u^{s_\phi} \phi(v), \quad \text{for } \ u < 1 \ \text{ and all } \ v > 0. \tag{5.26}$$

Similarly, if $\phi(t)$ is of lower type $i_\phi$ then there is a constant $C > 0$ such that

$$\phi(uv) \geq \frac{1}{C} u^{i_\phi} \phi(v), \quad \text{for } \ u \geq 1, \ \text{ and all } \ v > 0. \tag{5.27}$$

**Proposition 5.1.** *Given two functions $\phi(t)$ of lower type $\alpha$ and $\xi(t)$ of upper type $\beta \leq \alpha$, the function $\phi(t)/\xi(t)$ is quasi-increasing.*

The following proposition shows how to obtain regularizations of a quasi-increasing function of positive lower type.

**Proposition 5.2.** *If $\phi(t)$ is of positive lower type $\alpha$ and finite upper type $\beta$ then, for any $0 < \gamma < \alpha$, the function*

$$\psi(t) = t^\gamma \int_0^t \frac{\phi(u)}{u^{\gamma+1}} \, du$$

*is a strictly increasing and differentiable function equivalent to $\phi$. Moreover, its inverse $\psi^{-1}$ is of lower type $\beta^{-1}$ and of upper type $\alpha^{-1}$.*

Two applications of the previous statement that will be frequently used later are stated and proved next.

**Corollary 5.1.** *If $\phi$ is a quasi-increasing function of upper type $s_\phi < 1$ then there is an equivalent function $\tilde{\phi}$ such that $\tilde{\phi}(t)/t$ is a strictly decreasing and differentiable function on the positive real numbers.*

PROOF: Indeed, since $t/\phi(t)$ is of lower type $1 - s_\phi > 0$ and of upper type 1, we consider a strictly increasing differentiable function, $\psi(t)$, equivalent to $t/\phi(t)$, as given in Proposition 5.2. The function $\tilde{\phi} = t/\psi(t)$ satisfies our statement. $\square$

**Corollary 5.2.** *If $\phi(t)$ is function so that $t\phi(t)$ is of positive lower type and finite upper type then there exists a function $\hat{\phi}(t)$ equivalent to $\phi(t)$, such that $t\hat{\phi}(t)$ is a strictly increasing, differentiable function defined in $\mathbb{R}^+$.*

PROOF: The statement follows by defining $\hat{\phi}(t) = \hat{\psi}(t)/t$, where $\hat{\psi}(t)$ is a strictly increasing and differentiable function equivalent to $t\phi(t)$. $\square$

The proof of the next proposition follows by dyadic partition of the domain of integration.

**Proposition 5.3.** *Let $\phi_i(t)$ be a function of lower type $\alpha_i$ and of upper type $\beta_i$, $i = 1, 2$. The following inequalities hold for $x \in X$ and $r > 0$:*

$$\text{If } \alpha_1 > \beta_2 \text{ then } \int_{\delta(x,y)\le r} \frac{\phi_1(\delta(x,y))}{\phi_2(\delta(x,y))\delta(x,y)}\, d\mu(y) \le C\frac{\phi_1(r)}{\phi_2(r)}\,. \tag{5.28}$$

$$\text{If } \beta_1 < \alpha_2 \text{ then } \int_{\delta(x,y)\ge r} \frac{\phi_1(\delta(x,y))}{\phi_2(\delta(x,y))\delta(x,y)}\, d\mu(y) \le C\frac{\phi_1(r)}{\phi_2(r)}\,. \tag{5.29}$$

In the following paragraph we recall the Calderón-type reproduction formulas, stated by Han and Sawyer in the context of spaces of homogeneous type, which are the essential tool used to develop Littlewood-Paley characterizations of the Besov and Triebel-Lizorkin spaces and the theorems of continuity of our operators on those spaces. The proof of these formulas is given in [HS].

**Theorem 5.1.** *Let $(S_k)_{k\in\mathbb{Z}}$ be an approximation to the identity and set $D_k = S_k - S_{k-1}$. Then, there exist families of operators $(\tilde{D}_k)_{k\in\mathbb{Z}}$ and $(\hat{D}_k)_{k\in\mathbb{Z}}$ such that for all $f \in M^{(\beta,\gamma)}$*

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k f = \sum_{k=-\infty}^{\infty} D_k \hat{D}_k f,$$

*where the series converges in $M^{(\beta',\gamma')}$, for $\beta' < \beta$ and $\gamma' < \gamma$.*

Let us notice that if $(\tilde{D}_k)_{k\in\mathbb{Z}}$ and $(\hat{D}_k)_{k\in\mathbb{Z}}$ are as in Theorem 5.1 then their associated kernels $\tilde{D}_k(x,y)$ and $\hat{D}_k(x,y)$ are $(\epsilon',\epsilon')$-smooth molecules of width $(2A)^{-k}$,

as functions of the first and second variable respectively, for each $0 < \epsilon' < \epsilon$. Then $\tilde{D}_k^* f$ and $\hat{D}_k^* f \in M^{(\beta,\gamma)}$, whenever $f \in M^{(\beta,\gamma)}$, $0 < \beta, \gamma < \epsilon$.
Thus, for $h \in (M^{(\beta,\gamma)})'$ $\tilde{D}_k h$ and $\hat{D}_k h$ are defined as elements of $(M^{(\beta,\gamma)})'$ by $\langle \tilde{D}_k h, f\rangle = \langle h, \tilde{D}_k^* f\rangle$ and $\langle \hat{D}_k h, f\rangle = \langle h, \hat{D}_k^* f\rangle$. Therefore, the formulas in Theorem 5.1 will also hold true in the sense of distributions. More precisely,

**Theorem 5.2.** *Let $(D_k)_{k\in\mathbb{Z}}$, $(\tilde{D}_k)_{k\in\mathbb{Z}}$ and $(\hat{D}_k)_{k\in\mathbb{Z}}$ be as in Theorem 5.1. Then for all $f \in (M^{(\beta,\gamma)})'$,*

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k f = \sum_{k=-\infty}^{\infty} D_k \hat{D}_k f,$$

*in the sense of*

$$\langle f, g\rangle = \lim_{M\to\infty} \langle \sum_{|k|\le M} \tilde{D}_k D_k f, g\rangle = \lim_{M\to\infty} \langle \sum_{|k|\le M} D_k \hat{D}_k f, g\rangle$$

*for all $g \in M^{(\beta',\gamma')}$, with $\beta' > \beta$ and $\gamma' > \gamma$.*

The Calderón-type reproduction formulas allow us to prove that if the operators $D_k$ in the definitions of the norms of the spaces are replaced by $E_k = P_k - P_{k-1}$, where $(P_k)_{k\in\mathbb{Z}}$ is another approximation to the identity of order $\epsilon \le \theta$, the resulting norms are equivalent to those defined in (2.14) and (2.15) (for the proof of this fact see [HS] for the classical spaces or [H] for the generalized ones). The same result is true if the operators $D_k$ are replaced by $\tilde{D}_k$ or $\hat{D}_k$.

In the following two lemmas the main properties of the generalized Besov and Triebel-Lizorkin spaces are stated without proof, for the sake of briefness.

**Lemma 5.3.** *The classes $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$ are Banach spaces and their dual spaces are $\dot{B}_{p'}^{1/\psi,q'}$ and $\dot{F}_{p'}^{1/\psi,q'}$ respectively, with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.*

**Lemma 5.4.** *The molecular space $M^{(\beta,\gamma)}$ is embedded in $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$, when $s_1 < \beta$ and $s_2 < \gamma$. Moreover, $M^{(\epsilon',\epsilon')}$ is dense in $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$, for all $\epsilon'$, such that $\max(s_1, s_2) < \epsilon' < \epsilon$.*

In the setting of $\mathbb{R}^n$ and for $q = \infty$, unified approaches between Besov spaces of order $\xi$ related to a Banach space $E$ of functions (in our definitions $E = L^p$) and Lipschitz classes of distributions whose moduli of continuity in $E$ are dominated by $\xi$ are treated in [J] and [B]. See also [I] for the inhomogeneous case. The identification between the Sobolev space of fractional order $\dot{L}^{p,\alpha}$ and $\dot{F}_p^{\alpha,q}$ in the setting of spaces of homogeneous type is treated in [GV].

## 6. Main lemmas

Let now define two quasi-metrics associated to $\phi$ and equivalent to $\delta$ and obtain new representations of the kernels of $I_\phi$ and $D_\phi$ in terms of each quasi-metric.

Consider a quasi-increasing function $\phi$ of upper-type $s_\phi < 1$ and a fix function $\tilde{\phi}$, as given in Corollary 5.1. We define $\delta_\phi : X \times X \to \mathbb{R}$ in the following way: for every pair $(x, y) \in X$, $\delta_\phi(x, y)$ is the unique solution of

$$\begin{aligned} \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} &= \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)\,dt \quad \text{if} \quad x \neq y, \\ \delta_\phi(x,y) &= 0 \quad \text{if} \quad x = y. \end{aligned} \tag{6.30}$$

We then have that

$$K_\phi(x,y) = \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \quad \text{for} \quad x \neq y.$$

When $\phi(t) = t^\alpha$, $0 < \alpha < 1$, we can choose $\tilde{\phi} = \phi$ and then $\delta_\alpha := \delta_\phi$ is the quasi-metric associated to $I_\alpha$ defined in (1.1).

The next lemma proves that $K_\phi(x, y)$ is equivalent to $\phi(\delta(x, y))/\delta(x, y)$.

**Lemma 6.1.** *If $\phi$ is of upper type $s_\phi < 1$ then there are positive constants $C_1$ and $C_2$ such that for $\delta(x, y) > 0$,*

$$C_2 \frac{\phi(\delta(x,y))}{\delta(x,y)} \leq \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \leq C_1 \frac{\phi(\delta(x,y))}{\delta(x,y)}\,. \tag{6.31}$$

Proof: By (2.17) and (2.18), it holds that

$$\int_0^\infty \frac{\phi(t)}{t} s_t(x,y)\,dt \leq c_1 \int_{\delta(x,y)/b_1}^\infty \frac{\phi(t)}{t^2}\,dt.$$

The substitution $t = u\delta(x, y)/b_1$ and inequality (2.4) yield to

$$\int_0^\infty \frac{\phi(t)}{t} s_t(x,y)\,dt \leq \frac{c_1 b_1}{\delta(x,y)} \phi(\frac{\delta(x,y)}{b_1}) \int_1^\infty \frac{1}{u^{2-s_\phi}}\,du \leq C_1 \frac{\phi(\delta(x,y))}{\delta(x,y)} \tag{6.32}$$

since $s_\phi < 1$ and $\phi(s/b_1) \leq C \max(1, 1/b_1^{s_\phi})\phi(s)$ for all $s > 0$.

On the other hand, by (2.18) and the fact that $\phi$ is quasi-increasing, it follows that

$$\begin{aligned} \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)\,dt &\geq c_2 \int_{\delta(x,y)/b_2}^\infty \frac{\phi(t)}{t^2}\,dt \\ &\geq C c_2 \frac{\phi(\delta(x,y)/b_2)}{\delta(x,y)/b_2} \int_1^\infty \frac{1}{u^2}\,du = C_2 \frac{\phi(\delta(x,y))}{\delta(x,y)}, \end{aligned} \tag{6.33}$$

since $\phi(s/b_2) \geq C \min(1, 1/b_2^{s_\phi})\phi(s)$ for all $s > 0$. From definition (6.30) and the above inequalities then (6.31) follows. $\square$

As an immediate consequence of the previous lemma we have

$$0 < K_\phi(x,y) \leq C \frac{\phi(\delta(x,y))}{\delta(x,y)}. \tag{6.34}$$

**Lemma 6.2.** *If $\phi(t)$ is of upper type $s_\phi < 1$ then $\delta_\phi$ is a quasi-metric equivalent to $\delta$.*

PROOF: Since $s_\phi < 1$, from (6.31) and $\phi \simeq \tilde{\phi}$ it follows that

$$C_2' \frac{\tilde{\phi}(\delta(x,y))}{\delta(x,y)} \leq \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \leq C_1' \frac{\tilde{\phi}(\delta(x,y))}{\delta(x,y)}. \tag{6.35}$$

Nevertheless, since $\psi(t) = t/\tilde{\phi}(t)$ is increasing and invertible and its inverse function is of finite upper type, it follows that

$$C_1''\delta(x,y) \leq \delta_\phi(x,y) \leq C_2''\delta(x,y).$$

Clearly, from the above equivalence turns out that $\delta_\phi$ is a quasi-metric. $\square$

The next two lemmas state smoothness and cancellation properties of $K_\phi$.

**Lemma 6.3.** *Let $\phi$ be of upper type $s_\phi < 1$. Then*

$$|K_\phi(x,y) - K_\phi(x',y)| \leq C \left(\frac{\delta(x,x')}{\delta(x,y)}\right)^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)} \tag{6.36}$$

*whenever $\delta(x,y) \geq 2A\delta(x,x')$.*

PROOF: Let $a = b_1^{-1} \min\{\delta(x,y), \delta(x',y)\}$ where $b_1$ is that defined in (2.18). From (6.30), it follows that

$$|K_\phi(x,y) - K_\phi(x',y)| \leq \int_a^\infty \frac{\phi(t)}{t} |s_t(x,y) - s_t(x',y)|\, dt.$$

From the smoothness property (2.19) of $s_t$ it follows that

$$|K_\phi(x,y) - K_\phi(x',y)| \leq \int_a^\infty \frac{\phi(t)}{t} \frac{\delta(x,x')^\theta}{t^{1+\theta}}\, dt. \tag{6.37}$$

Since $s_\phi < 1$, Proposition 5.1 says that $\phi(t)/t$ is quasi-decreasing and then,

$$|K_\phi(x,y) - K_\phi(x',y)| \leq C\delta(x,x')^\theta \frac{\phi(a)}{a^{1+\theta}}. \tag{6.38}$$

Since $\delta(x,y) \geq 2A\delta(x,x')$ then $\delta(x,y) \leq 2A\delta(x',y)$ and thus, $\delta(x,y) \leq 2Ab_1 a$. But, since $\phi(t)/t^l$ is quasi-decreasing whenever $l > s_\phi$, it follows that

$$|K_\phi(x,y) - K_\phi(x',y)| \leq C\delta(x,x')^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}}$$

which is our statement. $\square$

**Lemma 6.4.** *Let $\phi$ be of upper type $s_\phi < \theta$. Then*

$$\int_X [K_\phi(x,y) - K_\phi(x',y)]\, d\mu(y) = 0, \tag{6.39}$$

*for every $x$ and $x' \in X$.*

Proof: First notice that the integral in (6.39) is absolutely convergent. Indeed, (6.30), (2.17) and (2.18) yield to

$$\begin{aligned} &\int_X \int_0^1 \frac{\phi(t)}{t} |s_t(x,y) - s_t(x',y)|\, dt\, d\mu(y) \\ &\leq C \int_0^1 \frac{\phi(t)}{t} \int_X (|s_t(x,y)| + |s_t(x',y)|)\, d\mu(y)\, dt \leq C \int_0^1 \frac{\phi(t)}{t}\, dt < \infty. \end{aligned} \tag{6.40}$$

Moreover, from (2.19) it follows that

$$\begin{aligned} &\int_X \int_1^\infty \frac{\phi(t)}{t} |s_t(x,y) - s_t(x',y)|\, dt\, d\mu(y) \\ &\leq C(\delta(x,x'))^\theta \int_1^\infty \frac{\phi(t)}{t^{2+\theta}} \int_{\delta(x,y)<b_1 t \vee \delta(x',y)<b_1 t} d\mu(y)\, dt \\ &\leq C(\delta(x,x'))^\theta \int_1^\infty \frac{\phi(t)}{t^{1+\theta}}\, dt \\ &\leq C(\delta(x,x'))^\theta \int_1^\infty \frac{1}{t^{1+\theta-s_\phi}}\, dt < \infty. \end{aligned} \tag{6.41}$$

Therefore, (6.39) is obtained by Fubini's theorem and (2.20). □

Consider now a quasi-increasing function $\phi$ of finite upper type and the function $\hat{\phi}$, as given by Corollary 5.2. We then define $\delta_{1/\phi} : X \times X \to \mathbb{R}$ such that $\delta_{1/\phi}(x,y)$ is the unique solution of the equation

$$\begin{gathered} \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} = \int_0^\infty \frac{1}{\phi(t)t} s_t(x,y)\, dt \quad \text{if} \quad x \neq y, \quad \text{and} \\ \delta_{1/\phi}(x,y) = 0 \quad \text{if} \quad x = y. \end{gathered} \tag{6.42}$$

Hence we have that

$$K_{1/\phi}(x,y) = \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} \quad \text{for} \quad x \neq y.$$

When $\phi(t) = t^\alpha$, $0 < \alpha < 1$, choosing $\hat{\phi} = \phi$ it turns out that $\delta_{-\alpha} := \delta_{t^{-\alpha}}$ is the quasi-metric associated to $D_\alpha$ defined in (1.2).

The following results are obtained in an analogous way to the case of $I_\phi$ and their proof is omitted for the sake of briefness. The first one states that $K_{1/\phi}(x,y)$ is equivalent to $1/(\phi(\delta(x,y))\delta(x,y))$.

**Lemma 6.5.** *If $\phi$ is quasi-increasing then there are positive constants $C_1$ and $C_2$ such that*

$$C_1 \frac{1}{\phi(\delta(x,y))\delta(x,y)} \leq \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} \leq C_2 \frac{1}{\phi(\delta(x,y))\delta(x,y)} . \tag{6.43}$$

*Moreover, $\delta_{1/\phi}$ is a quasi-metric equivalent to $\delta$.*

From the above lemma, the size estimate immediately follows

$$0 < K_{1/\phi}(x,y) < C \frac{1}{\phi(\delta(x,y))\delta(x,y)} . \tag{6.44}$$

**Lemma 6.6.** *If $\phi$ is a quasi-increasing function of finite upper type then*

$$\begin{aligned} |K_{1/\phi}(x,y) - K_{1/\phi}(x',y)| + |K_{1/\phi}(y,x) - K_{1/\phi}(y,x')| \\ \leq C \left( \frac{\delta(x,x')}{\delta(x,y)} \right)^{\theta} \frac{1}{\phi(\delta(x,y))\delta(x,y)} \end{aligned} \tag{6.45}$$

*for $\delta(x,y) \geq 2A\delta(x,x')$.*

## 7. Proof of Theorems 4.1 and 4.2

Proof of Theorem 4.1: Let us first see that $I_\phi f(x)$ is absolutely convergent for every $x \in X$. From (6.30), it follows that

$$\begin{aligned} \int_X |K_\phi(x,y)||f(y)|\, d\mu(y) &\leq C \int_X \frac{\phi(\delta(x,y))}{\delta(x,y)} |f(y)|\, d\mu(y), \\ &= C \left( \int_{\delta(x,y)\leq 1} + \int_{\delta(x,y)>1} \right) \frac{\phi(\delta(x,y))}{\delta(x,y)} |f(y)|\, d\mu(y) = I_1 + I_2. \end{aligned} \tag{7.46}$$

Applying (5.28) since $i_\phi > 0$, and the fact that $\xi$ is quasi-increasing it follows that

$$\begin{aligned} I_1 &\leq C \int_{\delta(x,y)\leq 1} \frac{\phi(\delta(x,y))}{\delta(x,y)} (|f(y) - f(x)| + |f(x)|)\, d\mu(y) \\ &\leq C|f|_\xi \int_{\delta(x,y)\leq 1} \frac{(\phi\xi)(\delta(x,y))}{\delta(x,y)}\, d\mu(y) \\ &\quad + C|f(x)| \int_{\delta(x,y)\leq 1} \frac{\phi(\delta(x,y))}{\delta(x,y)}\, d\mu(y) \\ &\leq C(\xi(1)|f|_\xi + |f(x)|) \\ &\leq C(|f|_\xi + |f(x)|). \end{aligned} \tag{7.47}$$

Furthermore, since $s_\phi < 1$, $\phi(t)/t$ is quasi-decreasing and

$$I_2 \leq C \int_{\delta(x,y)>1} \frac{\phi(\delta(x,y))}{\delta(x,y)} |f(y)|\, d\mu(y) \leq C\|f\|_1. \tag{7.48}$$

Inequalities (7.47) and (7.48) lead to the bound

$$|I_\phi f(x)| \leq \int_X |K_\phi(x,y)||f(y)|\, d\mu(y) < C(|f|_\xi + |f(x)| + \|f\|_1), \tag{7.49}$$

for every $x \in X$.

In order to prove that $\tilde{I}_\phi$ is well defined we follow the idea of the above proof. In fact, using (5.28) for $\delta(x,y) \leq 2A\delta(x,x_0)$ and, on the other hand, (6.36), the fact that $s_\phi + \beta < \theta$, and (5.29) for $\delta(x,y) \geq 2A\delta(x,x_0)$, it is not hard to prove that

$$\int_X |K_\phi(x,y) - K_\phi(x_0,y)||f(y)|\, d\mu(y) < C\phi(\delta(x,x_0))(\xi(\delta(x,x_0))|f|_\xi + |f(x)|),$$

for every $x, x_0 \in X$.

To prove that $|I_\phi f|_{\phi\xi} \leq C|f|_\xi$ and $|\tilde{I}_\phi f|_{\phi\xi} \leq C|f|_\xi$ it is enough to consider $x_1, x_2 \in X$, $x_1 \neq x_2$, set $r = \delta(x_1,x_2)$ and to show that there is a constant $C > 0$ such that

$$|\tilde{I}_\phi f(x_2) - \tilde{I}_\phi f(x_1)| = |I_\phi f(x_2) - I_\phi f(x_1)| \leq C|f|_\xi \xi(r)\phi(r), \tag{7.50}$$

where it must be understood that $f \in \Lambda^\xi$ for $\tilde{I}_\phi$ and $f \in \Lambda^\xi \cap L^1$ for $I_\phi$. By Lemma 6.4 we can write

$$\begin{aligned}\tilde{I}_\phi f(x_2) - \tilde{I}_\phi f(x_1) &= I_\phi f(x_2) - I_\phi f(x_1) \\ &= \int_X (f(y) - f(x_2))(K_\phi(x_2,y) - K_\phi(x_1,y))\, d\mu(y)\end{aligned}$$

and the right hand side in the above equalities is bounded by

$$\begin{aligned}&\int_{\delta(y,x_2)\leq 2Ar} \frac{\phi(\delta(y,x_2))}{\delta(y,x_2)} |f(y) - f(x_2)|\, d\mu(y) \\ &\quad + \int_{\delta(y,x_2)\leq 2Ar} \frac{\phi(\delta(y,x_1))}{\delta(y,x_1)} |f(y) - f(x_2)|\, d\mu(y) \\ &\quad + \int_{\delta(y,x_2)>2Ar} |K_\phi(x_2,y) - K_\phi(x_1,y)||f(y) - f(x_2)|\, d\mu(y) \\ &= J_1 + J_2 + J_3.\end{aligned} \tag{7.51}$$

Denote $B = B(x_2, 2Ar)$ and $B^c$ its complement. From the smoothness condition of $f$ and since $\xi\phi$ is of positive lower type it holds that

$$J_1 \le C|f|_\xi \int_B \frac{\phi(\delta(y,x_2))}{\delta(y,x_2)} \xi(\delta(y,x_2))\, d\mu(y) \le C|f|_{\xi,}\xi(r)\phi(r). \tag{7.52}$$

On the other, since $B \subset B(x_1, A(2A+1)r)$, $\xi$ is quasi-increasing and $\phi$ is of positive lower type, it holds that

$$J_2 \le C|f|_\xi \int_B \frac{\phi(\delta(y,x_1))}{\delta(y,x_1)} \xi(\delta(y,x_2))\, d\mu(y) \le C|f|_\xi \xi(r)\phi(r). \tag{7.53}$$

Finally, the smoothness conditions on the kernel and on $f$, the condition $\beta + s_\phi < \theta$ and Proposition 5.3 are used to get

$$J_3 \le C|f|_\xi r^\theta \int_{B^c} \frac{\phi(\delta(y,x_2))\xi(\delta(y,x_2))}{\delta(y,x_2)^{1+\theta}}\, d\mu(y) \le C|f|_\xi \xi(r)\phi(r). \tag{7.54}$$

□

**Remarks 7.1.** *From inequality* (7.49) *it also follows that $I_\phi$ is a linear continuous operator from $M^{(\beta_1,\gamma_1)}$ to $(M^{(\beta_2,\gamma_2)})'$, for every $\beta_1, \gamma_1, \beta_2$ and $\gamma_2 > 0$. More precisely, there is a finite constant $C$ such that $|\langle I_\phi f, g\rangle| \le C\|f\|_{M^{(\beta_1,\gamma_1)}}\|g\|_{M^{(\beta_2,\gamma_2)}}$, for every pair $f \in M^{(\beta_1,\gamma_1)}$ and $g \in M^{(\beta_2,\gamma_2)}$ and, moreover, it holds that*

$$\langle I_\phi f, g\rangle = \langle f, I_\phi g\rangle = \iint K_\phi(x,y) f(y) g(x)\, d\mu(y)\, d\mu(x). \tag{7.55}$$

Proof of Theorem 4.2: By (6.43) we have

$$\begin{aligned} \int_X |K_{1/\phi}(x,y)||f(y)-f(x)|\, d\mu(y) &\le C \int_{\delta(x,y)\le 1} \frac{|f(y)-f(x)|}{\phi(\delta(x,y))\delta(x,y)}\, d\mu(y) \\ &+ C \int_{\delta(x,y)>1} \frac{|f(y)-f(x)|}{\phi(\delta(x,y))\delta(x,y)}\, d\mu(y) = I_1 + I_2. \end{aligned} \tag{7.56}$$

Since $s_\phi < \alpha$, from (5.28) it follows that

$$I_1 \le C|f|_\xi \int_{\delta(x,y)\le 1} \frac{\xi(\delta(x,y))}{\phi(\delta(x,y))\delta(x,y)}\, d\mu(y) \le C|f|_\xi. \tag{7.57}$$

Furthermore, since $i_\phi > 0$ and $f \in L^\infty$, (5.29) leads to

$$I_2 \le 2C\|f\|_\infty \int_{\delta(x,y)>1} \frac{1}{\phi(\delta(x,y))\delta(x,y)}\, d\mu(y) \le C\|f\|_\infty, \tag{7.58}$$

and thus, from (7.57) and (7.58),

$$\int_X |K_{1/\phi}(x,y)||f(y)-f(x)|\,d\mu(y) \le C\|f\|_\xi \ \text{ for every } \ x\in X, \tag{7.59}$$

which implies that

$$\|D_\phi f\|_\infty \le C\|f\|_\xi \ \text{ for } \ s_\phi<\alpha. \tag{7.60}$$

To show that $\tilde{D}_\phi f(x)$ is absolutely convergent for $f\in\Lambda_\xi$ and $|\tilde{D}_\phi f|_{\xi/\phi} = |D_\phi f|_{\xi/\phi} \le C|f|_\xi$, it is enough to prove that

$$\begin{aligned}&\int_X |K_{1/\phi}(x,y)(f(y)-f(x)) - K_{1/\phi}(x_0,y)(f(y)-f(x_0))|\,d\mu(y)\\ &\quad\le C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}, \quad \text{for every } \ x,x_0\in X.\end{aligned} \tag{7.61}$$

Firstly, if $y\in B=B(x,2A\delta(x,x_0))$ then $\delta(y,x_0)\le A(2A+1)\delta(x,x_0)$, and proceeding as in (7.57), since $s_\phi<\alpha$, we have

$$\begin{aligned}&\int_B |K_{1/\phi}(x,y)(f(y)-f(x)) - K_{1/\phi}(x_0,y)(f(y)-f(x_0))|\,d\mu(y)\\ &\qquad\qquad\le C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}\,.\end{aligned} \tag{7.62}$$

Moreover, by reordering the integrand, it follows that

$$\begin{aligned}&\int_{B^c} K_{1/\phi}(x,y)(f(y)-f(x)) - K_{1/\phi}(x_0,y)(f(y)-f(x_0))|\,d\mu(y)\\ &\quad\le \int_{B^c} K_{1/\phi}(x,y)|f(x_0)-f(x)|\,d\mu(y)\\ &\qquad+\int_{B^c} |f(y)-f(x_0)||K_{1/\phi}(x,y)-K_{1/\phi}(x_0,y)|\,d\mu(y) = J_1+J_2.\end{aligned} \tag{7.63}$$

From (5.29) and $i_\phi>0$ it follows that

$$J_1 \le C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}\,.$$

On the other hand, Proposition 6.45, the facts that if $y\in B^c$ then $\delta(y,x_0)\le C\delta(x,y)$, $\xi$ is quasi-increasing and finally, inequality (5.29), since $\beta<\theta+i_\phi$, lead to the bound

$$J_2 \le C|f|_\xi \delta(x,x_0)^\theta \int_{B^c} \frac{\xi(\delta(x,y))}{\delta(x,y)^{1+\theta}\phi(\delta(x,y))}\,d\mu(y) \le C|f|_\xi \frac{\xi(\delta(x,x_0))}{\phi(\delta(x,x_0))}\,.$$

We then arrived to inequality (7.61). $\square$

**Remarks 7.2.** *Let $\xi_i$ be a function of lower type $\alpha_i$ and upper type $\beta_i$ for $i = 1, 2$ and let $s_\phi < \alpha_1$. Then*

$$\langle D_\phi f, g\rangle = \iint K_{1/\phi}(x,y)(f(y) - f(x))g(x)\, d\mu(x)\, d\mu(y), \tag{7.64}$$

*for any $f \in \Lambda^{\xi_1} \cap L^\infty$ and $g \in L^1$.*
*Furthermore, if $f \in \Lambda^{\xi_1} \cap L^\infty \cap L^1$, $g \in \Lambda^{\xi_2} \cap L^\infty \cap L^1$, and $s_\phi < \alpha_2$ then*

$$\langle D_\phi f, g\rangle = \langle D_\phi g, f\rangle. \tag{7.65}$$

Indeed, by (7.60), if $f \in \Lambda^{\xi_1} \cap L^\infty$, with $s_\phi < \alpha_1$, then $D_\phi f \in L^\infty$ and $\langle D_\phi f, g\rangle$ is well defined for $g \in L^1$ and the left side of (7.64) is absolutely convergent. The identity then follows from Fubini's theorem.
Moreover, we have that $|\langle D_\phi f, g\rangle| \le C\|f\|_\xi \|g\|_{L^1}$.
Furthermore, if $f \in \Lambda^{\xi_1} \cap L^\infty \cap L^1$, $g \in \Lambda^{\xi_2} \cap L^\infty \cap L^1$, and $s_\phi < \alpha_2$, the previous argument also leads to the identity

$$\langle D_\phi g, f\rangle = \iint K_{1/\phi}(x,y)(g(y) - g(x))f(x)\, d\mu(x)\, d\mu(y). \tag{7.66}$$

Therefore,

$$\begin{aligned}&\langle D_\phi f, g\rangle - \langle D_\phi g, f\rangle \\ &\qquad = \iint K_{1/\phi}(x,y)(f(y)g(x) - f(x)g(y))\, d\mu(x)\, d\mu(y) = 0\end{aligned} \tag{7.67}$$

since the integrand $h(x,y)$ satisfies the condition $h(x,y) = -h(y,x)$ and $\iint h(x,y)\, d\mu(x)\, d\mu(y)$ is absolutely convergent.

**Remarks 7.3.** *Since $M^{(\beta,\gamma)} \subset \Lambda^\beta \cap L^\infty \cap L^1$, for any $\beta$ and $\gamma > 0$, from Remark 7.2 follows that $D_\phi$ is a linear continuous operator from $M^{(\beta_1,\gamma_1)}$ in $(M^{(\beta_2,\gamma_2)})'$, for $s_\phi < \beta_1$, $\gamma_1, \gamma_2 > 0$ and $\beta_2 > 0$. Moreover, if also $s_\phi < \beta_2$ then $\langle D_\phi f, g\rangle = \langle D_\phi g, f\rangle$.*

## 8. Lemmas needed to prove Theorems 4.3, 4.4, 4.5 and 4.6

Seeking for the continuity of the operator $I_\phi$ on the generalized Besov and Triebel-Lizorkin spaces, a representation of the operator $I_\phi$ in terms of the Calderón-type reproduction formulas is needed.
Let consider an approximation to the identity $\{S_k\}_{k\in\mathbb{Z}}$ of order $\epsilon \le \theta$ and the family $\{D_k = S_k - S_{k-1}\}_{k\in\mathbb{Z}}$. Given $f \in M^{(\beta_1,\gamma_1)}$, with $0 < \beta_1 \le 1$ and $\gamma_1 > 0$, by Theorem 5.1 it follows that

$$f = \lim_{M\to\infty} \sum_{|j|\le M} D_j \hat{D}_j f, \tag{8.68}$$

where the series converges in $M^{(\beta',\gamma')}$ for every $\beta' < \beta_1$ and $\gamma' < \gamma_1$. Moreover, by Remark 7.1 $I_\phi$ is a linear continuous operator from $M^{(\beta',\gamma')}$ into $(M^{(\beta'',\gamma'')})'$, for every $\beta'' > 0$ and $\gamma'' > 0$. Then, for $g \in M^{(\beta_2,\gamma_2)}$, $0 < \beta_2 \leq 1$ and $\gamma_2 > 0$, it holds that

$$\langle I_\phi f, g\rangle = \lim_{M\to\infty} \sum_{|j|\leq M} \langle I_\phi D_j \hat{D}_j f, g\rangle. \tag{8.69}$$

Choosing $\beta'' < \beta_2$ and $\gamma'' < \gamma_2$ and now applying Theorem 5.2 it follows that

$$\begin{aligned}\langle I_\phi f, g\rangle &= \lim_{M\to\infty}\lim_{N\to\infty} \sum_{|k|\leq N}\sum_{|j|\leq M} \langle \tilde{D}_k D_k I_\phi D_j \hat{D}_j f, g\rangle \\ &= \lim_{M\to\infty}\lim_{N\to\infty} \sum_{|k|\leq N}\sum_{|j|\leq M} \langle D_k I_\phi D_j (\hat{D}_j f), \tilde{D}_k^* g\rangle.\end{aligned}$$

It is easy to check that the kernel associated to the operator $I_{\phi,kj} = D_k I_\phi D_j$ is defined by

$$\begin{aligned}K_{\phi,kj}(x,y) &= \langle D_k(x,.), I_\phi D_j(.,y)\rangle \\ &= \iint D_k(x,z) K_\phi(z,u) D_j(u,y)\, d\mu(u)\, d\mu(z),\end{aligned} \tag{8.70}$$

and it satisfies

$$K_{\phi,kj}(x,y) = K_{\phi,jk}(y,x), \quad \text{for every } \ x,y\in X \ \text{ and } \ k,j\in\mathbb{Z}. \tag{8.71}$$

In an analogous way, Remark 7.3 yields a representation of $D_\phi$ in terms of the Calderón-type reproduction formulas. Indeed, we get that

$$\begin{aligned}\langle D_\phi f, g\rangle &= \lim_{M\to\infty}\lim_{N\to\infty} \sum_{|k|\leq N}\sum_{|j|\leq M} \langle \tilde{D}_k D_k D_\phi D_j \hat{D}_j f, g\rangle \\ &= \lim_{M\to\infty}\lim_{N\to\infty} \sum_{|k|\leq N}\sum_{|j|\leq M} \langle D_k D_\phi D_j (\hat{D}_j f), \tilde{D}_k^* g\rangle,\end{aligned}$$

for every pair of functions $f \in M^{(\beta_1,\gamma_1)}$ and $g \in M^{(\beta_2,\gamma_2)}$, with $s_\phi < \beta_1$ and $\beta_2, \gamma_1, \gamma_2 > 0$. The kernel associated to the operator $D_{\phi,kj} = D_k D_\phi D_j$ is given by

$$\begin{aligned}K_{1/\phi,kj}(x,y) &= \langle D_\phi D_j(.,y), D_k(x,.)\rangle \\ &= \iint D_k(x,z) K_{1/\phi}(z,u)(D_j(u,y) - D_j(z,y))\, d\mu(u)\, d\mu(z),\end{aligned} \tag{8.72}$$

which is well defined by Remark 7.2. Moreover, since the kernels $D_k(x,z)$ and $D_j(u,y)$ are symmetric, from (7.65) it follows that

$$K_{1/\phi,kj}(x,y) = K_{1/\phi,jk}(y,x). \tag{8.73}$$

A sharp bound for $K_{\phi,kj}(x,y)$ will be obtained in the following lemma.

**Lemma 8.1.** *If $\phi$ is of positive lower type and of upper type $s_\phi < \epsilon \le \theta$ then the kernel $K_{\phi,kj}$ satisfies the inequality*

$$|K_{\phi,kj}(x,y)| \le C\phi((2A)^{-(k\vee j)})\frac{(2A)^{-(k\vee j)(\epsilon - s_\phi)}}{((2A)^{-(k\wedge j)} + \delta(x,y))^{1+(\epsilon - s_\phi)}} \tag{8.74}$$

*where $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.*

Proof: It is enough to consider the case $k \ge j$ since the other immediately follows from this by (8.71). From (8.70) and as $D_k$ has null mean in each variable, the kernel can be rewritten in the form

$$K_{\phi,kj}(x,y) = \iint D_k(x,z)[K_\phi(z,u) - K_\phi(x,u)]D_j(u,y)\,d\mu(u)\,d\mu(z)\,. \tag{8.75}$$

Let first consider the case $\delta(x,y) \le 4A^2C(2A)^{-j}$. Defining $\eta(t) \in \Lambda^\epsilon$, with $\eta(t) = 1$ if $|t| \le A$ and $\eta(t) = 0$ if $|t| > 4A^2$, by Lemma 6.4 it holds that

$$\begin{aligned} K_{\phi,kj}(x,y) = &\iint D_k(x,z)\left(K_\phi(z,u) - K_\phi(x,u)\right) \\ &\times \left(D_j(u,y) - D_j(x,y)\right)\eta(\frac{\delta(x,u)}{(2A)^{-k}})\,d\mu(u)\,d\mu(z) \\ &+ \iint D_k(x,z)\left(K_\phi(z,u) - K_\phi(x,u)\right)\left(D_j(u,y) - D_j(x,y)\right) \\ &\times \left(1 - \eta(\frac{\delta(x,u)}{(2A)^{-k}})\right)\,d\mu(u)\,d\mu(z) = D + B. \end{aligned}$$

The first term $D$ satisfies

$$\begin{aligned} |D| &\le C\int |D_k(x,z)| \int_{\delta(x,u)\le(2A)^{-k+2}} |K_\phi(z,u) - K_\phi(x,u)| \\ &\quad\times |D_j(u,y) - D_j(x,y)|\,d\mu(u)\,d\mu(z) \\ &\le C(2A)^{j(1+\epsilon)}\int |D_k(x,z)| \int_{\delta(x,u)\le C(2A)^{-k}} (K_\phi(z,u) \\ &\quad+ K_\phi(x,u))\delta(x,u)^\epsilon\,d\mu(u)\,d\mu(z), \end{aligned}$$

but if $\delta(x,z) \le C(2A)^{-k}$ and $\delta(x,u) \le C(2A)^{-k}$, then $\delta(z,u) \le C(2A)^{-k+1}$. Moreover, since $\phi$ is of positive lower type, inequality (5.28) and the size condition (6.34) can be used to get

$$\begin{aligned} |D| &\le C(2A)^j(2A)^{-(k-j)\epsilon} \\ &\quad\times\left(\int_{\delta(x,u)\le C(2A)^{-k}} K_\phi(x,u)\,d\mu(u) + \int_{\delta(z,u)\le C(2A)^{-k}} K_\phi(z,u)\,d\mu(u)\right) \\ &\le C(2A)^j(2A)^{-(k-j)\epsilon}\phi((2A)^{-k}). \end{aligned} \tag{8.76}$$

On the other hand,

$$
\begin{aligned}
&|B| \le \\
&\iint_{\delta(x,u)\ge A(2A)^{-k}} |D_k(x,z)||K_\phi(z,u)-K_\phi(x,u)||D_j(u,y)-D_j(x,y)|\,d\mu(u)\,d\mu(z) \\
&= \left(\iint_{C(2A)^{-k+1}\le\delta(x,u)\le C(2A)^{-j+1}} + \iint_{\delta(x,u)\ge C(2A)^{-j+1}}\right) \\
&\qquad |D_k(x,z)||K_\phi(z,u)-K_\phi(x,u)||D_j(u,y)-D_j(x,y)|\,d\mu(u)\,d\mu(z) \\
&= B_1 + B_2.
\end{aligned}
$$

As $\delta(x,u) \ge 2A\delta(x,z)$ for $u$ in the domain of $B_1$ and $B_2$, Lemma 6.3 can be applied. Moreover, denoting $C_i = \{C(2A)^{-k+i} \le \delta(x,u) \le C(2A)^{-k+i+1}\}$, $i = 1, 2, \ldots$, since $s_\phi > 0$ and $\phi(t)/t$ is quasi-decreasing and $\epsilon \le \theta$ then

$$
\begin{aligned}
B_1 &\le C(2A)^{j(1+\epsilon)} \int |D_k(x,z)| \sum_{i=1}^{k-j} \int_{C_i} |K_\phi(z,u)-K_\phi(x,u)|\delta(x,u)^\epsilon\,d\mu(u)\,d\mu(z) \\
&\le C(2A)^{j(1+\epsilon)} \int |D_k(x,z)| \sum_{i=1}^{k-j} \int_{C_i} \delta(x,z)^\epsilon \frac{\phi(\delta(x,u))}{\delta(x,u)}\,d\mu(u)\,d\mu(z) \\
&\le C(2A)^{j(1+\epsilon)}(2A)^{-k\epsilon} \sum_{i=1}^{k-j} \phi((2A)^{-k+i}) \\
&\le C(2A)^j \phi((2A)^{-k})(2A)^{-(k-j)\epsilon} \sum_{i=1}^{k-j} (2A)^{is_\phi} \\
&\le C(2A)^j \phi((2A)^{-k})(2A)^{-(k-j)(\epsilon-s_\phi)}.
\end{aligned}
$$

On the other side, since $s_\phi < \epsilon \le \theta$, from (5.29) and (6.36) it follows that

$$
\begin{aligned}
B_2 &\le C(2A)^j \int |D_k(x,z)| \\
&\quad \times \int_{\delta(x,u)\ge C(2A)^{-j+1}} |K_\phi(z,u)-K_\phi(x,u)|\,d\mu(u)\,d\mu(z) \\
&\le C(2A)^j \int |D_k(x,z)| \\
&\quad \times \int_{\delta(x,u)\ge C(2A)^{-j+1}} \frac{\delta(x,z)^\epsilon}{\delta(x,u)^\epsilon} \frac{\phi(\delta(x,u))}{\delta(x,u)}\,d\mu(u)\,d\mu(z) \\
&\le C(2A)^j (2A)^{-(k-j)\epsilon} \phi((2A)^{-j}) \\
&\le C(2A)^j \phi((2A)^{-k})(2A)^{-(k-j)(\epsilon-s_\phi)}.
\end{aligned}
\tag{8.77}
$$

Nevertheless, since $t^{\epsilon} < t^{\epsilon - s_\phi}$ for $t < 1$ and $(2A)^{j(1+\epsilon-s_\phi)} \leq C/((2A)^{-j} + \delta(x,y))^{1+\epsilon-s_\phi}$ for $\delta(x,y) \leq 4A^2(2A)^{-j}$, inequality (8.74) follows from (8.76), (8.77) and an estimate for $B_1$.

Let now consider the case $\delta(x,y) \geq 4A^2C(2A)^{-j}$. If $D_j(u,y) \neq 0$ then $\delta(u,y) < C(2A)^{-j}$ and thus, $\delta(x,u) \geq 2AC(2A)^{-j} > 2A\delta(x,z)$, moreover, the equivalence $\delta(x,u) \simeq (2A)^{-j} + \delta(x,y)$ holds. Therefore, using Lemma 6.3 and (5.26), from (8.75) it follows that

$$
\begin{aligned}
&|K_{\phi,kj}(x,y)| \\
&\quad \leq C \int |D_k(x,z)| \int_{\delta(u,y)<C(2A)^{-j}} \frac{\delta(x,z)^{\epsilon}}{\delta(x,u)^{1+\epsilon}} \phi(\delta(x,u)) |D_j(u,y)|\, d\mu(u)\, d\mu(z) \\
&\quad \leq C \int |D_k(x,z)| \int_{\delta(u,y)<C(2A)^{-j}} \frac{\delta(x,z)^{\epsilon-s_\phi}}{\delta(x,u)^{1+\epsilon-s_\phi}} \phi(\delta(x,z)) |D_j(u,y)|\, d\mu(u)\, d\mu(z) \\
&\quad \leq C\phi\left((2A)^{-k}\right) \frac{(2A)^{-k(\epsilon-s_\phi)}}{((2A)^{-j}+\delta(x,y))^{1+\epsilon-s_\phi}}.
\end{aligned}
$$

□

The next lemma follows easily from Lemma 8.1.

**Lemma 8.2.** *If $\phi$ is of positive lower type and of upper type $s_\phi < \epsilon$ then*

$$
\int |K_{\phi,kj}(x,y)|\, d\mu(x) + \int |K_{\phi,kj}(x,y)|\, d\mu(y) \leq C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}.
$$

An estimate of $I_{\phi,kj}$ in terms of the Hardy-Littlewood maximal operator follows from Lemma 8.1.

**Lemma 8.3.** *If $\phi$ is of positive lower type and of upper type $s_\phi < \epsilon$ then*

$$
|I_{\phi,kj}h(x)| \leq C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)} M|h|(x), \tag{8.78}
$$

*where $M$ denotes the Hardy-Littlewood maximal operator.*

Proof: As in the proof of Lemma 8.1, it is enough to consider the case $k \geq j$. From that lemma it follows that

$$
\begin{aligned}
&\int |K_{\phi,kj}(x,y)||h(y)|\, d\mu(y) \\
&\leq C\phi((2A)^{-k}) \Bigg( ((2A)^{-k(\epsilon-s_\phi)})(2A)^{j(\epsilon-s_\phi+1)} \int_{\delta(x,y)\leq 4A^2C(2A)^{-j}} |h(y)|\, d\mu(y) \\
&\quad + \int_{\delta(x,y)>4A^2C(2A)^{-j}} \frac{(2A)^{-k(\epsilon-s_\phi)}}{\delta(x,y)^{(\epsilon-s_\phi)+1}} |h(y)|\, d\mu(y) \Bigg) = I_1 + I_2.
\end{aligned}
$$

Clearly,

$$I_1 \leq C\phi((2A)^{-k})((2A)^{-(k-j)(\epsilon-s_\phi)})M(|h|)(x).$$

Finally, defining the sets $Q_i = \{y : C(2A)^{i-j} \leq \delta(x,y) \leq C(2A)^{i+1-j}\}$, $i = 2, 3, \dots$, since $s_\phi < \epsilon$ we have

$$\begin{aligned} I_2 &\leq C\phi((2A)^{-k}) \sum_{i=2}^{\infty} \int_{Q_i} \frac{(2A)^{-k(\epsilon-s_\phi)}}{\delta(x,y)^{1+(\epsilon-s_\phi)}} |h(y)|\, d\mu(y) \\ &\leq C\phi((2A)^{-k})(2A)^{-(k-j)(\epsilon-s_\phi)} \sum_{i=2}^{\infty} (2A)^{-i(\epsilon-s_\phi)}(2A)^{j-i} \\ &\quad \times \int_{\delta(x,y)\leq C(2A)^{i+1-j}} |h(y)|\, d\mu(y) \\ &\leq C\phi((2A)^{-k})(2A)^{-(k-j)(\epsilon-s_\phi)} M|h|(x). \end{aligned}$$

□

Corresponding results are obtained for the kernel $K_{1/\phi,kj}$ and the operator $D_{\phi,kj}$ in the following lemmas.

**Lemma 8.4.** *Let $\phi$ be of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$. Then, there is a constant $C > 0$ such that*

$$|K_{1/\phi,kj}(x,y)| \leq C \frac{1}{\phi((2A)^{-(k\vee j)})} \frac{(2A)^{-(k\vee j)\epsilon}}{((2A)^{-(k\wedge j)} + \delta(x,y))^{1+\epsilon}},$$

*where $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.*

Proof: It is enough to consider the case $k \geq j$ since the other one immediately follows from this by (8.73). Let first consider the case $\delta(x,y) \leq 4A^2(2A)^{-j}$. Since $D_k$ has null mean in the $z$ variable, the kernel defined in (8.72) can be rewritten as

$$\begin{aligned} K_{1/\phi,kj}(x,y) = \iint D_k(x,z)[&K_{1/\phi}(z,u)(D_j(u,y) - D_j(z,y)) \\ &- K_{1/\phi}(x,u)(D_j(u,y) - D_j(x,y))]\, d\mu(u)\, d\mu(z). \end{aligned}$$

Fix $\eta(t) \in \Lambda_0^\epsilon(\mathbb{R})$, such that $\eta(t) = 1$ for $|t| \leq A$ and $\eta = 0$, for $|t| \geq 2A$. Then,

$$\begin{aligned} &K_{1/\phi,kj}(x,y) \\ &= \iint D_k(x,z) \\ &\quad \times \Big(K_{1/\phi}(z,u)(D_j(u,y) - D_j(z,y)) - K_{1/\phi}(x,u)(D_j(u,y) - D_j(x,y))\Big) \end{aligned}$$

$$
\begin{aligned}
&\times \eta(\frac{\delta(x,u)}{(2A)^{-k}})\, d\mu(u)\, d\mu(z)\\
&+ \iint D_k(x,z)\\
&\times \Big(K_{1/\phi}(z,u)(D_j(u,y)-D_j(z,y)) - K_{1/\phi}(x,u)(D_j(u,y)-D_j(x,y))\Big)\\
&\times (1-\eta(\frac{\delta(x,u)}{(2A)^{-k}}))\, d\mu(u)\, d\mu(z) = D + B.
\end{aligned}
$$

First notice that if $\delta(x,z) \le C(2A)^{-k}$ and $\delta(x,u) \le C(2A)^{-k}$, then $\delta(z,u) \le CA(2A)^{-k}$. Therefore, from (2.10), applied to $D_j$, and (5.28) it follows that

(8.79)

$$
\begin{aligned}
|D| &\le \int |D_k(x,z)| \int_{\delta(z,u)\le CA(2A)^{-k}} |K_{1/\phi}(z,u)|(2A)^{j(1+\epsilon)}\delta(z,u)^{\epsilon}\, d\mu(u)\, d\mu(z)\\
&\quad + \int |D_k(x,z)| \int_{\delta(x,u)\le C(2A)^{-k}} |K_{1/\phi}(x,u)|(2A)^{j(1+\epsilon)}\delta(x,u)^{\epsilon}\, d\mu(u)\, d\mu(z)\\
&\le C(2A)^{j(1+\epsilon)}\frac{(2A)^{-k\epsilon}}{\phi((2A)^{-k})}.
\end{aligned}
$$

On the other hand, it holds that

$$
\begin{aligned}
|B| \le &\iint_{\delta(x,u)\ge (2A)^{-k+1}} |D_k(x,z)|\\
&\times \Big(|K_{1/\phi}(z,u) - K_{1/\phi}(x,u)||D_j(u,y)-D_j(x,y)|\\
&+ K_{1/\phi}(z,u)|D_j(x,y)-D_j(z,y)|\Big)\, d\mu(u)\, d\mu(z) = B_1 + B_2.
\end{aligned}
$$

But, if $D_k(x,z)\ne 0$ and $\delta(x,u)\ge (2A)^{-k+1}$ then $\delta(z,u)\ge C(2A)^{-k}$. Moreover, since $i_\phi > 0$, from (6.44), (2.10) and (5.29), we deduce that

$$
\begin{aligned}
B_2 &\le C(2A)^{j(1+\epsilon)} \int |D_k(x,z)|\\
&\quad\times \int_{\delta(z,u)\ge C(2A)^{-k}} \frac{\delta(z,x)^{\epsilon}}{\phi(\delta(z,u))\delta(z,u)}\, d\mu(u)\, d\mu(z)\\
&\le C(2A)^{j}(2A)^{-(k-j)\epsilon} \int |D_k(x,z)| \qquad (8.80)\\
&\quad\times \int_{\delta(z,u)\ge C(2A)^{-k}} \frac{1}{\phi(\delta(z,u))\delta(z,u)}\, d\mu(u)\, d\mu(z)\\
&\le C\frac{(2A)^{j(1+\epsilon)}(2A)^{-k\epsilon}}{\phi((2A)^{-k})}.
\end{aligned}
$$

We now split $B_1$ in the form

$$
\begin{aligned}
B_1 \leq & \left( \iint_{(2A)^{-k+1} \leq \delta(x,u) \leq (2A)^{-j+1}} + \iint_{\delta(x,u) \geq (2A)^{-j+1}} \right) \\
& \times |D_k(x,z)||K_{1/\phi}(z,u) - K_{1/\phi}(x,u)||D_j(u,y) - D_j(x,y)|\, d\mu(u)\, d\mu(z) \\
= & B_{1,1} + B_{1,2}.
\end{aligned}
$$

Since $\delta(x,z) \leq C(2A)^{-k}$ and $\delta(x,u) \geq 2A\delta(x,z)$, smoothness conditions (6.45) and (2.10), (5.29) and, also, (2.3) lead to the bound

(8.81)

$$
\begin{aligned}
B_{1,1} & \leq C(2A)^{j(1+\epsilon)} \int |D_k(x,z)| \\
& \quad \times \int_{(2A)^{-k+1} \leq \delta(x,u) \leq (2A)^{-j+1}} \delta(x,u)^{\epsilon} \frac{\delta(x,z)^{\epsilon}}{\delta(x,u)^{1+\epsilon}} \frac{1}{\phi(\delta(x,u))}\, d\mu(u)\, d\mu(z) \\
& \leq C(2A)^{j}(2A)^{-(k-j)\epsilon} \int_{(2A)^{-k+1} \leq \delta(x,u)} \frac{1}{\phi(\delta(x,u))\delta(x,u)}\, d\mu(u) \\
& \leq C \frac{(2A)^{j}(2A)^{-(k-j)\epsilon}}{\phi((2A)^{-k})}.
\end{aligned}
$$

On the other hand, using (6.45), (2.9), (5.29) and the fact that $\phi$ is quasi-increasing, we obtain

(8.82)

$$
\begin{aligned}
B_{1,2} & \leq C(2A)^{j} \int |D_k(x,z)| \int_{\delta(x,u) \geq (2A)^{-j+1}} \frac{\delta(x,z)^{\epsilon}}{\delta(x,u)^{1+\epsilon}} \frac{1}{\phi(\delta(x,u))}\, d\mu(u)\, d\mu(z) \\
& \leq C(2A)^{j}(2A)^{-k\epsilon} \frac{1}{(2A)^{-j\epsilon}\phi((2A)^{-j})} \leq C(2A)^{j}(2A)^{-(k-j)\epsilon} \frac{1}{\phi((2A)^{-k})}.
\end{aligned}
$$

From inequalities (8.79), (8.80), (8.81) and (8.82), we conclude that if $\delta(x,y) \leq 4A^2(2A)^{-j}$ then

$$
\begin{aligned}
|K_{1/\phi,kj}(x,y)| & \leq C(2A)^{j(1+\epsilon)}(2A)^{-k\epsilon} \frac{1}{\phi((2A)^{-k})} \\
& \leq C \frac{(2A)^{-k\epsilon}}{((2A)^{-j} + \delta(x,y))^{1+\epsilon}} \frac{1}{\phi((2A)^{-k})}.
\end{aligned}
$$

To finish the proof, we consider the case $\delta(x,y) \geq C4A^2(2A)^{-j}$.
Notice that if $\delta(x,z) \leq C(2A)^{-k}$ then $\delta(z,y) \geq CA(2A)^{-j}$ and therefore $D_j(z,y) = 0$. Moreover, the condition $\int D_k(x,z)\, d\mu(z) = 0$ enables us to rewrite the kernel in (8.72) in the form

$$
K_{1/\phi,kj}(x,y) = \int D_k(x,z) \int (K_{1/\phi}(z,u) - K_{1/\phi}(x,u)) D_j(u,y)\, d\mu(u)\, d\mu(z).
$$

But, also, since $\delta(u,y) \le C(2A)^{-j}$ then $\delta(x,u) \ge C(2A)^{-j} \ge C(2A)^{-k} \ge 2A\delta(x,z)$ and $\delta(x,u) \ge C(\delta(x,y)+(2A)^{-j})$. Therefore, from (6.45) and the fact that $\phi(t)$ is quasi-increasing, we deduce that

$$\begin{aligned}
&|K_{1/\phi,kj}(x,y)| \\
&\quad\le \int |D_k(x,z)| \int_{\delta(u,y)<(2A)^{-j}} \frac{\delta(x,z)^{\epsilon}|D_j(u,y)|}{\delta(x,u)^{1+\epsilon}\phi(\delta(x,u))}\, d\mu(u)\, d\mu(z) \\
&\quad\le C\frac{(2A)^{-k\epsilon}}{((2A)^{-j}+\delta(x,y))^{1+\epsilon}} \frac{1}{\phi((2A)^{-k})} \int |D_k(x,z)| \int |D_j(u,y)|\, d\mu(u)\, d\mu(z) \\
&\quad\le C\frac{(2A)^{-k\epsilon}}{((2A)^{-j}+\delta(x,y))^{1+\epsilon}} \frac{1}{\phi((2A)^{-k})}\,.
\end{aligned}$$

□

The proofs of the following two lemmas are similar to those given for the integral of order $\phi$ and so they will be omitted.

**Lemma 8.5.** *If $\phi$ is of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$, then there is a constant $C > 0$ such that*

$$\int |K_{1/\phi,kj}(x,y)|\, d\mu(x) + \int |K_{1/\phi,kj}(x,y)|\, d\mu(y) \le C\frac{(2A)^{-|k-j|\epsilon}}{\phi((2A)^{-(k\vee j)})}\,. \tag{8.83}$$

**Lemma 8.6.** *If $\phi$ is of lower type $i_\phi > 0$ and upper type $s_\phi < \epsilon$, then there is a constant $C > 0$ such that*

$$|D_{\phi,kj}h(x)| \le C\frac{(2A)^{-|k-j|\epsilon}}{\phi((2A)^{-(k\vee j)})} M|h|(x), \tag{8.84}$$

*where $M$ denotes the Hardy-Littlewood maximal operator.*

## 9. Proof of Theorems 4.3, 4.4, 4.5, and 4.6

If $\max(s_1,s_2) < \epsilon$ then the space $M^{(\epsilon,\epsilon)}$ is dense in $\dot{F}_p^{\psi,q}$ and $\dot{B}_p^{\psi,q}$ and hence, in all the theorems, it is enough to prove the boundedness of the operators on such molecules.

Proof of Theorem 4.3: For $f \in M^{(\epsilon,\epsilon)}$, by using (8.69) we obtain

$$\|I_\phi f\|_{\dot{F}_p^{\phi\psi,q}} = \left\| \left( \sum_{k\in\mathbb{Z}} \left( \frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})} |D_k I_\phi f| \right)^q \right)^{1/q} \right\|_p$$

$$\leq \left\| \left( \sum_{k\in\mathbb{Z}} \left( \frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})} \sum_{j\in\mathbb{Z}} |D_k I_\phi D_j(\hat{D}_j f)| \right)^q \right)^{1/q} \right\|_p$$

$$\leq \left\| \left( \sum_{k\in\mathbb{Z}} \left( \frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})} \sum_{j\leq k} |I_{\phi,kj}(\hat{D}_j f)| \right)^q \right)^{1/q} \right.$$

$$\left. + \left( \sum_{k\in\mathbb{Z}} \left( \frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})} \sum_{j>k} |I_{\phi,k,j}(\hat{D}_j f)| \right)^q \right)^{1/q} \right\|_p = \|S_1 + S_2\|_p.$$

First notice that as $\psi_2$ is quasi-increasing and $\psi_1$ is of upper-type $s_1$, for $k \geq j$ it holds that

(9.85)
$$\frac{1}{\psi((2A)^{-k})} = \frac{\psi_2((2A)^{-k})}{\psi_1((2A)^{-k})} \leq C(2A)^{(k-j)s_1} \frac{\psi_2((2A)^{-j})}{\psi_1((2A)^{-j})} = C\frac{(2A)^{(k-j)s_1}}{\psi((2A)^{-j})}.$$

Also, since $\psi_1$ is quasi-increasing and $\psi_2$ is of upper-type $s_2$ then, for $k < j$,

(9.86)
$$\frac{1}{\psi((2A)^{-k})} = \frac{\psi_2((2A)^{-k})}{\psi_1((2A)^{-k})} \leq C(2A)^{(j-k)s_2} \frac{\psi_2((2A)^{-j})}{\psi_1((2A)^{-j})} = C\frac{(2A)^{(j-k)s_2}}{\psi((2A)^{-j})}.$$

Therefore, applying (8.78) and then (9.85) it follows that

$$\begin{aligned} S_1(x) &\leq \left( \sum_{k\in\mathbb{Z}} \left( \sum_{j\leq k} \frac{(2A)^{-(k-j)(\epsilon - s_\phi - s_1)}}{\psi((2A)^{-j})} M|\hat{D}_j f|(x) \right)^q \right)^{1/q} \\ &= \left( \sum_{k\in\mathbb{Z}} \left( \sum_{j\geq 0} \frac{(2A)^{-j(\epsilon - s_\phi - s_1)}}{\psi((2A)^{-(k-j)})} M|\hat{D}_{k-j} f|(x) \right)^q \right)^{1/q}. \end{aligned} \tag{9.87}$$

On the other hand, using (8.78), (9.86) and inequality

$$\phi((2A)^{-j}) \leq C(2A)^{-(j-k)i_\phi}\phi((2A)^{-k}), \quad \text{for} \quad j > k, \tag{9.88}$$

it follows that

$$S_2(x) \leq \left( \sum_{k\in\mathbb{Z}} \left( \sum_{j>k} \frac{(2A)^{-(j-k)(\epsilon - s_\phi + i_\phi - s_2)}}{\psi((2A)^{-j})} M|\hat{D}_j f|(x) \right)^q \right)^{1/q}. \tag{9.89}$$

From Minkowski's inequality and the hypothesis $s_\phi + s_1 < \epsilon$ for (9.87), and $s_\phi - i_\phi + s_2 < \epsilon$ for (9.89), it follows that

$$S_1(x) + S_2(x) \leq C \left( \sum_{k\in\mathbb{Z}} \left( \frac{M|\hat{D}_k f|(x)}{\psi((2A)^{-k})} \right)^q \right)^{1/q} \tag{9.90}$$

for every $x \in X$. Since $1 < p, q < \infty$, we are able to apply the Fefferman-Stein vector valued maximal inequality to get that

$$\|S_1 + S_2\|_p \leq C \| \left( \sum_{k\in\mathbb{Z}} \left( \frac{|\hat{D}_k f|}{\psi((2A)^{-k})} \right)^q \right)^{1/q} \|_p \leq C\|f\|_{\dot{F}_p^{\psi,q}}.$$

□

Proof of Theorem 4.4: For $f \in M^{(\epsilon,\epsilon)}$, by (8.69), it holds that

$$\begin{aligned}
\|I_\phi f\|_{\dot{B}_p^{\phi\psi,q}} &\leq \left( \sum_{k\in\mathbb{Z}} \left( \frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})} \sum_{j\in\mathbb{Z}} \|D_k I_\phi D_j(\hat{D}_j f)\|_p \right)^q \right)^{1/q} \\
&\leq \left( \sum_{k\in\mathbb{Z}} \left( \frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})} \sum_{j\leq k} \|I_{\phi,k,j}\|_{p,p} \|(\hat{D}_j f)\|_p \right)^q \right)^{1/q} \\
&\quad + \left( \sum_{k\in\mathbb{Z}} \left( \frac{1}{\phi((2A)^{-k})\psi((2A)^{-k})} \sum_{j>k} \|I_{\phi,k,j}\|_{p,p} \|(\hat{D}_j f)\|_p \right)^q \right)^{1/q} \\
&= S_1 + S_2,
\end{aligned}$$

where $\|T\|_{p,p}$ denotes $\|T\|_{L^p\to L^p}$.
Nevertheless, from Lemma 8.2, it follows that

$$\|I_{\phi,k,j}\|_{p,p} \leq C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon - s_\phi)}. \tag{9.91}$$

In fact, for $1 < p < \infty$, it holds that

$$\begin{aligned}
\|I_{\phi,kj} h\|_p &\leq \left( \int \left( \int |K_{\phi,kj}(x,y)||h(y)|\, d\mu(y) \right)^p d\mu(x) \right)^{1/p} \\
&\leq \left( \int \left( \int |K_{\phi,kj}(x,y)|\, d\mu(y) \right)^{p/p'} \right. \\
&\quad \times \left. \left( \int |K_{\phi,kj}(x,y)||h(y)|^p\, d\mu(y) \right) d\mu(x) \right)^{1/p};
\end{aligned} \tag{9.92}$$

and, for $p = 1$,

$$\|I_{\phi,kj}h\|_1 \le \int\int |K_{\phi,kj}(x,y)||h(y)|\,d\mu(y)\,d\mu(x). \tag{9.93}$$

Then applying Lemma 8.2 in (9.92) and (9.93), it follows that

$$\begin{aligned}\|I_{\phi,kj}h\|_p &\le C\left(\phi((2A)^{-(k\vee j)}(2A)^{-|k-j|(\epsilon-s_\phi)})\right)^{1/p'}\\ &\quad\times\left(\iint |K_{\phi,kj}(x,y)||h(y)|^p\,d\mu(y)\,d\mu(x)\right)^{1/p}\\ &\le C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}\|h\|_p,\end{aligned}$$

for $1 < p < \infty$, and

$$\begin{aligned}\|I_{\phi,kj}h\|_1 &\le C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}\int |h(y)|\,d\mu(y)\\ &= C\phi((2A)^{-(k\vee j)})(2A)^{-|k-j|(\epsilon-s_\phi)}\|h\|_1.\end{aligned}$$

for $p = 1$. Thus inequality (9.91) follows. Substituting it in $S_1$ and using (9.85), it follows that

$$S_1 \le C\left(\sum_{k\in\mathbb{Z}}\left(\sum_{j\le k}(2A)^{-(k-j)(\epsilon-s_\phi-s_1)}\frac{\|(\hat{D}_jf)\|_p}{\psi((2A)^{-j})}\right)^q\right)^{1/q}. \tag{9.94}$$

On the other hand, using (9.91), (9.85) and (9.88) it follows that

$$S_2 \le C\left(\sum_{k\in\mathbb{Z}}\left(\sum_{j>k}(2A)^{-(j-k)(\epsilon-s_\phi+i_\phi-s_2)}\frac{\|(\hat{D}_jf)\|_p}{\psi((2A)^{-j})}\right)^q\right)^{1/q}. \tag{9.95}$$

For $1 \le q < \infty$, Minkowski's inequality and conditions $s_\phi + s_1 < \epsilon$ for (9.94) and $s_\phi - i_\phi + s_2 < \epsilon$ for (9.95) lead to the bound

$$S_1 + S_2 \le C\|f\|_{\dot{B}_p^{\psi,q}}.$$

□

Proof of Theorem 4.5: For $f \in M^{(\epsilon,\epsilon)}$, proceeding as in the above proofs and applying (8.72) it follows that

$$\begin{aligned}\|D_\phi f\|_{\dot{F}_p^{\psi/\phi,q}} &\le \left\|\left(\sum_{k\in\mathbb{Z}}\left(\frac{\phi((2A)^{-k})}{\psi((2A)^{-k})}\sum_{j\le k}|D_{\phi,kj}(\hat{D}_jf)|\right)^q\right)^{1/q}\right.\\ &\quad\left.+\left(\sum_{k\in\mathbb{Z}}\left(\frac{\phi((2A)^{-k})}{\psi((2A)^{-k})}\sum_{j>k}|D_{\phi,kj}(\hat{D}_jf)|\right)^q\right)^{1/q}\right\|_p\\ &= \|S_1(x) + S_2(x)\|_p.\end{aligned}$$

Using (8.84) and (9.85) it follows that

$$S_1(x) \leq C \left( \sum_{k\in\mathbb{Z}} \left( \sum_{j\leq k} (2A)^{-(k-j)(\epsilon-s_1)} \frac{M|\hat{D}_j f|(x)}{\psi((2A)^{-j})} \right)^q \right)^{1/q}. \tag{9.96}$$

On the other side, again using (8.84) and inequalities (9.86) and

$$\phi((2A)^{-k}) \leq C(2A)^{(j-k)s_\phi}\phi((2A)^{-j}), \quad \text{for} \quad k < j,$$

we conclude

$$S_2(x) \leq C \left( \sum_{k\in\mathbb{Z}} \left( \sum_{j> k} (2A)^{-(j-k)(\epsilon-s_\phi-s_2)} \frac{M|\hat{D}_j f|(x)}{\psi((2A)^{-j})} \right)^q \right)^{1/q}. \tag{9.97}$$

From Minkowski's inequality and the hypothesis $s_1 < \epsilon$ for (9.96) and $s_\phi + s_2 < \epsilon$ for (9.97) it follows that

$$S_1(x) + S_2(x) \leq C \left( \sum_{k\in\mathbb{Z}} \left( \frac{M|\hat{D}_k f|(x)}{\psi((2A)^{-k})} \right)^q \right)^{1/q}.$$

From the Fefferman-Stein vector valued maximal inequality, for $1 < p, q < \infty$, it follows that

$$\|S_1 + S_2\|_p \leq C \| \left( \sum_{k\in\mathbb{Z}} \left( \frac{|\hat{D}_k f|(x)}{\psi((2A)^{-k})} \right)^q \right)^{1/q} \|_p \leq C\|f\|_{\dot{F}_p^{\psi,q}}.$$

$\square$

Since the proof of Theorem 4.6 is similar to the previous ones, it is omitted.

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