On transitivity of pronormality

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Abstract. This article is dedicated to soluble groups, in which pronormality is a transitive relation. Complete description of such groups is obtained.

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The class of groups in which normality is a transitive relation (T-groups), is well studied now. These groups and some their generalizations has been investigated by many authors ([BT], [Z], [G], [AK], [R1], [R2], [R3], [R4], [E1],[E2], [H], [HL], [SK], [FdeG], [GK], [M1], [M2] and others). The most valuable results about T-groups have been obtained in [R1]. This work with the works mentioned above established the wide variety of interesting and important results placing T-groups in the main stream of study of the normal structure of groups.

In the article [SK] the relationship of T-groups with some groups saturated with pronormal subgroups has been investigated. As it has been established in this article, soluble T-groups could be characterized as the groups with all pronormal cyclic subgroups. Recall, that a subgroup A is called *pronormal* in a group G (we will denote it A pr G) if for every $x \in G$ subgroups A^x and A are conjugate in $\langle A^x, A\rangle$. Also a subgroup A is called *abnormal* in a group G (A ab G) if $g \in \langle A, A^g\rangle$ for each element $g \in G$. It follows immediately from the definition that abnormal subgroups are self-normalizing, and every subgroup containing an abnormal subgroup is also abnormal. Therefore we can consider abnormality as some kind of opposite to normality. Pronormal subgroups are very useful and natural generalizations of normal and abnormal subgroups. Many articles have been dedicated to study of pronormal subgroups and their influence on the normal structure of a group (see, for example, [P], [Mu], [Ro], [W], [KS1], [KS2], [KS3], [SK], [S1], [deFKS], [Ce], [CD], [deGV], [V], [V1], and others).

From this point of view the question about the groups, in which pronormality is transitive (briefly TP-groups) seems very natural and actual. In this article we investigate the soluble TP-groups.

Following [R1], a group in which all subgroups are T-groups will be called a $\bar{\mathrm{T}}$-group, and a group in which all subgroups are TP-groups a $\bar{\mathrm{T}}$P-group.

1. Preliminary results

We will use the following definitions from [B].

Let G be a group, D and K subgroups of G. We say that K is an intermediate subgroup for D if $K \geq D$.

We say that a subgroup D of a group G satisfies the Frattini property if for any intermediate subgroups K and L such that $D \leq K \trianglelefteq L \leq G$, we have $L = (L \cap N_G(D))K$.

In the following lemma we just collected all needed properties of pronormal and abnormal subgroups.

Lemma 1 (see for example [BB]). *Let G be a group, F a subgroup of G.*

I. *The following statements hold.*

(1) *If A pr G and $A \leq F$, then A pr F.*
(2) *If $N \trianglelefteq G$, and $N \leq A$, then A/N pr G/N if an only if A pr G.*
(3) *If $N \trianglelefteq G$ and A pr G, then AN/N pr G/N.*
(4) *If $N \trianglelefteq G$ and A pr G, then $N_G(AN) = N_G(A)N$.*
(5) *If A pr G and A is subnormal in G, then $A \trianglelefteq G$.*
(6) *If A pr G, then $N_G(A)$ ab G.*
(7) *If A pr G and $B \trianglelefteq G$, then AB pr G.*
(8) *If G is a soluble group, then A pr G if and only if it satisfies the Frattini Property. In particular, if A pr G, then for any $N \trianglelefteq G$ such that $A \leq N$ we have $G = N_G(A)N$.*
(9) *If A ab G, then A is abnormal in any intermediate subgroup of G.*

II. *The following statements are equivalent.*

(1) *H ab G.*
(2) *H pr G and $N_G(H) = H$.*
(3) *H pr G and $N_G(K) = K$ for any intermediate subgroup K for H.*

III. *Let H be a subgroup of a group G and $N \trianglelefteq G$ such that $N \leq H$; then H ab G if and only if H/N ab G/N.*

Corollary 1. *An arbitrary TP-group is a T-group. An arbitrary $\bar{T}$P-group is a $\bar{T}$-group.*

We say that a subgroup A is quasicentral in a group G if each subgroup of A is normal in G.

Lemma 2. *Let G be a periodic group and $G = A \leftthreetimes B$, where A is a quasicentral Hall subgroup of G. Then B pr G.*

Proof: Let $g = ab$, $a \in A$, $b \in B$ be an arbitrary element of G. It follows that $B^g = B^a$. The cyclic subgroup $\langle a \rangle$ could be presented as a direct product $\langle a \rangle = \langle f \rangle \times \langle c \rangle$ of two Hall cyclic subgroups $\langle f \rangle$ and $\langle c \rangle$, such that $f \in C_G(b)$

and $\langle c\rangle \cap C_G(b) = 1$, $2 \nmid |c|$. It is clear that $B^a = B^{c_1}$ where $c_1 \in \langle c\rangle$. Since $[B, \langle c_1\rangle] = \langle c_1\rangle$, $c_1 \in \langle B^{c_1}, B\rangle$. Hence $B^{c_1} = B^g$ and Lemma 2 is proved. □

The following lemma is almost obvious.

Lemma 3. *All factor-groups and all pronormal subgroups of a TP-group are TP-groups. All factor groups of a $\bar{\text{T}}$P-group are $\bar{\text{T}}$P-groups.*

2. Periodic soluble TP-groups

Theorem 1. *Locally soluble $\bar{\text{T}}$P-groups become exhausted by the locally soluble groups in which all subgroups are pronormal.*

Proof: Let G be a locally soluble $\bar{\text{T}}$P-group. Corollary 1 implies that G is a $\bar{\text{T}}$-group. Group G is soluble (see [KS4, Theorem 1]). Theorem 6.1.1 from [R1] implies that if G is non-periodic, then G is an abelian group. Assume that G is a periodic group. Let H be some non-pronormal subgroup of G. Theorem 6.1.1 from [R1] implies that in this case G is an extension of quasicentral Hall abelian $2'$-subgroup A coinciding with the nilpotent residual of G by a Dedekind group. Consider the subgroup $R = AH$. Since G/A is a Dedekind group, $R \trianglelefteq G$. Denote $D = A \cap H$. It is clear that $D \neq H$ and $D \trianglelefteq G$. From Lemma 1(2) it follows that H/D is not a pronormal subgroup in G/D. Denote $\bar{A} = A/D$, $\bar{H} = H/D$, $\bar{G} = G/D$, $\bar{R} = R/D$. It is clear, that $\pi(\bar{A}) \cap \pi(\bar{H}) = \emptyset$. By Lemma 2 $\bar{H}$ pr $\bar{G}$. This contradiction proves our theorem. □

Lemma 4. *An arbitrary periodic soluble TP-group splits over the odd component of the last term of its lower central series.*

Proof: Let G be a periodic soluble TP-group. By Corollary 1 G is a T-group. By Theorem 4.2.2 from [R1] there exists the odd component A of the last terms of the lower central series of G, which is an abelian Hall quasicentral $2'$-subgroup in G, and the derived group of the factor G/A is a 2-subgroup. Section 2 of [R1] implies that the set of the elements of odd orders of a soluble T-group forms a normal subgroup. It follows from this that G/A is a direct product of its Sylow prime subgroups. Using common notations we can say that G is a s-group. It is following from Section 3 of [R1] that such p-subgroups for $p > 2$ are abelian, and for $p = 2$ their structure has been described precisely (see, [R1, Section 3.2]). Let H be a π-subgroup of G, $\pi \subseteq \pi(G/A)\backslash\{2\}$. Consider the group $M = A \leftthreetimes H$. Since all Sylow p-subgroups of the s-group G/A with $p \neq 2$ are abelian and $2 \notin \pi$, M is a normal subgroup in G. Subgroup H is pronormal in M by Lemma 2. Since G is a TP-group, H is pronormal in G. Let P be a Sylow 2-subgroup of G. Since G/A is a Dedekind group, $R = A \leftthreetimes P \trianglelefteq G$. Subgroup P is pronormal in R by Lemma 2. Since G is a TP-group, P is pronormal in G. It follows that all subgroups of Sylow 2-subgroup P are pronormal in G.

From Lemma 1(8) it follows that $G = (A \leftthreetimes P)N_G(P)$. From the imbedding $P \leq R \cap N_G(P)$ and the isomorphism $G/R \cong N_G(P)/R \cap N_G(P)$ it follows that there is no 2-element in the factor-group G/R. Let B be a Sylow $\pi(G/R)$-subgroup of $N_G(P)$. As above, B is pronormal in G. Since $G_1 = RB \trianglelefteq G$, $G = G_1 N_G(B)$ by Lemma 1(8). Since B is the only normal Sylow π-subgroup in its normalizer and $B \leq G_1 \cap N_G(B)$, the factor-group $G/G_1 \cong N_G(B)/G_1 \cap N_G(B)$ does not have π-elements. Moreover G/G_1 does not also have $\pi(A)$-elements. In addition, G/G_1 does not have an involution. Therefore $G = G_1 = A \leftthreetimes (B \times P)$ and we proved the lemma. □

Using [R1, Lemma 4.2.1 and Section 3.2] we can obtain the following result.

Corollary 2. *Under the conditions of Lemma 4 the group G has the decomposition $G = A \leftthreetimes (B \times P)$, where A is the odd component of the last term of the lower central series of G, A an abelian Hall quasicentral $2'$-subgroup, B an abelian $2'$-subgroup, P is either trivial or a 2-T-group. The derived group G' is a direct product $G' = A \times P'$ and quasicentral in G.*

The next corollary follows from [KS4, Lemma 4] and the results above.

Corollary 3. *In an arbitrary periodic soluble T-group G, which is decomposed in a direct product of its primary components, all pronormal subgroups are normal. In particular G is a TP-group.*

Corollary 4. *The class of periodic soluble TP-groups, which are decomposed in a direct product of their primary components, coincides with the class of periodic soluble T-groups decomposed in a direct product of their Sylow p-subgroups.*

The next lemma is following from Lemmas 1, 2, and 4.

Lemma 5. *Let G be a periodic soluble TP-group.*

1. *All its $2'$-subgroups and all its subgroups containing some Sylow 2-subgroup of G are pronormal in G.*
2. *Any Sylow $\pi(G/A)$-subgroup of G complements the odd component A of the last term of the lower central series of G.*

Lemma 6. *Let G be a periodic group of the form $G = A \leftthreetimes (B \times P)$, where A is an abelian Hall quasicentral in G $2'$-subgroup, B an abelian 2-subgroup, P a 2-group with the T-property (a 2-T-group), $G' = A \times P$, and any Sylow $\pi(B \times P)$-subgroup of G complements A in G. Then every pronormal subgroup K of G is decomposed in the product $K = A_1 \leftthreetimes (B_1 \times P_1)$ where $A_1 = K \cap A$, $B_1 \leq B$, $P_2 \leq P$, and every Sylow $\pi(B_1 \times P_1)$-subgroup of K complements A_1 in K.*

Proof: By Lemma 1 $G_1 = AK$ is pronormal in G. It is clear that G/A is a soluble T-group decomposed into a direct product of its Sylow p-subgroups. By Lemma 1 AK/A is pronormal in G/A, and AK/A is normal in G/A by Corollary 3. Thus $G_1 \trianglelefteq G$. The subgroup $B \times P$ complements A in G. It follows from this that $G_1 = A \leftthreetimes (B_1 \times P_1)$ where $B_1 \times P_1 = (B \times P) \cap G_1$. Let $B_2 \times P_2 = R_2$ be a Sylow $\pi(B \times P)$-subgroup of K. Since every Sylow $\pi(B \times P)$-subgroup of G complements

A, we can assume without loss of generality that $B_2 \times P_2 \leq B_1 \times P_1 \leq B \times P$. Consider the group $G_2 = A \leftthreetimes (B_2 \times P_2)$. We shall prove that $G_1 = G_2$. Let x be an arbitrary element of the derived subgroup P_1' of the Sylow 2-subgroup P_1 of the group G_1. Since $G_1 = AK$, $x = ak$, where $a \in A$, $k \in K$. Then $k = a^{-1}x$. It follows from $P_1' \leq P'$, $[P', A] = 1$, and $2 \in \pi(\langle a \rangle)$ that $k^{|a|} = (a^{-1}x)^{|a|} = x^{|a|} \in K$. Therefore $x \in K$. Since G' is quasicentral, $\langle x \rangle \trianglelefteq G$. Hence x belongs to any 2-Sylow subgroup of K, in particular $x \in P_2$. Therefore $P_2 \geq P_1'$ and $G_2 \geq G_1'$. It follows that $G_2 \trianglelefteq G_1$. Let $c = b_1p_1$ be an element of $B_1 \times P_1 \setminus B_2 \times P_2$, $b_1 \in B_1$, $p_1 \in P_1$. Since $P_2 \geq P_1'$, $P_2 \trianglelefteq P_1$, and $(B_2 \times P_2)^c = (B_2 \times P_2)^{p_1} = B_2 \times P_2$. Therefore $B_2 \times P_2 \trianglelefteq B_1 \times P_1$.

We shall show that $R_2 = B_2 \times P_2$ is pronormal in G_1. In fact, for every $g \in G_1$ we can write $g = pba$, where $p \in P_1$, $b \in B_1$, $a \in A$. Therefore $(B_2 \times P_2)^{pba} = (B_2 \times P_2)^a$. Consider the group $R = \langle a \rangle \leftthreetimes (B_2 \times P_2)$. By Lemma 2 R_2 is pronormal in R, and therefore in G_1.

Since $G_2 \trianglelefteq G_1$, $G_1 = G_2 N_G(R_2)$ by Lemma 1. The factor-group $G_1/G_2 \cong N_G(R_2)/G_2 \cap N_G(R_2)$ does not have $\pi(G_1/A)$-elements (R_2 is a normal Sylow $\pi(G_1/A)$-subgroup in $N_G(R_2)$, and $R_2 \leq G_2$). Since G_1/G_2 does not have $\pi(A)$-elements, $G_1 = G_2$. Hence $G_1 = A \leftthreetimes R_2$ where $R_2 = B_2 \times P_2$ is an arbitrary Sylow $\pi(G/A)$-subgroup of K, and R_2 is complemented in K by the subgroup $A_1 = A \cap K$. Our proof is completed. □

Lemma 7. *The group G from Lemma 6 is a TP-group.*

PROOF: Let L be a pronormal subgroup of a pronormal subgroup K of G. By Lemma 6 $K = A_1 \leftthreetimes (A_2 \times P_2)$ where $A_1 = A \cap K$, $B_2 \times P_2$ is a Sylow $\pi(B \times P)$-subgroup of K. This subgroup $B_2 \times P_2$ belongs to some complement $B \times P$ to A in G, and $B_2 \times P_2 = (B \times P) \cap K$ by Lemma 5. Proving Lemma 6 we particularly proved that for pronormal subgroup K the subgroup $G_1 = AK$ is normal in G. This implies that $AK \cap (B \times P) = B_2 \times P_2 \trianglelefteq B \times P$. By Lemma 6 $L = A_2 \leftthreetimes (B_3 \times P_3)$, where $A_2 = L \cap A_1$, $B_3 \times P_3 = L \cap (B_2 \times P_2)$, and $B_3 \times P_3 \trianglelefteq B_2 \times P_2$. Since $B \times P$ is a T-group, $B_3 \times P_3 \trianglelefteq B \times P$. Let $g = bpa$ be an arbitrary element of $G = A \leftthreetimes (B \times P)$. It is clear that $(B_3 \times P_3)^g = (B_3 \times P_3)^a$. Consider the group $R = \langle a \rangle \leftthreetimes (B_3 \times P_3)$. It is easy to show that there exists an element $\bar{a}$ of R, such that $(B_3 \times P_3)^a = (B_3 \times P_3)^{\bar{a}}$, and $\bar{a} \in \langle (B_3 \times P_3)^{\bar{a}}, B_3 \times P_3 \rangle$. It means that $B_3 \times P_3$ is pronormal in G. By Lemma 1 $L = A_2 \leftthreetimes (B_3 \times P_3)$ is pronormal in G. Lemma is proved. □

The next theorem follows from the results above.

Theorem 2. *A periodic soluble group G satisfies the transitive condition on pronormal subgroups if and only if it decomposed into a semidirect product $G = A \leftthreetimes (B \times P)$ where A and B are abelian Hall subgroups in G, $2 \notin \pi(A)$, P is either a trivial group or a 2-T-group, the derived subgroup G' is an abelian quasicentral subgroup of G decomposed into a direct product of A and the derived subgroup P' of P, and any Sylow $\pi(B \times P)$-subgroup of G complements A in G.*

Corollary 5. *A periodic soluble TP-group is a semidirect product of a Hall normal subgroup in which all subgroups are pronormal and either a 2-T-group (in which only normal subgroups are pronormal) or trivial.*

Using Theorem 2 and [R1, Section 3.2] we can obtain the following corollary.

Corollary 6. *Periodic soluble TP-groups become exhausted by groups of the following types.*

I. $G = A \leftthreetimes B$, *where A is an abelian quasicentral Hall $2'$-subgroup, B is a Dedekind group, $G' = A \times B'$, and any Sylow $\pi(B)$-subgroup of G complements A in G.*

II–V. $G = G_1 \leftthreetimes P$ *where $G_1 = A \leftthreetimes B$ is a Hall normal subgroup of type I, P is a 2-group of one of the following types of groups:*

1. $G = (C \leftthreetimes \langle z \rangle) \times D$ *where C is a divisible non-trivial abelian group,* $\exp D \leq 2$, $z^2 = 1$ *or* $z^4 = 1$, $z^2 \in Z(G)$, $c^z = c^{-1}$ *for each* $c \in C$;
2. $G = (C \leftthreetimes \langle z, s \rangle) \times D$ *where C and D are the same groups as in* 1, $\langle z, s \rangle$ *is a quaternion group,* $c^z = c^{-1}$, $c^s = s$, *for each* $c \in C$;
3. $G = (C \leftthreetimes K\langle s \rangle) \times D$ *where D is the same group as in* 1, *C is a divisible abelian group, K is a Prüfer* 2*-group,* $z^4 = 1$, $z^2 \in K$, $[K, C] = 1$, $c^z = c^{-1}$, $k^z = k^{-1}$ *for each* $c \in C$ *and* $k \in K$.

In addition, $[P, B] = 1$, $G' = A \times P'$ *and any Sylow $\pi(B \times P)$-subgroup of G complements A in G.*

Now we consider the following two cases: the first one is about soluble TP-groups, in which the centralizer of the derived subgroup contains an element of infinite order; and the second case is about non-periodic soluble TP-groups with the periodic centralizer of the derived subgroup. This classification has been introduced and successfully applied by D.J.S. Robinson [R1] for non-periodic soluble T-groups. Using his results we can describe the non-periodic soluble TP-groups. We will begin with the case of the non-periodic centralizer $C = C_G(G')$.

3. TP-groups with a non-periodic centralizer of the derived group

The next corollary follows directly from [R1, Theorem 3.1.1].

Corollary 7. *Torsion-free soluble TP-groups are Abelian.*

Theorem 3. *Let G be a soluble group with the non-periodic centralizer $C = C_G(G')$. Then G is a TP-group if and only if it is a T-group.*

Proof: Let G be a soluble TP-group with the non-periodic centralizer $C = C_G(G')$ of the derived subgroup. By Corollary 1 G is a T-group.

If G is a soluble T-group with a non-periodic centralizer of the derived subgroup $C = C_G(G')$, then by [R1, Theorem 3.2.1] G is either an abelian group (in this case our theorem is obvious), or G is decomposed into a direct product of a group

B of exponent ≤ 2 and a group A belonging to one of the following types of groups.

1. $A = A_0 \leftthreetimes \langle z \rangle$ where A_0 is a non-periodic abelian group, $A_0^2 = A_0$, z an element of order 2 or 4, and $a_0^z = a_0^{-1}$ for any $a_0 \in A_0$.
2. $A = A_0 \leftthreetimes \langle z, s \rangle$ where A_0 is a non-periodic abelian group, $A_0^2 = A_0$, $\langle z, s \rangle$ is a quaternion group, $a_0^z = a_0^{-1}$, $a_0^s = a_0$ for any $a_0 \in A_0$.
3. $A = A_0 \leftthreetimes \langle z, s_\infty \rangle$ where A_0 is a non-periodic abelian group, $A_0^2 = A_0$, $\langle z, s_\infty \rangle$ is a locally quaternion group with a *Prüfer* subgroup $\langle s_\infty \rangle$, $[\langle s_\infty \rangle, A_0] = 1$, $a_0^z = a_0^{-1}$ for any $a_0 \in A_0$.

Considering all these cases we will prove that G satisfies the transitive condition for pronormal subgroups. We will investigate the structure of such subgroups in every particular case.

Consider the group $G = (A_0 \leftthreetimes \langle z \rangle) \times B$ of the type 1. Without loss of generality we can assume that any abnormal subgroup F of G coincides with $(A_1 \leftthreetimes \langle z \rangle) \times B$ where $A_1 = A_0 \cap F$ and the factor A_0/A_1 is a periodic $2'$-group. Indeed, by [deFKS, Lemma 4] F supplements the derived subgroup A_0 in G. Therefore, we can assume that $F \supseteq \langle z \rangle \times B$ and this subgroup $\langle z \rangle \times B$ is complemented in G by A_0.

We will prove that for such subgroup $F = (A_1 \leftthreetimes \langle z \rangle) \times B$ the factor A_0/A_1 is a periodic $2'$-group. In fact, $\bar{G} = G/A_1$ could be represented as a semidirect product $\bar{G} = \bar{A} \leftthreetimes \bar{D}$ where $\bar{A} = A_0/A_1$, $\bar{D}$ is isomorphic with $\langle z \rangle \times B$, and $\bar{D}$ is abnormal in $\bar{G}$ as an image of an abnormal subgroup F of the group G. If $\bar{A}$ would have an element of infinite order or a 2-element (denoted by $\bar{x}$ in both cases), then in the group $\langle \bar{x} \rangle \leftthreetimes \bar{D}$ the subgroup $\bar{D}$ would be abnormal. In this case $\bar{D}$ would supplement the derived subgroup $\langle \bar{x}^2 \rangle$ in $\langle \bar{x} \rangle \leftthreetimes \bar{D}$, which is impossible.

We shall prove that $A_1^2 = A_1$. Indeed, if $A_1^2 \neq A_1$, the factor-group A_0/A_1^2 has a non-trivial Sylow 2-subgroup, which is an elementary abelian 2-group. It follows from here that A_0 has an elementary abelian 2-factor-group, which is impossible ($A_0^2 = A_0$).

Note that any subgroup $R = (A_1 \leftthreetimes \langle z \rangle) \times B$, where $A_1 < A_0$ defines the periodic $2'$-factor-group A_0/A_1, $A_1^2 = A_1$, is abnormal in G. Indeed, the derived subgroup of any intermediate subgroup containing R is contained into A_0, and (if the appropriate factor-group $\bar{A}$ is a periodic $2'$-subgroup) R will supplement it. It means that all abnormal subgroups of G have the same structure as G does.

Because A_0 is non-periodic it always has a proper subgroup A_1 with the periodic $2'$-factor-group A_0/A_1. In fact, if A_0 is not divisible, then there exists an odd number n, such that $A_0^n \neq A_0$ and A_0/A_0^n is a periodic $2'$-group. If A_0 is divisible, then it is a direct product of *Prüfer* groups and groups are isomorphic to the additive group of rational numbers. Since A_0 is non-periodic, there exists at least one subgroup Q like that. We can consider the factor group by the complement R to Q. It is isomorphic to Q. Let x be an element of Q. The factor-group $Q/\langle x \rangle$

is periodic. Consider the factor group L of this group by its Sylow 2-subgroup. It is a $2'$-group. Now it is easy to build the appropriate normal subgroup of A_0, determining a factor-group isomorphic with L.

We will show that our group G of type 1 is a TP-group. By Lemma 1(6) any proper pronormal subgroup F of G has an abnormal in G normalizer, which has the same structure as G does, i.e. without loss of generality we can assume that $R = N_G(F) = (A_1 \leftthreetimes \langle z\rangle) \times B$, where $A_1 < A_0$ defines the periodic $2'$-factor-group A_0/A_1, and $A_1^2 = A_1$. Let M be a normal subgroup in R. There are two following possibilities: (1) M contains an element $m = axb$, where $a \in A_1$, $x \in \langle z\rangle$, $b \in B$, $x \notin \langle z^2\rangle$; and (2) M contains an element $m = axb$, where $a \in A_1$, $x \in \langle z\rangle$, $b \in B$, $x \in \langle z^2\rangle$. In the second case, $M \trianglelefteq G$; moreover, M is quasicentral in G. It means that in this case all subgroups of M are pronormal in G. Consider the first case. Without loss of generality we can assume that $x = z$. Since $[A_1, z] = A_1^2 = A_1$, $A_1 \le M$. It follows that $M = (A_1 \leftthreetimes \langle z\rangle) \times B_1$ where $B_1 = M \cap B$.

Let us show that the subgroup $K = A_1 \leftthreetimes \langle z\rangle$ is pronormal in G. Let y be an element of G. We can write $y = atb$ where $a \in A_0$, $t \in \langle z\rangle$, $b \in B$. Hence $K^y = K^a$. Since $[a, z] = a^{-2}$ and A_0/A_1 is a $2'$-group, $a \in \langle K^a, K\rangle$. It means that $K = A_1 \leftthreetimes \langle z\rangle$ is pronormal in G, and by Lemma 1(7) $M = KB_1$ pr G.

Any pronormal subgroup F in M either is quasicentral in G or has the same structure as M does. It means that we can use the arguments from the paragraph above to show that F pr G. This way we prove that all groups of the type 1 are TP-groups.

Consider the type 2: $G = A \times B = (A_0 \leftthreetimes \langle z, s\rangle) \times B$, A_0 is a non-periodic abelian group, $A_0^2 = A_0$, $\langle z, s\rangle$ is a quaternion group, $a_0^z = a_0^{-1}$, $a_0^s = a_0$ for any $a_0 \in A_0$. In this case the subgroup $H = \langle z^2\rangle \times B$ as a central subgroup belongs to any abnormal subgroup of G. By Lemma 1(III) every abnormal subgroup of G is a preimage of an abnormal subgroup of the factor-group G/H. Note, that this subgroup contains $\langle s\rangle H/H$ as a central subgroup of G/H. It means that an abnormal subgroup F of G could be written as $F = (A_1 \leftthreetimes \langle z, s\rangle) \times B$, where $A_1 < A_0$ defines the periodic $2'$-factor-group A_0/A_1, and $A_1^2 = A_1$. Any pronormal subgroup M of G is a normal subgroup of a subgroup of this type by Lemma 1(6). Without loss of generality we can consider the following two possibilities: (1) M contains an element $m = axyb$ where $a \in A_1$, $x \in \langle z\rangle$, $y \in \langle s\rangle$, $b \in B$, $x \notin \langle z^2\rangle$; and (2) M contains an element $m = axyb$ where $a \in A_1$, $y \in \langle s\rangle$, $b \in B$, $x \in \langle z^2\rangle$. In the second case, $M \trianglelefteq G$, moreover, M is quasicentral in G. It means that in this case all subgroups of M are pronormal in G. Consider the first case. Without loss of generality we can assume that $x = z$. Since $[A_1, z] = A_1^2 = A_1$, $A_1 \le M$. It follows that M contains the subgroup $T = (A_1 \leftthreetimes \langle z\rangle) \times B_1$, where $B_1 = M \cap B$. In this case it is easy to show that $M^x = M^a$ for any $x \in G$, $x = atb$, $a \in A_0$, $t \in \langle z, s\rangle$, $b \in B$. Note, that $a \in \langle M^x, M\rangle = \langle M^a, M\rangle$. It means that M is pronormal in G. Any pronormal subgroup F of M has the same structure as M does, and using the above arguments we can show that F is pronormal in G.

Considering the type 3 where $A = A_0 \leftthreetimes \langle z, s_\infty\rangle$, A_0 is a non-periodic abelian group, $A_0^2 = A_0$, $\langle z, s_\infty\rangle$ is a locally quaternion group with a *Prüfer* subgroup $\langle s_\infty\rangle$, $[\langle s_\infty\rangle, A_0] = 1$, $a_0^z = a_0^{-1}$ for any $a_0 \in A_0$, we will prove that any abnormal subgroup contains $\langle z, s_\infty\rangle$. First of all note that any abnormal subgroup F of G contains some element $x = astb$, where $a \in A_0$, $s \in \langle s_\infty\rangle$, $\langle t\rangle = \langle z\rangle$, $b \in B$. Indeed, in the opposite case, when for all elements of F in the decomposition above $t \in \langle z^2\rangle$, F is normal in G, i.e. $G = F$, which is impossible. Thus, $\langle t\rangle = \langle z\rangle$, and without loss of generality we can assume that $z \in F$. If such F does not contain an element $s \in \langle s_\infty\rangle$, we will assume that s^2 belongs to F. Consider the subgroup $K = \langle s\rangle F$. By Lemma 1(9) F is abnormal in K. However, $F \geq K'$, and $F = K$, which is impossible. Hence, $F \geq \langle s_\infty\rangle$. Considering the factor-group of G by $\langle z^2, s_\infty\rangle$ we come to the type 1 and using arguments similar to those above we can prove that every abnormal subgroup F of G could be represented as $F = (A_1 \leftthreetimes \langle s_\infty, z\rangle) \times B$ where $A_1 < A_0$ defines the periodic $2'$-factor-group A_0/A_1, and $A_1^2 = A_1$. It means that only normal subgroups M of such a group F could be pronormal in G (Lemma 1(6)). Without loss of generality we can consider the following two possibilities: (1) M contains an element $m = axyb$, where $a \in A_1$, $x \in \langle z\rangle$, $y \in \langle s_\infty\rangle$, $b \in B$, $x \notin \langle z^2\rangle$; and (2) M contains an element $m = axyb$, where $a \in A_1$, $y \in \langle s_\infty\rangle$, $b \in B$, $x \in \langle z^2\rangle$. In the second case, $M \trianglelefteq G$; moreover, M is quasicentral in G. It means that in this case all subgroups of M are pronormal in G. Consider the first case. Without loss of generality we can assume that $x = z$. Since $[A_1, z] = A_1^2 = A_1$ and $[\langle s_\infty\rangle, z] = \langle s_\infty\rangle$, $A_1 \times \langle s_\infty\rangle \leq M$. It follows that M contains the subgroup $T = (A_1 \leftthreetimes \langle s_\infty, z\rangle) \times B_1$ where $B_1 = M \cap B$. Now it is easy to show that $M^x = M^a$ for any $x \in G$, $x = atb$, $a \in A_0$, $t \in \langle z, s_\infty\rangle$, $b \in B$. Note, that $a \in \langle M^x, M\rangle = \langle M^a, M\rangle$. It means that M is pronormal in G. Any pronormal subgroup F of M has the same structure as M does, and using the above arguments we can show that F is pronormal in G. Hence G is a TP-group. Our theorem is proved. □

4. Non-periodic TP-groups with a periodic centralizer of the derived group

Lemma 8. *Let G be a soluble non-periodic T-group with the periodic centralizer $C = C_G(G')$ of the derived group G'. Then there exists the hypercentral residual L in G, which is the direct product of all Sylow p-subgroups V of G' such that $V \cap Z(G) = 1$, and each of these Sylow subgroups is complemented in G by an abnormal subgroup of G.*

Proof: Consider the direct product L of all Sylow p-subgroups V of G' such that $V \cap Z(G) = 1$, and the direct product Y of all Sylow p-subgroups W of G' such that $W \cap Z(G) \neq 1$. It is obvious that $G' = L \times Y$ and $2 \notin \pi(L)$. It follows from [R1, Theorem 4.3.1] that L is a direct product of *Prüfer* subgroups having trivial intersection with the center of G and the factor-group G/L is hypercentral. Let G/Z be another hypercentral factor-group of G. By Remak theorem the factor $G/(L \cap Z)$ is also hypercentral. Since $L \cap Z$ is a periodic normal subgroup of G from [R1, Theorem 4.3.1] it follows that $L \cap Z = L$. It means that L is the hypercentral residual of L.

If V is a Sylow p-subgroup of G' such that $V \cap Z(G) = 1$, then V is contained in the hypercentral residual L. We have $G' = V \times W$ where W is a p'-component of G'. In the factor group G/W the hypercentral residual $G'/W \cong V$ is its derived group. From [R1, Theorem 4.3.1] it follows that in the T-group G/W there is an element Wx such that $[Wx, G'/W] = G'/W$ and $(G'/W)\langle Wx\rangle \trianglelefteq G/W$. By Lemma 7 from [Z] $G/W = (G'/W)N_{G/W}(\langle Wx\rangle)$. It follows that $G = G'N$ where N is the preimage of $N_{G/W}(\langle Wx\rangle)$. Since $V \cap N_G(\langle x\rangle) = 1$, it means that $G = (V \times W)N = V(WN) = V(WN_G(\langle x\rangle)) = V \leftthreetimes WN_G(\langle x\rangle)$. Thus we have proved that every such Sylow p-subgroup of L is complemented in G.

Let us show that any supplement X to V in G is abnormal. In fact, let Y be an intermediate subgroup for V. Then $Y = UX$, where $U = Y \cap V$. It is obvious that $U \leq Y'$. Since G is a soluble group, X is abnormal in G by [deFKS, Lemma 4]. We proved the lemma. □

There is a natural question about existence of a complement for L in G. The answer is negative. The article [KS5] contains an example of a non-periodic T-group with a periodic centralizer of its derived group and with a non-complemented hypercentral residual.

Theorem 4. *Let G be a soluble non-periodic group with a periodic centralizer of the derived group. The group G is a TP-group if and only if it is a hypercentral T-group.*

Proof: It is sufficient to prove that an arbitrary TP-group G with a periodic centralizer of its derived subgroup is hypercentral. Assume that G is a non-hypercentral group of this type. By Lemma 8 $G = V \leftthreetimes WN_G(\langle x\rangle)$ where V is a Sylow p-subgroup of the hypercentral residual of G. From [R1, Theorem 4.3.1] it follows that V is a direct product of Prüfer p-subgroups. Let R be a Prüfer p-subgroup from this product, K a complement to R in V. By [R1, Theorem 4.3.1] $K \trianglelefteq G$. The factor group $\bar{G} = G/(K \times W)$ is a TP-group by Lemma 3. Obviously, $\bar{G} = \bar{V} \leftthreetimes \bar{N}$ where $\bar{V} = V \times W/(K \times W) \cong R, \bar{N} = (K \times W)N_G(\langle x\rangle)/(K \times W)$, $\bar{N}$ is a non-periodic abelian group, and $[\bar{V}, \bar{N}] = \bar{V}$. Hence there is an element of infinite order of $\bar{N}$ non-centralizing the lower layer of $\bar{V}$. Without loss of generality we will denote this element by $\bar{x}$. By Lemma 3 the subgroup $\bar{M} = \bar{V} \leftthreetimes \langle\bar{x}\rangle$ is a TP-group. By [deFKS, Lemma 4] $\langle\bar{x}\rangle ab\ \bar{M}$. It means that all subgroups of $\langle\bar{x}\rangle$ are pronormal in $\bar{M}$. Consider the lower layer $\langle\bar{v}\rangle$ of $\bar{V}$. This is a normal subgroup in $\bar{M}$. In the subgroup $\langle\bar{v}\rangle \leftthreetimes \langle\bar{x}\rangle$ the centralizer of $\langle\bar{v}\rangle$ has finite index. It follows that there exists an integer m such that the subgroup $\bar{V} \leftthreetimes \langle\bar{x}^m\rangle$ is hypercentral. Any hypercentral TP-group has no abnormal proper subgroup [KS4, Lemma 4].

Since $\langle \bar{x}^m \rangle$ is pronormal in $\bar{M}$ and therefore in $\bar{V} \leftthreetimes \langle \bar{x}^m \rangle$, it follows that $\langle \bar{x}^m \rangle$ is a normal subgroup in $\bar{V} \leftthreetimes \langle \bar{x}^m \rangle$, i.e. $\bar{V} \leftthreetimes \langle \bar{x}^m \rangle = \bar{V} \times \langle \bar{x}^m \rangle$. The later means that $\bar{M}$ is a group with a non-periodic centralizer of its derived group $\bar{V}$. We obtain a contradiction with Theorem 3. This contradiction proves our theorem. □

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