Conditions of Prodi-Serrin's type for local regularity of suitable weak solutions to the Navier-Stokes equations

Zdeněk Skalák

Abstract. In the context of suitable weak solutions to the Navier-Stokes equations we present local conditions of Prodi-Serrin's type on velocity $\mathbf{v}$ and pressure p under which $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$ is a regular point of $\mathbf{v}$. The conditions are imposed exclusively on the outside of a sufficiently narrow space-time paraboloid with the vertex $(\mathbf{x}_0, t_0)$ and the axis parallel with the t-axis.

Keywords: Navier-Stokes equations, suitable weak solutions, local regularity
Classification: 35Q10, 35B65

Let Ω be either $\mathbb{R}^3$ or a bounded domain in $\mathbb{R}^3$ with $C^{2+\mu}$ $(\mu > 0)$ boundary $\partial\Omega$, $T > 0$ and $Q_T = \Omega \times (0,T)$. Consider the Navier-Stokes equations describing the evolution of velocity $\mathbf{v}$ and pressure p in Q_T:

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \mathbf{0}, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega\times(0,T), \tag{3}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \tag{4}$$

where $\nu > 0$ is the viscosity coefficient and the initial data $\mathbf{v}_0$ satisfy the compatibility conditions $\mathbf{v}_0|_{\partial\Omega} = \mathbf{0}$ and $\nabla\cdot\mathbf{v}_0 = 0$. The pair $(\mathbf{v}, p)$ is called a suitable weak solution to (1)–(4) if $\mathbf{v}$ and p are measurable functions on Q_T, $\mathbf{v} \in L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,W_0^{1,2}(\Omega))$,

$$\int_0^T \int_\Omega [\mathbf{v}\cdot\frac{\partial\phi}{\partial t} - \mathbf{v}\cdot\nabla\mathbf{v}\cdot\phi - \nu\nabla\mathbf{v}\cdot\nabla\phi]\, d\mathbf{x}\, dt = -\int_\Omega \mathbf{v}_0\cdot\phi(\mathbf{x},0)\, d\mathbf{x}$$

for every $\phi \in C_0^\infty(\Omega\times\langle 0,T))$ such that $\nabla\cdot\phi = 0$ in Q_T, $p \in L^{5/4}(Q_T)$ and $(\mathbf{v}, p)$ satisfies the so called generalized energy inequality

$$2\nu\int_0^T\int_\Omega |\nabla\mathbf{v}|^2\phi\, d\mathbf{x}\, dt \le \int_0^T\int_\Omega \Big[|\mathbf{v}|^2\Big(\frac{\partial\phi}{\partial t} + \nu\Delta\phi\Big) + (|\mathbf{v}|^2 + 2p)\mathbf{v}\cdot\nabla\phi\Big]\, d\mathbf{x}\, dt$$

This research was supported by the Research plan of the Czech Ministry of Education No. J04/98/210000010, by the Grant Agency of the Academy of Sciences of the Czech Republic through the grant A2060803 and by the Institute of Hydrodynamics (project No. 5436).

for every non-negative real-valued function $\phi \in C_0^\infty(Q_T)$. Further, a point $(\mathbf{x}_0, t_0) \in Q_T$ is called a regular point of $\mathbf{v}$ if there exists a space-time neighborhood U of $(\mathbf{x}_0, t_0)$ in Q_T such that $\mathbf{v} \in L^\infty(U)^3$. Points of Q_T which are not regular are called singular. For the concept of regular and singular points and suitable weak solutions, see [1].

In [6] J. Neustupa proved the following theorem:

Theorem 1. *There exists an absolute constant $\epsilon_0 > 0$ such that if $\mathbf{v}$ is a suitable weak solution to the problem (1)–(4), $(\mathbf{x}_0, t_0) \in Q_T$ and*

$$\lim_{r \to 0_+} \liminf_{t \to t_0^-} \left(\int_{B_r(\mathbf{x}_0)} |\mathbf{v}(\mathbf{x}, t)|^3 \, d\mathbf{x} \right)^{1/3} < \epsilon_0,$$

then $(\mathbf{x}_0, t_0)$ is a regular point of $\mathbf{v}$.

As was stressed in [6], Theorem 1 shows that if $(\mathbf{x}_0, t_0)$ is a singular point of $\mathbf{v}$ then the L^3 norm of $\mathbf{v}$ must necessarily concentrate in an amount greater than or equal to ϵ_0 in smaller and smaller neighborhoods of $\mathbf{x}_0$ as $t \longrightarrow t_0-$.

This paper was inspired by a theorem (Theorem 2 below) also proved in [6] which says that under certain conditions on $\mathbf{v}$ and p the region of concentration of L^3 norm of $\mathbf{v}$ does not lie inside a sufficiently narrow paraboloid in Q_T with its axis parallel with the t-axis and with the vertex $(\mathbf{x}_0, t_0)$. Let $\rho > 0$, $r > 0$ and $\sigma_0 = r^2/\rho^2$. Denote

$$U_r^\rho = \{(\mathbf{x}, t) \in Q_T; t_0 - \sigma_0 < t < t_0, \rho\sqrt{t_0 - t} < |\mathbf{x} - \mathbf{x}_0| < r\}, \tag{5}$$

$$V_r^\rho = \{(\mathbf{x}, t) \in Q_T; t_0 - \sigma_0 < t < t_0, |\mathbf{x} - \mathbf{x}_0| < \rho\sqrt{t_0 - t}\}, \tag{6}$$

$$Q_r^\rho = \{(\mathbf{x}, t) \in Q_T; t_0 - r^2/\rho^2 < t < t_0, |\mathbf{x} - \mathbf{x}_0| < r\}. \tag{7}$$

Theorem 2. *Suppose that $(\mathbf{v}, p)$ is a suitable weak solution to (1)–(4), $(\mathbf{x}_0, t_0) \in Q_T$, $\rho \in (0, \sqrt{2\nu})$ and*

$$|\mathbf{v}(\mathbf{x}, t)| \le c, \quad |p(\mathbf{x}, t)| \le c \ \text{ in } \ U_r^\rho \tag{8}$$

for some c and $r > 0$. Then $(\mathbf{x}_0, t_0)$ is a regular point of $\mathbf{v}$.

It was shown in [8] that Theorem 2 can be further generalized:

Theorem 3. *Suppose that $(\mathbf{v}, p)$ is a suitable weak solution to (1)–(4), $(\mathbf{x}_0, t_0) \in Q_T$, $\rho > 0$ is sufficiently small, $r > 0$ and*

$$|\mathbf{v}(\mathbf{x}, t)| \le \frac{1}{|\mathbf{x} - \mathbf{x}_0|^\alpha} \quad \text{in } \ U_r^\rho, \qquad p \in L^{\beta,\gamma}(V_r^{\rho+\kappa} \setminus V_r^\rho), \tag{9}$$

where $\alpha \in (0,1)$, $\beta, \gamma \ge 1$, $2/\beta + 3/\gamma < 3 - \alpha$ and $\kappa > 0$. Then $(\mathbf{x}_0, t_0)$ is a regular point of $\mathbf{v}$.

The following theorem was proved in [7].

Theorem 4. *Let* $\Omega = \mathbb{R}^3$. *Suppose that* $(\mathbf{v}, p)$ *is a suitable weak solution to* (1), (2) *and* (4), $(\mathbf{x}_0, t_0) \in Q_T$, $\rho > 0$ *and* $r > 0$. *Let*

(10) $\mathbf{v} \in L^{a,b}(U_r^\rho),\ 2/a + 3/b = 1,\ a \geq 3,\ b > 3$ *or* $\|\mathbf{v}\|_{L^{\infty,3}(U_r^\rho)} < \epsilon_1$ *and*

(11) $P \in L^{\alpha,\beta}(V_r^\rho), 2/\alpha + 3/\beta = 2,\ \alpha \geq 1,\ \beta > 3/2$ *or* $\|P\|_{L^{\infty,3/2}(V_r^\rho)} < \epsilon_2$,

where P *denotes the negative part of pressure* p : $P = 0$ *if* $p \geq 0$, $P = -p$ *if* $p < 0$ *and* ϵ_1, ϵ_2 *are sufficiently small. Then* $(\mathbf{x}_0, t_0)$ *is a regular point of* $\mathbf{v}$.

Theorem 4 does not need any assumption on integrability of $\mathbf{v}$ inside the paraboloid. It is compensated by assumptions on a certain integrability of the negative part of pressure P (conditions (11)).

The main goal of this paper is to prove the two following theorems:

Theorem 5. *Suppose that* $(\mathbf{v}, p)$ *is a suitable weak solution to* (1)–(4), $(\mathbf{x}_0, t_0) \in Q_T$, $\rho > 0$ *is sufficiently small,* $r > 0, \kappa > 0$ *and*

$$\mathbf{v} \in L^{a,b}(U_r^\rho), \quad 2/a + 3/b = 1,\ a \geq 3,\ b > 3, \tag{12}$$

$$p \in L^{\alpha,\beta}(V_r^{\rho+\kappa} \setminus V_r^\rho), \quad 2/\alpha + 3/\beta = 2,\ \alpha \geq a/(a-1),\ \beta > 3/2. \tag{13}$$

Then $\mathbf{v} \in L^\infty(Q_\eta^1)$ *for some* $\eta > 0$. *Moreover, if* $\Omega = \mathbb{R}^3$ *then* $(\mathbf{x}_0, t_0)$ *is a regular point of* $\mathbf{v}$.

Theorem 6. *Suppose that* $(\mathbf{v}, p)$ *is a suitable weak solution to* (1)–(4), $(\mathbf{x}_0, t_0) \in Q_T$, $\rho > 0$ *is sufficiently small,* $r > 0, \kappa > 0$ *and*

$$\mathbf{v} \in L^{\tilde{a},\tilde{b}}(U_r^{\rho+\kappa}), \quad 2/\tilde{a} + 3/\tilde{b} = 1,\ \tilde{a} \geq 2,\ \tilde{b} > 3, \tag{14}$$

$$\mathbf{v} \in L^{a,b}(V_r^{\rho+\kappa} \setminus V_r^\rho), \quad 2/a + 3/b = 1,\ a \geq 3,\ b > 3, \tag{15}$$

$$p \in L^{\alpha,\beta}(V_r^{\rho+\kappa} \setminus V_r^\rho), \quad 2/\alpha + 3/\beta = 2,\ \alpha \geq a/(a-1),\ \alpha \geq 5/4,\ \beta > 3/2. \tag{16}$$

Then $\mathbf{v} \in L^\infty(t_0 - \eta^2, t_0, W^{1,2}(B_\eta(\mathbf{x}_0)))$ *for some* $\eta > 0$. *Moreover, if* $\Omega = \mathbb{R}^3$ *then* $(\mathbf{x}_0, t_0)$ *is a regular point of* $\mathbf{v}$.

In Theorem 5 the conditions on velocity $\mathbf{v}$ (12) are imposed only on U_r^ρ. They are not the usual Prodi-Serrin's conditions, since $a \geq 3$ instead of usually used $a \geq 2$. In Theorem 6 this restrictive assumption is removed and the usual Prodi-Serrin's conditions with $\tilde{a} \geq 2$ are used on $U_r^{\rho+\kappa}$. However, an additional assumption $\alpha \geq 5/4$ for pressure is prescribed on an arbitrarily narrow strip $V_r^{\rho+\kappa} \setminus V_r^\rho$.

Before proving Theorem 5 and Theorem 6, we present a few definitions and considerations. For the sake of simplicity, we use the notation $L^p(A)$ throughout the paper instead of $L^p(A)^3$ (similarly $W^{m,p}(A)$ instead of $W^{m,p}(A)^3$ and so on) if spaces of vector functions are considered. As in [6] define new coordinates

$$\mathbf{x}' = \frac{\mathbf{x} - \mathbf{x}_0}{\sqrt{t_0 - t}}, \quad t' = \ln \frac{\sigma_0}{t_0 - t}. \tag{17}$$

Then

$$t = t_0 - \sigma_0 e^{-t'} \quad \text{and} \quad \mathbf{x} = \mathbf{x}_0 + \sqrt{\sigma_0} e^{-t'/2} \mathbf{x}'. \tag{18}$$

If we denote

$$\begin{aligned} U_r'^{\rho} &= \{(\mathbf{x}', t') \in \mathbb{R}^3 \times (0, \infty); t' > 0, \rho < |\mathbf{x}'| < \rho e^{t'/2}\}, \\ V_r'^{\rho} &= \{(\mathbf{x}', t') \in \mathbb{R}^3 \times (0, \infty); t' > 0, |\mathbf{x}'| < \rho\}, \end{aligned}$$

then we have

$$(\mathbf{x}, t) \in U_r^{\rho} \Longleftrightarrow (\mathbf{x}', t') \in U_r'^{\rho}, \quad (\mathbf{x}, t) \in V_r^{\rho} \Longleftrightarrow (\mathbf{x}', t') \in V_r'^{\rho}. \tag{19}$$

Define functions $\mathbf{v}'$, p' by the equations

$$\mathbf{v}'(\mathbf{x}', t') = \sqrt{t_0 - t}\ \mathbf{v}(\mathbf{x}, t), \quad p'(\mathbf{x}', t') = (t_0 - t)\ p(\mathbf{x}, t). \tag{20}$$

Then $(\mathbf{v}', p')$ is a suitable weak solution of the problem

$$\begin{aligned} \frac{\partial \mathbf{v}'}{\partial t'} - \nu \Delta' \mathbf{v}' + \mathbf{v}' \cdot \nabla' \mathbf{v}' + \nabla' p' &= -\mathbf{v}'/2 - \mathbf{x}' \cdot \nabla' \mathbf{v}'/2, \\ \nabla' \cdot \mathbf{v}' &= 0 \end{aligned}$$

in $\{(\mathbf{x}', t') \in \mathbb{R}^3 \times (0, \infty); t' > 0, |\mathbf{x}'| < \rho e^{t'/2}\}$ and satisfies the generalized energy inequality

$$\begin{aligned} 2\nu \int_0^\infty \int_{\mathbb{R}^3} |\nabla' \mathbf{v}'|^2 \phi\ d\mathbf{x}'\ dt' \leq \int_0^\infty \int_{\mathbb{R}^3} \Big[|\mathbf{v}'|^2 \Big(\frac{\partial \phi}{\partial t'} + \nu \Delta' \phi \Big) \\ + (|\mathbf{v}'|^2 + 2p') \mathbf{v}' \cdot \nabla' \phi + |\mathbf{v}'|^2 \phi/2 + (\mathbf{x}' \cdot \nabla' \phi) |\mathbf{v}'|^2/2 \Big]\ d\mathbf{x}'\ dt' \end{aligned} \tag{21}$$

for every non-negative real-valued function $\phi \in C_0^\infty(\{(\mathbf{x}', t') \in \mathbb{R}^3 \times (0, \infty); t' > 0, |\mathbf{x}'| < \rho e^{t'/2}\})$. Moreover, it follows from (17)–(20) that

$$\|\mathbf{v}\|_{L^{a,b}(U_r^{\rho})} = \|\mathbf{v}'\|_{L^{a,b}(U_r'^{\rho})}, \quad \|p\|_{L^{\alpha,\beta}(V_r^{\rho+\kappa} \setminus V_r^{\rho})} = \|p'\|_{L^{\alpha,\beta}(V_r'^{\rho+\kappa} \setminus V_r'^{\rho})}, \tag{22}$$

if $a \geq 2$, $b \geq 3$, $2/a + 3/b = 1$ and $\alpha \geq 1$, $\beta \geq 3/2$, $2/\alpha + 3/\beta = 2$.

Lemma 1. *Let $\vartheta \in (0,1)$ and $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$. Then there exist absolute constants $\epsilon_1 > 0$ and $C_0 > 0$ with the following property. Suppose that $(\mathbf{v}, p)$ is a suitable weak solution to the Navier-Stokes equations on $Q_r^1 = Q_r^1(\mathbf{x}, t) = \{(\mathbf{y}, \tau); |\mathbf{x} - \mathbf{y}| < r, t - r^2 < \tau < t\}$, $r > 0$. Suppose further that*

$$\frac{1}{r^2} \int \int_{Q_r} (|\mathbf{v}|^3 + |\mathbf{v}||p|)\ d\mathbf{y}\ d\tau + \frac{1}{r^{13/4}} \int_{t-r^2}^t (\int_{|\mathbf{x}-\mathbf{y}|<r} |p|\ d\mathbf{y})^{5/4}\ d\tau \leq \epsilon \tag{23}$$

for some $\epsilon \in (0, \epsilon_1\rangle$. *Then*

$$|\mathbf{v}| \le C_0 \epsilon^{2/3}/r \tag{24}$$

Lebesgue-almost-everywhere on $Q^1_{\vartheta r}(\mathbf{x}, t)$.

Lemma 1 was firstly declared and proved in [1] — see Proposition 1, Corollary 1 and the proof on page 789. In fact, Lemma 1 differs slightly from Proposition 1, Corollary 1. Firstly, we have $\mathbf{f} \equiv \mathbf{0}$. Secondly, Proposition 1 was proved for $\vartheta = 1/2$. However, it can be seen easily that the proof does not change if $\vartheta \in (0, 1)$. Of course, ϵ_1 and C_0 may then possibly depend on ϑ. Finally and most importantly, we have that $|\mathbf{v}| \le C_0 \epsilon^{2/3}/r$ Lebesgue-almost-everywhere on $Q_{\vartheta r}(\mathbf{x}, t)$ in Lemma 1 (C_0 independent of ϵ), which means that $\|\mathbf{v}\|_{L^\infty(Q_{\vartheta r}(\mathbf{x},t))}$ depends on ϵ. This fact is not particularly stressed in [1], but it follows directly from the proof of Proposition 1 and Corollary 1 (see Step 3 of the proof — page 792 and the final remark in the proof). Thus, the smaller ϵ we take the smaller the L^∞ norm of $\mathbf{v}$ we have and this fact will be used in the proof of Theorem 6.

Remark 1. *Let* $(\mathbf{y}_0, \tau_0) \in Q_T$ *be a regular point of* $\mathbf{v}$. *It is known (see for instance* [2]*) that there exist* $\epsilon > 0$ *and* $\delta > 0$ *such that* $D^\gamma_{\mathbf{x}} \frac{\partial \mathbf{v}}{\partial t}$, $D^\gamma_{\mathbf{x}} p \in L^\alpha(\tau_0 - \epsilon, \tau_0 + \epsilon, L^\infty(B_{\delta_1}(\mathbf{y}_0)))$ *for every multi-index* $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, *where* $D^\gamma_{\mathbf{x}} = \frac{\partial^{|\gamma|}}{\partial \mathbf{x}_1^{\gamma_1} \cdots \partial \mathbf{x}_3^{\gamma_3}}$, $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$, *every* $\delta_1 \in (0, \delta)$ *and* $\alpha \in \langle 1, 2)$. *In the case* $\Omega = \mathbb{R}^3$, α *can be even taken from the interval* $\langle 1, \infty)$. *We will use this fact at the end of the proof of Theorem 6. It will enable us to conclude that* $(\mathbf{x}_0, t_0)$ *is a regular point of* $\mathbf{v}$. *Unfortunately, in the case of* Ω *being a bounded domain in* $\mathbb{R}^3$ *(and thus* $\alpha < 2$*) we are not sure whether the same procedure can be used or not and therefore cannot deduce the regularity of* $(\mathbf{x}_0, t_0)$.

The following lemma (see e.g. [5]) will be useful in connection with the cut-off function technique.

Lemma 2. *Let* $D \subset \mathbb{R}^3$ *be a bounded Lipschitz domain,* $r \in \langle 1, \infty)$ *and* $m \in N \cup \{0\}$. *Then there exists a linear operator* R *from* $W^{m,r}_0(D)$ *into* $W^{m+1,r}_0(D)^3$ *such that for every* $f \in W^{m,r}_0(D)$

$$\begin{aligned} &\operatorname{div}\ Rf = f, \quad \textit{if} \int_D f\, d\mathbf{x} = 0, \\ &\|\nabla^{m+1} Rf\|_{L^r(D)} \le c \|\nabla^m f\|_{L^r(D)}. \end{aligned} \tag{25}$$

In addition, if f *has a compact support in* D *then also* Rf *has a compact support in* D.

Proof of Theorem 5: The proof is based on the generalized energy inequality (21). We choose a suitable test function ϕ, estimate the right hand side of (21) and obtain the inequality (42). Using then standard embedding theorems we get the estimate (48) for velocity $\mathbf{v}$ which together with the analogical estimate for pressure (49) leads to the use of the famous Lin's result (see the paragraph around (56)) and the proof is then easily completed.

Thus, let $t'_1 \geq 2$, $t'_2 > 2t'_1$ and $\epsilon > 0$ and suppose without loss of generality that $\kappa \leq \rho/2$. We use the generalized energy inequality (21) with the function $\phi(\mathbf{x}', t') = \xi(t')\chi(|\mathbf{x}'|)e^{-t'/2}$, where χ is an infinitely differentiable function on $\langle 0, \infty)$, $\chi(s) = 1$ for $0 \leq s \leq \rho + \kappa/3$, $\chi(s) = 0$ for $s \geq \rho + 2\kappa/3$ and χ is decreasing on $(\rho + \kappa/3, \rho + 2\kappa/3)$. ξ is defined on $(0, \infty)$ in the following way: $\xi(t') = 0$ on $(0, t'_1/2 - e^{-3t'_1/2}\rangle \cup \langle t'_2 + \epsilon, \infty)$, $\xi(t') = t' - t'_1/2 + e^{-3t'_1/2}$ on $\langle t'_1/2 - e^{-3t'_1/2}, t'_1/2\rangle$, $\xi(t') = e^{t'-2t'_1}$ on $\langle t'_1/2, 2t'_1\rangle$, $\xi(t') = 1$ on $\langle 2t'_1, t'_2\rangle$, $0 \leq \xi(t') \leq 1$ on $\langle t'_2, t'_2 + \epsilon\rangle$, ξ is decreasing on $(t'_2, t'_2 + \epsilon)$ and infinitely differentiable on $(2t'_1, \infty)$. To justify the use of (non-smooth) function ϕ in (21), it is possible to find a suitable sequence of functions $\xi_n \in C_0^\infty((0, \infty))$ such that (21) holds for $\phi_n(\mathbf{x}', t') = \xi_n(t')\chi(|\mathbf{x}'|)e^{-t'/2}$, $n \in \mathbb{N}$ and letting $n \longrightarrow \infty$ we get the validity of the generalized energy inequality also for $\phi(\mathbf{x}', t') = \xi(t')\chi(|\mathbf{x}'|)e^{-t'/2}$.

Firstly, we will estimate the terms on the right hand side of (21).

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} |\mathbf{v}'|^2 \Big(\frac{\partial \phi}{\partial t'} + \phi/2\Big)\, d\mathbf{x}'\, dt' = \int_0^\infty \int_{\mathbb{R}^3} |\mathbf{v}'|^2 \Big(-\frac{1}{2}\xi(t')\chi(|\mathbf{x}'|)e^{-t'/2} \\
&+ \xi'(t')\chi(|\mathbf{x}'|)e^{-t'/2} + \frac{1}{2}\xi(t')\chi(|\mathbf{x}'|)e^{-t'/2}\Big)\, d\mathbf{x}'\, dt' \qquad (26)\\
&= \int_{t'_1/2 - e^{-3t'_1/2}}^{t'_2+\epsilon} \xi'(t')e^{-t'/2} \int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|)\, d\mathbf{x}'\, dt'.
\end{aligned}$$

Further, we will use the inequality

$$\int_{B_1(0)} |\mathbf{u}|^2\, d\mathbf{x} \leq k_1 \Big(\int_{B_1(0)} |\nabla \mathbf{u}|^2\, d\mathbf{x} + \int_{\partial B_1(0)} |\mathbf{u}|^2\, dS \Big), \qquad (27)$$

which holds for every $\mathbf{u} \in W^{1,2}(B_1(0))$ and where k_1 is an absolute constant. It follows from (27) that

$$\int_{B_r(0)} |\mathbf{u}|^2\, d\mathbf{x} \leq k_1 r \Big(r \int_{B_r(0)} |\nabla \mathbf{u}|^2\, d\mathbf{x} + \int_{\partial B_r(0)} |\mathbf{u}|^2\, dS \Big), \qquad (28)$$

for every $\mathbf{u} \in W^{1,2}(B_r(0))$ and $r > 0$. Using (28) and the Hölder inequality we get for almost every $t' \in (0, \infty)$ that

$$\begin{aligned}
&\int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|)\, d\mathbf{x}' \leq \int_{B_{\rho+r_1(t')}(0)} |\mathbf{v}'|^2\, d\mathbf{x}' + \int_{|\mathbf{x}'|>\rho} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|)\, d\mathbf{x}' \\
&\leq k_1 \Big(\rho + r_1(t')\Big)^2 \int_{B_{\rho+r_1(t')}(0)} |\nabla' \mathbf{v}'|^2\, d\mathbf{x}' + k_1 \Big(\rho + r_1(t')\Big) \qquad (29)\\
&\times\ c_1 \Big(\int_{\partial B_{\rho+r_1(t')}(0)} |\mathbf{v}'|^b\, dS' \Big)^{2/b} +\ c_1 \Big(\int_{\rho \leq |\mathbf{x}'| \leq \rho+\kappa} |\mathbf{v}'|^b\, d\mathbf{x}' \Big)^{2/b},
\end{aligned}$$

where $r_1(t')$ is such a number from $\langle 0, \kappa/3\rangle$ that

$$\int_{\partial B_{\rho+r_1(t')}(0)} |\mathbf{v}'(\cdot,t')|^b \, dS' = \inf_{r\in\langle 0,\kappa/3\rangle} \int_{\partial B_{\rho+r}(0)} |\mathbf{v}'(\cdot,t')|^b \, dS'.$$

It follows from the continuity of $\mathbf{v}'(\cdot,t')$ in space coordinates that $r_1(t')$ is well defined for almost every $t' \in (0,\infty)$. We have from (12), (26), (29), the definition of ξ and (17)–(20) that

(30)

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} |\mathbf{v}'|^2 \Big(\frac{\partial \phi}{\partial t'} + \phi/2\Big) \, d\mathbf{x}' \, dt' \\
&\leq \int_{t_1'/2-e^{-3t_1'/2}}^{t_1'/2} e^{-t'/2} \int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \, d\mathbf{x}' \, dt' \\
&+ \int_{t_2'}^{t_2'+\epsilon} \xi'(t') e^{-t'/2} \int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|) \, d\mathbf{x}' \, dt' \\
&+ \int_{t_1'/2}^{2t_1'} e^{t'/2-2t_1'} \Bigg[k_1 \Big(\rho + r_1(t')\Big)^2 \int_{B_{\rho+r_1(t')}(0)} |\nabla' \mathbf{v}'|^2 \, d\mathbf{x}' \\
&+ k_1 c_1 \Big(\rho + r_1(t')\Big) \Big(\int_{\partial B_{\rho+r_1(t')}(0)} |\mathbf{v}'|^b \, dS' \Big)^{2/b} \\
&+ c_1 (\int_{\rho\leq|\mathbf{x}'|\leq\rho+\kappa} |\mathbf{v}'|^b \, d\mathbf{x}')^{2/b} \Bigg] \, dt' \leq \frac{\|\mathbf{v}\|^2_{L^\infty(t_0-\sigma_0,t_0,L^2(\Omega))}}{\sqrt{\sigma_0}} \\
&\times \ [-\ln(t_0-t)]_{t=t_0-\sigma_0 e^{-t_1'/2+e^{-3t_1'/2}}}^{t_0-\sigma_0 e^{-t_1'/2}} \\
&+ \int_{t_2'}^{t_2'+\epsilon} \xi'(t') e^{-t'/2} \int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|) \, d\mathbf{x}' \, dt' \\
&+ k_1 (\rho+\kappa)^2 \int_{t_1'/2}^{2t_1'} e^{t'/2-2t_1'} \int_{B_{\rho+\kappa/3}(0)} |\nabla' \mathbf{v}'|^2 \, d\mathbf{x}' \, dt' + c_2 e^{-2t_1'} \\
&\times \Bigg[\Big(\int_{t_1'/2}^{2t_1'} (\int_{\partial B_{\rho+r_1(t')}(0)} |\mathbf{v}'|^b \, dS')^{a/b} \, dt' \Big)^{2/a} \\
&+ \Big(\int_{t_1'/2}^{2t_1'} (\int_{\rho\leq|\mathbf{x}'|\leq\rho+\kappa} |\mathbf{v}'|^b \, d\mathbf{x}')^{a/b} \, dt' \Big)^{2/a} \Bigg] \Big(\int_{t_1'/2}^{2t_1'} e^{\frac{at'}{2(a-2)}} \, dt' \Big)^{(a-2)/a} \\
&\leq \int_{t_2'}^{t_2'+\epsilon} \xi'(t') e^{-t'/2} \int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|) \, d\mathbf{x}' \, dt'
\end{aligned}$$

$$+\ k_1(\rho+\kappa)^2 \int_{t_1'/2}^{2t_1'} e^{t'/2-2t_1'} \int_{B_{\rho+\kappa/3}(0)} |\nabla'\mathbf{v}'|^2 \, d\mathbf{x}' \, dt' + c_3 e^{-t_1'} c_1(t_1'),$$

where

$$c_1(t_1') = \frac{\|\mathbf{v}\|_{L^\infty(t_0-\sigma_0,t_0,L^2(\Omega))}}{\sqrt{\sigma_0}} e^{-t_1'/2} \tag{31}$$
$$+ \Big(\int_{t_1'/2}^{2t_1'} (\int_{\partial B_{\rho+r_1(t')}(0)} |\mathbf{v}'|^b \, dS')^{a/b} \, dt' \Big)^{2/a}$$
$$+ \Big(\int_{t_1'/2}^{2t_1'} (\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^b \, d\mathbf{x}')^{a/b} \, dt' \Big)^{2/a}.$$

We now show that

$$\lim_{t_1'\to\infty} c_1(t_1') = 0. \tag{32}$$

Obviously, $\lim_{t_1'\to\infty} \big(\int_{t_1'/2}^{2t_1'} (\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^b \, d\mathbf{x}')^{a/b} \, dt' \big)^{2/a} = 0$, as follows from (12) and (22). Check on the second term of $c_1(t_1')$:

$$\int_1^\infty (\int_{\partial B_{\rho+r_1(t')}(0)} |\mathbf{v}'|^b \, dS')^{a/b} \, dt'$$
$$= \int_1^\infty \Big(\frac{3}{\kappa} \int_0^{\kappa/3} (\int_{\partial B_{\rho+r_1(t')}(0)} |\mathbf{v}'|^b \, dS') \, dr \Big)^{a/b} \, dt' \tag{33}$$
$$\le \int_1^\infty \Big(\frac{3}{\kappa} \int_0^{\kappa/3} (\int_{\partial B_{\rho+r}(0)} |\mathbf{v}'|^b \, dS') \, dr \Big)^{a/b} \, dt'$$
$$\le \int_1^\infty \Big(\frac{3}{\kappa} \int_{\rho\le|\mathbf{x}'|\le\rho+\kappa/3} |\mathbf{v}'|^b \, d\mathbf{x}' \Big)^{a/b} \, dt'$$
$$\le \Big(\frac{3}{\kappa}\Big)^{a/b} \|\mathbf{v}'\|^a_{L^{a,b}(U_r'^{\rho})} < \infty.$$

Therefore, the second term of $c_1(t_1')$ goes to zero if t_1' goes to infinity and (32) is proved.

Further, we can use integration by parts and get

$$\int_{t_2'}^{t_2'+\epsilon} \xi'(t') e^{-t'/2} \int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|) \, d\mathbf{x}' \, dt'$$
$$= \Big[\xi(t') e^{-t'/2} \int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|) \, d\mathbf{x}' \Big]_{t'=t_2'}^{t_2'+\epsilon}$$
$$- \int_{t_2'}^{t_2'+\epsilon} \xi(t') \frac{d}{dt'} \Big(e^{-t'/2} \int_{B_{\rho+\kappa}(\mathbf{x}_0)} |\mathbf{v}'|^2 \chi(|\mathbf{x}'|) \, d\mathbf{x}' \Big) \, dt'$$

for almost every $t_2' \in (2t_1', \infty)$. Therefore,

$$\lim_{\epsilon\to 0}\int_{t_2'}^{t_2'+\epsilon} \xi'(t')e^{-\frac{t'}{2}}\int_{B_{\rho+\kappa}(0)} |\mathbf{v}'|^2\chi(|\mathbf{x}'|)\,d\mathbf{x}'\,dt' = -e^{-\frac{t_2'}{2}}\int_{B_{\rho+\kappa}(0)} |\mathbf{v}'(\mathbf{x}',t_2')|^2\chi(|\mathbf{x}'|)\,d\mathbf{x}'. \tag{34}$$

If we suppose that ρ is such a small number that $k_1(\rho+\kappa)^2 \le \nu$, we get from (30) and (34) that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^3} |\mathbf{v}'|^2\Big(\frac{\partial\phi}{\partial t'} + \phi/2\Big)\,d\mathbf{x}'\,dt' \\
&\le -e^{t_2'/2}\int_{B_{\rho+\kappa}(0)} |\mathbf{v}'(\mathbf{x}',t_2')|^2\chi(|\mathbf{x}'|)\,d\mathbf{x}' \\
&+\ \nu\int_{t_1'/2}^{2t_1'} e^{t'/2-2t_1'}\int_{B_{\rho+\kappa/3}(0)} |\nabla'\mathbf{v}'|^2\,d\mathbf{x}'\,dt' + c_3c_1(t_1')e^{-t_1'},
\end{aligned} \tag{35}$$

which holds for every $t_2' > 2t_1'$ since $\mathbf{v}'$ is weakly continuous as a function from $(2t_1', \infty)$ into $L^2(B_{\rho+\kappa}(0))$.

It follows from the definition of ϕ that $\mathbf{x}'\cdot\nabla'\phi \le 0$. Therefore

$$\int_0^\infty\int_{\mathbb{R}^3} (\mathbf{x}'\cdot\nabla'\phi)|\mathbf{v}'|^2/2\,d\mathbf{x}'\,dt' \le 0. \tag{36}$$

Further, using (12), the Hölder inequality gives

$$\begin{aligned}
&\Big|\int_0^\infty\int_{\mathbb{R}^3} |\mathbf{v}'|^2\mathbf{v}'\cdot\nabla'\phi\,d\mathbf{x}'\,dt'\Big| \\
&\le c_4\int_{t_1'/2-e^{-3t_1'/2}}^{t_1'/2} e^{-(3t_1'+t')/2}\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^3\,d\mathbf{x}'\,dt' \\
&+\ c_4\int_{t_1'/2}^{2t_1'} e^{(t'-4t_1')/2}\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^3\,d\mathbf{x}'\,dt' \\
&+ c_4\int_{2t_1'}^{\infty} e^{-t'/2}\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^3\,d\mathbf{x}'\,dt' \\
&\le c_5e^{-3t_1'/2}e^{-t_1'/4}\Big(\int_{t_1'/2-e^{-3t_1'/2}}^{t_1'/2}\Big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^b\,d\mathbf{x}'\Big)^{a/b}\,dt'\Big)^{3/a} \\
&+\ c_6e^{-2t_1'}e^{t_1'}\Big(\int_{t_1'/2}^{2t_1'}\Big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^b\,d\mathbf{x}'\Big)^{a/b}\,dt'\Big)^{3/a} \\
&+\ c_7e^{-t_1'}\Big(\int_{2t_1'}^{\infty}\Big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa} |\mathbf{v}'|^b\,d\mathbf{x}'\Big)^{a/b}\,dt'\Big)^{3/a} \le c_8c_2(t_1')e^{-t_1'},
\end{aligned} \tag{37}$$

where

$$
(38)\qquad
\begin{aligned}
c_2(t_1') = e^{-3t_1'/4}\Big(&\int_{t_1'/2-e^{-3t_1'/2}}^{t_1'/2}\big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|\mathbf{v}'|^b\,d\mathbf{x}'\big)^{a/b}\,dt'\Big)^{3/a}\\
&+\Big(\int_{t_1'/2}^{2t_1'}\big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|\mathbf{v}'|^b\,d\mathbf{x}'\big)^{a/b}\,dt'\Big)^{3/a}\\
&+\Big(\int_{2t_1'}^{\infty}\big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|\mathbf{v}'|^b\,d\mathbf{x}'\big)^{a/b}\,dt'\Big)^{3/a}
\end{aligned}
$$

and by (12) and (22) $\lim_{t_1'\to\infty} c_2(t_1') = 0$. Analogically, using (12) and (13)

$$
(39)\qquad
\begin{aligned}
&\Big|\int_0^\infty\int_{\mathbb{R}^3} 2p'\mathbf{v}'\cdot\nabla'\phi\,d\mathbf{x}'\,dt'\Big|\\
&\le c_4\int_{t_1'/2-e^{-3t_1'/2}}^{t_1'/2} e^{-(3t_1'+t')/2}\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|p'\mathbf{v}'|\,d\mathbf{x}'\,dt'\\
&+\ c_4\int_{t_1'/2}^{2t_1'} e^{(t'-4t_1')/2}\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|p'\mathbf{v}'|\,d\mathbf{x}'\,dt'\\
&+c_4\int_{2t_1'}^{\infty} e^{-t'/2}\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|p'\mathbf{v}'|\,d\mathbf{x}'\,dt' \le c_9c_3(t_1')e^{-t_1'},
\end{aligned}
$$

where

$$
(40)\qquad
\begin{aligned}
c_3(t_1') = c_{10}e^{-t_1'}&\Big(\int_{t_1'/2-e^{-3t_1'/2}}^{\infty}\big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|p'|^\beta\,d\mathbf{x}'\big)^{\alpha/\beta}\,dt'\Big)^{1/\alpha}\\
\times\ &\Big(\int_{t_1'/2-e^{-3t_1'/2}}^{\infty}\big(\int_{\rho\le|\mathbf{x}'|\le\rho+\kappa}|\mathbf{v}'|^b\,d\mathbf{x}'\big)^{a/b}\,dt'\Big)^{1/a}
\end{aligned}
$$

and by (12), (13) and (22) $\lim_{t_1'\to\infty} c_3(t_1') \longrightarrow 0$. To estimate the term

$$
(41)\qquad \nu\int_0^\infty\int_{\mathbb{R}^3}|\mathbf{v}'|^2\Delta'\phi\,d\mathbf{x}'\,dt'
$$

we proceed in the same way as above and get a similar estimate as in (37). It can be concluded from (21) and (35)–(41) that

$$
(42)\qquad
\begin{aligned}
\nu\int_0^\infty\int_{\mathbb{R}^3}|\nabla'\mathbf{v}'|^2\phi\,d\mathbf{x}'\,dt' + e^{-t_2'/2}\int_{B_{\rho+\kappa}(0)}|\mathbf{v}'(\mathbf{x}',t_2')|^2\chi(|\mathbf{x}'|)\,d\mathbf{x}'&\\
\le c_{11}c_4(t_1')e^{-t_1'},&
\end{aligned}
$$

where $\lim_{t_1'\to\infty} c_4(t_1') \longrightarrow 0$ and c_{11} is an absolute constant independent of t_1 and t_2.

Secondly, let $\delta \in (0,r)$ be sufficiently small and let $\tau = 2\ln(r/\delta)$. Put $t_1' = \tau/2$. If $\frac{2}{\bar a} + \frac{3}{\bar b} > \frac{3}{2}$ and $\bar b \in (2,6)$ then by (17)–(20), (42) and by classical embedding theorems

$$
\begin{aligned}
&\int_{t_0-\delta^2/\rho^2} \Big(\int_{B_{\rho\sqrt{t_0-t}}} |\mathbf{v}(\mathbf{x},t)|^{\bar b}\, d\mathbf{x}\Big)^{\bar a/\bar b}\, dt \\
&= \int_\tau^\infty e^{\bar a t'(1-2/\bar a-3/\bar b)/2}\Big(\int_{B_\rho} |\mathbf{v}'(\mathbf{x}',t')|^{\bar b}\, d\mathbf{x}'\Big)^{\bar a/\bar b}\, dt' \\
&\le \int_\tau^\infty e^{\bar a t'(1-\frac{2}{\bar a}-\frac{3}{\bar b})/2}\Big(\int_{B_\rho} |\mathbf{v}'(\mathbf{x}',t')|^2\, d\mathbf{x}'\Big)^{\bar a(3/\bar b-1/2)/2} \\
&\times \Big[\Big(\int_{B_\rho} |\nabla'\mathbf{v}'(\mathbf{x}',t')|^2\, d\mathbf{x}'\Big)^{\bar a(3/2-3/\bar b)/2}\, dt' \\
&+ \Big(\int_{B_\rho} |\mathbf{v}'(\mathbf{x}',t')|^2\, d\mathbf{x}'\Big)^{\bar a(3/2-3/\bar b)/2}\Big]\, dt' \\
&\le (c_4(t_1')e^{-t_1'})^{\bar a(3/\bar b-1/2)/2}\int_\tau^\infty e^{\bar a t'(\frac{3}{2}-\frac{2}{\bar a}-\frac{3}{\bar b})/2}\Big(e^{-t'/2} \\
&\times \int_{B_\rho} |\nabla'\mathbf{v}'(\mathbf{x}',t')|^2\, d\mathbf{x}'\Big)^{\bar a(3/2-3/\bar b)/2}\, dt' \\
&+ \int_\tau^\infty e^{\bar a t'(1-\frac{2}{\bar a}-\frac{3}{\bar b})/2}\Big(\int_{B_\rho} |\mathbf{v}'(\mathbf{x}',t')|^2\, d\mathbf{x}'\Big)^{\bar a/2}\, dt' \\
&\le (c_4(t_1')e^{-t_1'})^{\bar a(3/\bar b-1/2)/2} e^{\bar a t_1'(3/2-2/\bar a-3/\bar b)} \\
&\times \Big(\int_\tau^\infty e^{-t'/2}\int_{B_\rho} |\nabla'\mathbf{v}'(\mathbf{x}',t')|^2\, d\mathbf{x}'\, dt'\Big)^{(3\bar a\bar b-6\bar a)/4\bar b} \\
&+ \int_\tau^\infty e^{\bar a t'(\frac{3}{2}-\frac{2}{\bar a}-\frac{3}{\bar b})/2}\Big(e^{-t'/2}\int_{B_\rho} |\mathbf{v}'(\mathbf{x}',t')|^2\, d\mathbf{x}'\Big)^{\bar a/2}\, dt' \\
&\le (c_4(t_1')e^{-t_1'})^{\bar a(3/\bar b-1/2)/2} e^{\bar a t_1'(3/2-2/\bar a-3/\bar b)} (c_4(t_1')e^{-t_1'})^{(3\bar a\bar b-6\bar a)/4\bar b} \\
&+ (c_4(t_1')e^{-t_1'})^{\bar a/2}\int_\tau^\infty e^{\bar a t'(\frac{3}{2}-\frac{2}{\bar a}-\frac{3}{\bar b})/2}\, dt' \le c_4(t_1')^{\bar a/2}\delta^{\bar a(2/\bar a+3/\bar b-1)}.
\end{aligned}
\tag{43}
$$

Consequently,

$$
\lim_{\delta\to 0_+} \frac{1}{\delta^{\bar a(2/\bar a+3/\bar b-1)}}\int_{t_0-\delta^2/\rho^2} \Big(\int_{B_{\rho\sqrt{t_0-t}}} |\mathbf{v}(\mathbf{x},t)|^{\bar b}\, d\mathbf{x}\Big)^{\bar a/\bar b}\, dt \le \lim_{\delta\to 0_+} c_4\Big(\ln(r/\delta)\Big)^{\bar a/2} = 0.
\tag{44}
$$

We could prove in the same way that for every $\omega \in \langle 0,1)$

$$\lim_{\delta\to 0_+} \frac{1}{\delta^{\bar a(2/\bar a+3/\bar b-1)}} \int_{t_0-\delta^2/(\rho+\omega\kappa)^2} \Big(\int_{B_{(\rho+\omega\kappa)\sqrt{t_0-t}}} |\mathbf{v}(\mathbf{x},t)|^{\bar b}\ d\mathbf{x}\Big)^{\bar a/\bar b}\ dt = 0. \tag{45}$$

If we put $\bar a = \bar b = 3$, (45) gives

$$\lim_{\delta\to 0_+} \frac{1}{\delta^2} \int\int_{V_\delta^{\rho+\omega\kappa}} |\mathbf{v}(\mathbf{x},t)|^3\ d\mathbf{x}\ dt = 0. \tag{46}$$

Further,

$$\begin{aligned}
&\frac{1}{\delta^2}\int\int_{U_\delta^\rho} |\mathbf{v}(\mathbf{x},t)|^3\ d\mathbf{x}\ dt\\
&\le \frac{c_{13}}{\delta^2}\int_{t_0-\delta^2/\rho^2}^{t_0}\Big(\int_{\rho\sqrt{t_0-t}<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}(\mathbf{x},t)|^b\ d\mathbf{x}\Big)^{3/b}\delta^{3(b-3)/b}\ dt\\
&\le \frac{c_{14}}{\delta^2}\Big(\int_{t_0-\delta^2/\rho^2}^{t_0}\Big(\int_{\rho\sqrt{t_0-t}<|\mathbf{x}-\mathbf{x}_0|<\delta} |\mathbf{v}(\mathbf{x},t)|^b\ d\mathbf{x}\Big)^{a/b}\ dt\Big)^{3/a}\delta^{3(b-3)/b+2(a-3)/a}\\
&= c_{14}\|\mathbf{v}\|^3_{L^{a,b}(U_\delta^\rho)} \longrightarrow 0, \text{ if } \delta\to 0.
\end{aligned} \tag{47}$$

It follows from (46) and (47) that

$$\lim_{\delta\to 0_+} \frac{1}{\delta^2} \int\int_{Q_\delta^1} |\mathbf{v}(\mathbf{x},t)|^3\ d\mathbf{x}\ dt = 0. \tag{48}$$

Derive now that

$$\lim_{\delta\to 0_+} \frac{1}{\delta^2} \int\int_{Q_\delta^1} |p(\mathbf{x},t)|^{3/2}\ d\mathbf{x}\ dt = 0. \tag{49}$$

We present a proof which was used in [7]. Let $\theta > 2$. It is possible to prove that for almost every $t \in ((t_0-(\delta/\theta)^2, t_0)$

$$\begin{aligned}
\int_{B_{\delta/\theta}(\mathbf{x}_0)} |p|^{3/2}\ d\mathbf{x} \le c_{15}\int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3\ d\mathbf{x} + \frac{c_{16}}{\theta^3}\int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}|^3\ d\mathbf{x}\\
+ \frac{c_{16}}{\theta^3}\int_{B_\delta(\mathbf{x}_0)} |p|^{3/2}\ d\mathbf{x},
\end{aligned} \tag{50}$$

where c_{15}, c_{16} are independent of t. Integrating (50) with respect to t on $(t_0 - (\delta/\theta)^2, t_0)$ and dividing then the inequality by $(\delta/\theta)^2$, we obtain

$$\begin{aligned}
\frac{\theta^2}{\delta^2}\int_{Q^1_{\delta/\theta}} |p|^{3/2}\ d\mathbf{x}\ dt \le \Big(c_{15}\,\theta^2 + \frac{c_{16}}{\theta}\Big)\frac{1}{\delta^2}\int_{Q^1_\delta} |\mathbf{v}|^3\ d\mathbf{x}\ dt\\
+ \frac{c_{16}}{\theta}\frac{1}{\delta^2}\int_{Q^1_\delta} |p|^{3/2}\ d\mathbf{x}\ dt.
\end{aligned} \tag{51}$$

Denoting

$$h(\delta)=\frac{1}{\delta^2}\int_{Q^1_\delta}|p|^{3/2}\;d\mathbf{x}\;dt,\quad g(\delta)=\frac{1}{\delta^2}\int_{Q^1_\delta}|\mathbf{v}|^3\;d\mathbf{x}\;dt,$$

equation (51) can be written as

$$h(\delta/\theta)\le\Big(c_{15}\,\theta^2+\frac{c_{16}}{\theta}\Big)g(\delta)+\frac{c_{16}}{\theta}h(\delta). \tag{52}$$

If we further denote

$$\lambda=1/\delta,\quad \tilde{h}(\lambda)=h(1/\lambda),\quad \tilde{g}(\lambda)=g(1/\lambda),$$

we get from (52) that

$$\tilde{h}(\theta\lambda)\le\Big(c_{15}\,\theta^2+\frac{c_{16}}{\theta}\Big)\tilde{g}(\lambda)+\frac{c_{16}}{\theta}\tilde{h}(\lambda). \tag{53}$$

To prove (49), it suffices to show that $\lim_{\lambda\to+\infty}\tilde{h}(\lambda)=0$. Verify first the boundedness of $\tilde{h}$. Without loss of generality we can suppose that $c_{16}>1$. If we put $\tilde{h}_1(\lambda)=\max\{\tilde{h}(\lambda);\eta\}$, where η is a fixed positive number and $\theta=2c_{16}$ then

$$\tilde{h}(\theta\lambda)\le\Big[\Big(c_{13}\,\theta^2+1/2\Big)\frac{\tilde{g}(\lambda)}{\eta}+1/2\Big]\tilde{h}_1(\lambda). \tag{54}$$

Since $\lim_{\lambda\to\infty}\tilde{g}(\lambda)=0$, there exists λ_0 such that $\big[\big(c_{13}\,\theta^2+1/2\big)\tilde{g}(\lambda)/\eta+1/2\big]\le 1$, $\forall\,\lambda\ge\lambda_0$, i.e.

$$\tilde{h}(\theta\lambda)\le\tilde{h}_1(\lambda),\ \forall\,\lambda\ge\lambda_0. \tag{55}$$

Further, there exists $L\ge\eta$ such that $\tilde{h}_1(\lambda)\le L$ on the interval $\langle\lambda_0,2c_{16}\lambda_0\rangle$, as follows from the fact that $p\in L^{3/2}\big((\delta^*,T)\times\Omega\big)$ for any positive δ^* (see [3]). Thus, as a result of (55), $\tilde{h}$ is bounded by L on $\langle 2c_{16}\lambda_0,4c_{16}^2\lambda_0\rangle$ and therefore by the definition also $\tilde{h}_1$ is bounded by L on $\langle 2c_{16}\lambda_0,4c_{16}^2\lambda_0\rangle$. Proceeding further in this way we get that $\tilde{h}(\lambda)\le L$, $\forall\,\lambda\ge\lambda_0$.

If we return to (53) and use the boundedness of $\tilde{h}$ we get that $\limsup_{\lambda\to\infty}\tilde{h}(\theta\lambda)\le c_{16}L/\theta$. Since θ can be chosen arbitrarily large, we have $\lim_{\lambda\to+\infty}\tilde{h}(\lambda)=0$ and (49) follows immediately.

To finish the proof of Theorem 5 we use the result proved by F. Lin in [4]: There exists a positive constant ϵ_3 such that if

$$\frac{1}{\delta^2}\int\int_{Q^1_\delta}(|\mathbf{v}(\mathbf{x},t)|^3+|p(\mathbf{x},t)|^{3/2})\;d\mathbf{x}\;dt\le\epsilon_3 \tag{56}$$

for some $\delta > 0$ then $\mathbf{v} \in L^\infty(Q^1_{\delta/2})$. The first part of Theorem 5 thus follows from (48), (49) and (56). The regularity of $(\mathbf{x}_0, t_0)$ can be now proved exactly in the same way as is done in detail at the end of the proof of Theorem 6. The proof of Theorem 5 is complete. □

Proof of Theorem 6: We proceed in the same way as in the proof of Theorem 5 until the relation (46). Unfortunately, (47) was proved under the assumption that $a \geq 3$, which is not the case now (we have only $\tilde{a} \geq 2$). Therefore, we are not able to obtain equation (48) which is key for the use of the Lin's result mentioned earlier. Thus, we proceed in the following way. It holds for almost every $t \in (t_0 - r^2/(\rho+\kappa/2)^2, t_0)$ and every $\mathbf{x} \in B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)$ that

$$p(\mathbf{x}, t) = p^I(\mathbf{x}, t) + p^{II}(\mathbf{x}, t), \tag{57}$$

where

$$|p^{II}(\mathbf{x}, t)| \leq \frac{c}{\kappa^3 (t_0 - t)^{3/2}} \int_{d(t)} (|\mathbf{v}|^2 + |p|)\, d\mathbf{y}, \tag{58}$$

$d(t) = B_{(\rho+\kappa)\sqrt{t_0-t}}(\mathbf{x}_0) \setminus B_{\rho\sqrt{t_0-t}}(\mathbf{x}_0)$ and

$$\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p^I|^q \, d\mathbf{x} \leq c(q) \int_{B_{(\rho+3\kappa/4)\sqrt{t_0-t}}(\mathbf{x}_0)} |\mathbf{v}|^{2q} \, d\mathbf{x}, \quad q > 1. \tag{59}$$

For more detailed description of these facts see [1, p. 782] and [7, Lemma 1]. Prove now that

$$\begin{aligned} \lim_{\delta \to 0_+} \Big(& \frac{1}{\delta^2} \int \int_{V_\delta^{\rho+\kappa/2}} (|\mathbf{v}(\mathbf{x}, t)|^3 + |\mathbf{v}||p|) \, d\mathbf{x}\, dt \\ & + \frac{1}{\delta^{13/4}} \int_{t_0 - \delta^2/(\rho+\kappa/2)^2}^{t_0} \Big(\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}} |p| \, d\mathbf{x} \Big)^{5/4} dt \Big) = 0. \end{aligned} \tag{60}$$

We begin with the term

$$\lim_{\delta \to 0_+} \frac{1}{\delta^2} \int \int_{V_\delta^{\rho+\kappa/2}} |\mathbf{v}||p| \, d\mathbf{x}\, dt = 0. \tag{61}$$

Thus, we have using (57), (58) and (59) and taking $q \in (3, 18/5)$, $\bar{a} > \alpha/(\alpha-1)$ and $\bar{b} > \beta/(\beta - 1)$

(62)

$$\frac{1}{\delta^2} \int \int_{V_\delta^{\rho+\kappa/2}} |\mathbf{v}||p| \, d\mathbf{x}\, dt \leq \frac{1}{\delta^2} \int \int_{V_\delta^{\rho+\kappa/2}} |\mathbf{v}|(|p^{II}| + |p^I|) \, d\mathbf{x}\, dt$$

$$\leq \frac{1}{\delta^2}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} (\frac{c}{\kappa^3(t_0-t)^{3/2}}\int_{d(t)}(|\mathbf{v}|^2+|p|)\ d\mathbf{x})$$
$$\times (\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |\mathbf{v}|\ d\mathbf{x})\ dt$$
$$+\frac{1}{\delta^2}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} |\mathbf{v}|_q(\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p^I|^{q/2}\ d\mathbf{x})^{2/q}(t_0-t)^{3(q-3)/2q}\ dt$$
$$\leq \frac{1}{\delta^2}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} c(t_0-t)^{3[(\beta-1)/\beta+(\bar{b}-1)/\bar{b}-1]/2}$$
$$\times |p|_{\beta,d(t)}|\mathbf{v}|_{\bar{b},B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)}\ dt$$
$$+\frac{1}{\delta^2}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} c(t_0-t)^{3[(b-2)/b+(\bar{b}-1)/\bar{b}-1]/2}$$
$$\times |\mathbf{v}|^2_{b,d(t)}|\mathbf{v}|_{\bar{b},B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)}\ dt$$
$$+\frac{1}{\delta^2}(\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} |\mathbf{v}|^3_{q,B_{(\rho+3\kappa/4)\sqrt{t_0-t}}(\mathbf{x}_0)}\ dt)\delta^{3(q-3)/q}$$
$$\leq (|p|_{L^{\alpha,\beta}(V^{\rho+\kappa}_{\delta(\rho+\kappa)/(\rho+\kappa/2)})}+|\mathbf{v}|^2_{L^{a,b}(V^{\rho+\kappa}_{\delta(\rho+\kappa)/(\rho+\kappa/2)})})$$
$$\times (\frac{1}{\delta^{\bar{a}(2/\bar{a}+3/\bar{b}-1)}}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0}(\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |\mathbf{v}|^{\bar{b}}\ d\mathbf{x})^{\bar{a}/\bar{b}}\ dt)^{1/\bar{a}}$$
$$+\frac{1}{\delta^{3(3/q-1/3)}}(\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} |\mathbf{v}|^3_{q,B_{(\rho+3\kappa/4)\sqrt{t_0-t}}(\mathbf{x}_0)}\ dt).$$

(61) now follows from (45), where we use firstly the pair $(\bar{a},\bar{b})$ and then $(3,q)$. Show now that

$$\lim_{\delta\to 0_+}\frac{1}{\delta^{13/4}}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0}(\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p|\ d\mathbf{x})^{5/4}\ dt=0. \tag{63}$$

It follows again from (57), (58) and (59) that

$$\frac{1}{\delta^{13/4}}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0}(\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p|\ d\mathbf{x})^{5/4}\ dt$$
$$\leq \frac{1}{\delta^{13/4}}\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0}((\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p^I|\ d\mathbf{x})^{5/4}$$
$$+(\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p^{II}|\ d\mathbf{x})^{5/4})\ dt$$

$$\leq \frac{1}{\delta^{13/4}} \int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} (\int_{d(t)} (|\mathbf{v}|^2 + |p|)\ d\mathbf{x})^{5/4}\ dt$$

$$+ \frac{1}{\delta^{13/4}} \int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} ((\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p^I|^{q/2}\ d\mathbf{x})^{2/q}$$

$$\times (t_0-t)^{3(q/2-1)/q})^{5/4}\ dt$$

$$\leq \frac{1}{\delta^{13/4}} \int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} ((t_0-t)^{15(\beta-1)/8\beta} |p|_{\beta,d(t)}^{5/4}$$

$$+ (t_0-t)^{15(b-2)/8b} |\mathbf{v}|_{b,d(t)}^{5/2})\ dt$$

$$+ \frac{1}{\delta^{13/4}} \int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} (t_0-t)^{15(q/2-1)/4q} |\mathbf{v}|_{q,B_{(\rho+3\kappa/4)\sqrt{t_0-t}}(\mathbf{x}_0)}^{5/2}\ dt$$

$$\leq \frac{1}{\delta^{13/4}} |p|_{L^{\alpha,\beta}(V^{\rho+\kappa}_{\delta(\rho+\kappa)/(\rho+\kappa/2)})}^{5/4} (\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0}$$

$$\times (t_0-t)^{15\alpha(\beta-1)/2\beta(4\alpha-5)}\ dt)^{(4\alpha-5)/4\alpha}$$

$$+ \frac{1}{\delta^{13/4}} |\mathbf{v}|_{L^{a,b}(V^{\rho+\kappa}_{\delta(\rho+\kappa)/(\rho+\kappa/2)})}^{5/2}$$

$$\times (\int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} (t_0-t)^{30a(b-2)/8b(2a-5)}\ dt)^{(2a-5)/2a}$$

$$+ \frac{1}{\delta^{(15/q-1)/2}} \int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} |\mathbf{v}|_{q,B_{(\rho+3\kappa/4)\sqrt{t_0-t}}(\mathbf{x}_0)}^{5/2}\ dt$$

$$= |p|_{L^{\alpha,\beta}(V^{\rho+\kappa}_{\frac{\delta(\rho+\kappa)}{\rho+\kappa/2}})}^{5/4} + |\mathbf{v}|_{L^{a,b}(V^{\rho+\kappa}_{\frac{\delta(\rho+\kappa)}{\rho+\kappa/2}})}^{5/2}$$

$$+ \frac{1}{\delta^{(15/q-1)/2}} \int_{t_0-\delta^2/(\rho+\kappa/2)^2}^{t_0} |\mathbf{v}|_{q,B_{(\rho+3\kappa/4)\sqrt{t_0-t}}(\mathbf{x}_0)}^{5/2}\ dt$$

and (63) follows from (15), (16) and (45). In the previous paragraph we used the assumption that $\alpha \geq 5/4$. (60) now follows now from (46), (61) and (63).

Due to (60) we have

(64)

$$\frac{1}{\delta^2[1+(\rho+\kappa/2)^2]} \int\int_{V^{\rho+\kappa/2}_{\delta\sqrt{1+(\rho+\kappa/2)^2}}} (|\mathbf{v}(\mathbf{x},t)|^3 + |\mathbf{v}(\mathbf{x},t)||p(\mathbf{x},t)|)\ d\mathbf{x}\ dt$$

$$+ \frac{1}{(\delta\sqrt{[1+(\rho+\kappa/2)^2]})^{13/4}} \int_{t_0-\frac{\delta^2[1+(\rho+\kappa/2)^2]}{(\rho+\kappa/2)^2}}^{t_0}$$

$$(\int_{B_{(\rho+\kappa/2)\sqrt{t_0-t}}(\mathbf{x}_0)} |p(\mathbf{x},t)|\ d\mathbf{x})^{5/4}\ dt$$

$$= c(\delta\sqrt{1+(\rho+\kappa/2)^2}) \longrightarrow 0, \text{ if } \delta \to 0.$$

Since

$$Q_\delta(\mathbf{x}_0, t_0 - \delta^2/(\rho+\kappa/2)^2) \subset V^{\rho+\kappa/2}_{\delta\sqrt{1+(\rho+\kappa/2)^2}},$$

it follows from (64) that

$$\begin{aligned}
&\frac{1}{\delta^2}\int\int_{Q_\delta(\mathbf{x}_0,t_0-\delta^2/(\rho+\kappa/2)^2)} (|\mathbf{v}(\mathbf{x},t)|^3 + |\mathbf{v}(\mathbf{x},t)||p(\mathbf{x},t)|)\ d\mathbf{x}\ dt \\
&+\frac{1}{\delta^{13/4}}\int_{t_0-\frac{\delta^2[1+(\rho+\kappa/2)^2]}{(\rho+\kappa/2)^2}}^{t_0-\frac{\delta^2}{(\rho+\kappa/2)^2}} (\int_{B_\delta(\mathbf{x}_0)} |p(\mathbf{x},t)|\ d\mathbf{x})^{5/4}\ dt \qquad (65)\\
&\le [1+(\rho+\kappa/2)^2]^{13/8}\ c(\delta\sqrt{1+(\rho+\kappa/2)^2}).
\end{aligned}$$

Choosing now δ so small that the right hand side of (65) is smaller than ϵ_1 from Lemma 1, it follows from Lemma 1 that

$$|\mathbf{v}| \le C_0\{[1+(\rho+\kappa/2)^2]^{13/8}\ c(\delta\sqrt{1+(\rho+\kappa/2)^2})\}^{2/3}/\delta = \tilde{c}(\delta)/\delta, \qquad (66)$$

almost everywhere on $Q_{\delta\rho/(\rho+\kappa/2)}(\mathbf{x}_0, t_0-\delta^2/(\rho+\kappa/2)^2)$ with $\lim_{\delta\to 0_+}\tilde{c}(\delta)=0$. Further, we use the fact that $\mathbf{v}(\cdot,t)$ is a smooth function for almost every t and obtain that for every such $t = t_0 - \frac{\delta^2}{(\rho+\kappa/2)^2}$ we have $|\mathbf{v}(\mathbf{x}, t_0-\delta^2/(\rho+\kappa/2)^2)| \le \tilde{c}(\delta)/\delta$ for every $\mathbf{x}\in B_{\rho\delta/(\rho+\kappa/2)}(\mathbf{x}_0)$. As a result of this we get that for $r>0$ sufficiently small

$$|\mathbf{v}(\mathbf{x},t)| \le \tilde{c}((\rho+\kappa/2)\sqrt{t_0-t})/((\rho+\kappa/2)\sqrt{t_0-t}) = c^*(\sqrt{t_0-t})/\sqrt{t_0-t}, \qquad (67)$$

almost everywhere in V^ρ_r and $\lim_{s\to 0_+} c^*(s)=0$. Therefore, we can write

$$\begin{aligned}
&(\int_{B_{\rho\sqrt{t_0-t}}} |\mathbf{v}(\mathbf{x},t)|^3\ d\mathbf{x})^{1/3} \\
&\qquad\le (\int_{B_{\rho\sqrt{t_0-t}}} |c^*(\sqrt{t_0-t})/\sqrt{t_0-t}|^3\ d\mathbf{x})^{1/3} = \rho c^*(\sqrt{t_0-t}),
\end{aligned}$$

which means that

$$\|\mathbf{v}\|_{L^{\infty,3}(V^\rho_r)} \le \rho c^*(r/\rho). \qquad (68)$$

for every $r>0$ sufficiently small.

It follows from (14), (15) and (68) that if δ is sufficiently small then $\mathbf{v}$ on Q^1_δ can be written as the sum

$$\mathbf{v} = \mathbf{v}^1 + \mathbf{v}^2 + \mathbf{v}^3,$$
$$\mathbf{v}^1 \in L^{\infty,3}(Q^1_\delta),$$
$$\|\mathbf{v}^1\|_{L^{\infty,3}(Q^1_\delta)} \text{ can be made arbitrarily small by making } \delta \text{ sufficiently small,} \tag{69}$$
$$\mathbf{v}^2 \in L^{a,b}(Q^1_\delta), \quad 2/a + 3/b \le 1,\ a \ge 3,\ b > 3, \tag{70}$$
$$\mathbf{v}^3 \in L^{\tilde{a},\tilde{b}}(Q^1_\delta), \quad 2/\tilde{a} + 3/\tilde{b} \le 1,\ \tilde{a} \ge 2,\ \tilde{b} > 3. \tag{71}$$

We use the fact that $(\mathbf{v}, p)$ is a suitable weak solution to (1)–(4). Then as was explained in detail in [5] there exist δ_1, δ_2 such that $\delta/2 < \delta_1 < \delta_2 < \delta$ and the set $(\overline{B}_{\delta_2}(\mathbf{x}_0) \setminus B_{\delta_1}(\mathbf{x}_0)) \times (0, T)$ does not contain any singular point of $\mathbf{v}$. It follows from [2] that if $D = (B_{\delta_2}(\mathbf{x}_0) \setminus \overline{B}_{\delta_1}(\mathbf{x}_0))$ then

$$D^\gamma_{\mathbf{x}}\mathbf{v} \in L^\infty(t_0 - \delta^2, t_0 + \delta^2, L^\infty(D)), \tag{72}$$
$$D^\gamma_{\mathbf{x}}\frac{\partial \mathbf{v}}{\partial t}, D^\gamma_{\mathbf{x}}p \in L^\alpha(t_0 - \delta^2, t_0 + \delta^2, L^\infty(D)), \tag{73}$$

for every multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, $D^\gamma_{\mathbf{x}} = \frac{\partial^{|\gamma|}}{\partial \mathbf{x}_1^{\gamma_1} \ldots \partial \mathbf{x}_3^{\gamma_3}}$, $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$ and $\alpha \in \langle 1, 2)$. Moreover, if $\Omega = \mathbb{R}^3$, then α can be taken from the interval $\langle 1, \infty)$. Let $\delta_3 \in (\delta_1, \delta_2)$. Multiplying (1) by $-\Delta\mathbf{v}$ and integrating this equation over $B_{\delta_3}(\mathbf{x}_0)$ we obtain for almost every $t \in (t_0 - \delta^2, t_0)$

$$\frac{1}{2}\frac{d}{dt}|\nabla\mathbf{v}|_2^2 + \nu|\Delta\mathbf{v}|_2^2 \le \int_{B_{\delta_3}(\mathbf{x}_0)} |\mathbf{v}||\nabla\mathbf{v}||\Delta\mathbf{v}|\ d\mathbf{x} + \int_{\partial B_{\delta_3}(\mathbf{x}_0)} (|\frac{\partial \mathbf{v}}{\partial t}||\nabla\mathbf{v}| + |p||\Delta\mathbf{v}|)\ dS. \tag{74}$$

Obviously,

$$\int_{B_{\delta_3}(\mathbf{x}_0)} |\mathbf{v}||\nabla\mathbf{v}||\Delta\mathbf{v}|\ d\mathbf{x} \le \sum_{i=1}^3 \int_{B_{\delta_3}(\mathbf{x}_0)} |\mathbf{v}^i||\nabla\mathbf{v}||\Delta\mathbf{v}|\ d\mathbf{x} \tag{75}$$

and we will now estimate the terms on the right hand side of (75).

$$\int_{B_{\delta_3}(\mathbf{x}_0)} |\mathbf{v}^2||\nabla\mathbf{v}||\Delta\mathbf{v}|\ d\mathbf{x} \le \frac{\nu}{8}|\Delta\mathbf{v}|_2^2 + \frac{8}{\nu}\int_{B_{\delta_3}(\mathbf{x}_0)} |\mathbf{v}^2|^2|\nabla\mathbf{v}|^2\ d\mathbf{x} \tag{76}$$

$$\begin{aligned}
&\le \frac{\nu}{8}|\Delta\mathbf{v}|_2^2 + \frac{8}{\nu}\Big(\int_{B_{\delta_3}(\mathbf{x}_0)} |\mathbf{v}^2|^b \; d\mathbf{x}\Big)^{2/b}\Big(\int_{B_{\delta_3}(\mathbf{x}_0)} |\nabla\mathbf{v}|^{\frac{2b}{b-2}} \; d\mathbf{x}\Big)^{\frac{b-2}{b}} \\
&\le \frac{\nu}{8}|\Delta\mathbf{v}|_2^2 + \frac{8}{\nu}|\mathbf{v}^2|_b^2 |\nabla\mathbf{v}|_2^{\frac{2(b-3)}{b}} |\nabla\mathbf{v}|_6^{\frac{6}{b}} \\
&\le \frac{\nu}{8}|\Delta\mathbf{v}|_2^2 + \frac{8}{\nu}|\mathbf{v}^2|_b^2 |\nabla\mathbf{v}|_2^{\frac{2(b-3)}{b}} c_{17}(|\Delta\mathbf{v}|_2 + c_{18})^{\frac{6}{b}} \\
&\le \frac{\nu}{8}|\Delta\mathbf{v}|_2^2 + c_{19}c(\nu)|\mathbf{v}^2|_b^{\frac{2b}{b-3}} |\nabla\mathbf{v}|_2^2 + \frac{\nu}{16}(|\Delta\mathbf{v}|_2 + c_{18})^2 \\
&\le \frac{\nu}{4}|\Delta\mathbf{v}|_2^2 + c_{19}c(\nu)|\mathbf{v}^2|_b^a |\nabla\mathbf{v}|_2^2 + c_{20}.
\end{aligned}$$

Analogically,

$$\int_{B_{\delta_3}(\mathbf{x}_0)} |\mathbf{v}^3||\nabla\mathbf{v}||\Delta\mathbf{v}| \; d\mathbf{x} \le \frac{\nu}{4}|\Delta\mathbf{v}|_2^2 + c_{19}c(\nu)|\mathbf{v}^3|_{\tilde{b}}^{\tilde{a}} |\nabla\mathbf{v}|_2^2 + c_{20}. \tag{77}$$

Similarly,

$$\begin{aligned}
&\int_{B_\delta(\mathbf{x}_0)} |\mathbf{v}^1||\nabla\mathbf{v}||\Delta\mathbf{v}| \; d\mathbf{x}| \le \frac{\nu}{8}|\Delta\mathbf{v}|_2^2 + \frac{8}{\nu}|\mathbf{v}^1|_3^2 |\nabla\mathbf{v}|_6^2 \le \frac{\nu}{8}|\Delta\mathbf{v}|_2^2 \\
&+ \frac{8}{\nu}|\mathbf{v}^1|_3^2 c_{17}(|\Delta\mathbf{v}|_2 + c_{18})^2 \le \frac{\nu}{4}|\Delta\mathbf{v}|_2^2 + c_{21}.
\end{aligned} \tag{78}$$

The last inequality follows from (69) for δ sufficiently small and the fact that c_{17} does not depend on δ. Denote $h_1(t) = 2c_{19}c(\nu)(|\mathbf{v}^2(t)^2|_b^a + |\mathbf{v}^3(t)|_{\tilde{b}}^{\tilde{a}})$ and $h_2(t) = 2\int_{\partial B_{\delta_3}(\mathbf{x}_0)}(|\frac{\partial\mathbf{v}}{\partial t}||\nabla\mathbf{v}| + |p||\Delta\mathbf{v}|) \; dS + 2c_{20} + 2c_{21}$. Using (74)–(78) we have for almost every $t \in (t_0 - \delta^2, t_0)$

$$\frac{d}{dt}|\nabla\mathbf{v}|_2^2 \le h_1(t)|\nabla\mathbf{v}|_2^2 + h_2(t), \tag{79}$$

from which we get

$$\frac{d}{dt}(|\nabla\mathbf{v}|_2^2 h_3(t)) \le h_2(t)h_3(t), \tag{80}$$

where we denoted $h_3(t) = e^{-\int_{t_0-\delta^2}^t h_1(s) \; ds}$. Then for every $t_1 \in (t_0 - \delta^2, t_0)$

(81)

$$\begin{aligned}
&|\nabla\mathbf{v}(t_1)|_2^2 h_3(t_1) - |\nabla\mathbf{v}(t_0 - \delta^2)|_2^2 h_3(t_0 - \delta^2) \\
&= \int_{B_{\delta_3}(\mathbf{x}_0)} (|\nabla\mathbf{v}(\mathbf{x}, t_1)|^2 h_3(t_1) - |\nabla\mathbf{v}(\mathbf{x}, t_0 - \delta^2)|^2 h_3(t_0 - \delta^2)) \; d\mathbf{x}
\end{aligned}$$

$$= \int_{B_{\delta_3}(\mathbf{x}_0)} \int_{t_0-\delta^2}^{t_1} \frac{\partial}{\partial t}(|\nabla \mathbf{v}(\mathbf{x},t)|^2 h_3(t))\ dt\ d\mathbf{x}$$
$$= \int_{t_0-\delta^2}^{t_1} \int_{B_{\delta_3}(\mathbf{x}_0)} \frac{\partial}{\partial t}(|\nabla \mathbf{v}(\mathbf{x},t)|^2 h_3(t))\ d\mathbf{x}\ dt$$
$$= \int_{t_0-\delta^2}^{t_1} \frac{d}{dt}(\int_{B_{\delta_3}(\mathbf{x}_0)} |\nabla \mathbf{v}(\mathbf{x},t)|^2 h_3(t)\ d\mathbf{x})\ dt = \int_{t_0-\delta^2}^{t_1} \frac{d}{dt}(|\nabla \mathbf{v}|_2^2 h_3(t))\ dt.$$

We used the fact that for every $\mathbf{x} \in B_{\delta_3}(\mathbf{x}_0)$ the function $t \mapsto D^{\gamma}_{\mathbf{x}}\mathbf{v}(\mathbf{x},t)$ is absolutely continuous (see [2]), then (72) and (73) and Fubini theorem. (80) can now be integrated over $(t_0-\delta^2, t_1)$ and using (81) we get

$$(82)\quad |\nabla \mathbf{v}(t_1)|_2^2 \le |\nabla \mathbf{v}(t_0-\delta^2)|_2^2 e^{\int_{t_0-\delta^2}^{t_1} h_1(s)\ ds} + c\int_{t_0-\delta^2}^{t_1} h_2(t) e^{\int_t^{t_1} h_1(s)\ ds}\ dt$$

for every $t_1 \in (t_0-\delta^2, t_0)$. Since $h_1, h_2 \in L^1(t_0-\delta^2, t_0)$, (82) gives that

$$(83)\qquad \mathbf{v} \in L^\infty(t_0-\delta^2, t_0, W^{1,2}(B_{\delta_3}(\mathbf{x}_0))).$$

The proof of Theorem 6 is almost complete now. It suffices to prove that if $\Omega = \mathbb{R}^3$ then $(\mathbf{x}_0, t_0)$ is a regular point of $\mathbf{v}$. Let $\delta_4 \in (\delta_1, \delta_3)$. Let η be an infinitely differentiable function on $\mathbb{R}^3$ with its values in $\langle 0,1\rangle$, $\eta \equiv 1$ on $B_{\delta_1}(\mathbf{x}_0)$ and $\eta \equiv 0$ outside $B_{\delta_4}(\mathbf{x}_0)$. Set $\mathbf{V}(\cdot,t) = R(\nabla\eta\cdot\mathbf{v}(\cdot,t))$ for every $t \in (t_0-\delta^2, t_0+\delta^2)$, where R is the operator from Lemma 2. Put $\mathbf{w} = \eta\mathbf{v} - \mathbf{V}$. It follows from (83), reflexivity of $W^{1,2}(B_{\delta_3}(\mathbf{x}_0))$ and the weak continuity of $\mathbf{w}$ as a function from $(t_0-\delta^2, t_0+\delta^2)$ to $L^2(B_{\delta_3}(\mathbf{x}_0))$ that $\mathbf{w}(\cdot,t_0) \in W_0^{1,2}(B_{\delta_3}(\mathbf{x}_0))$ and $\nabla\cdot\mathbf{w}(\cdot,t_0) = 0$ in $B_{\delta_3}(\mathbf{x}_0)$. Further, $\mathbf{w}$ is a weak solution to the following system in $B_{\delta_3}(\mathbf{x}_0)\times(t_0-\delta^2, t_0+\delta^2)$:

$$(84)\qquad \begin{aligned} \frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{w} + \nabla(\eta p) &= \mathbf{g},\\ \nabla\cdot\mathbf{w} &= 0,\\ \mathbf{w} &= \mathbf{0} \quad \text{on } \partial B_{\delta_3}(\mathbf{x}_0)\times(t_0-\delta^2, t_0+\delta^2),\\ \mathbf{w}|_{t=t_0} &= \mathbf{w}(\cdot,t_0), \end{aligned}$$

where

$$(85)\qquad \begin{aligned} \mathbf{g} = &-\nu\Delta\eta\mathbf{v} - 2\nu\nabla\eta\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\eta\mathbf{v} - p\nabla\eta\\ &-\frac{\partial \mathbf{V}}{\partial t} + \nu\Delta\mathbf{V} - \mathbf{v}\cdot\nabla\mathbf{V} + [(\eta-1)\mathbf{v} - \mathbf{V}]\cdot\nabla\mathbf{w}. \end{aligned}$$

It follows from the definition of $\mathbf{w}$ and $\mathbf{V}$, Lemma 2 and (72) and (73) that

$$(86)\qquad \mathbf{g} \in L^\infty(t_0-\delta^2, t_0+\delta^2, L^\infty(B_{\delta_3}(\mathbf{x}_0)))$$

and due to [9, Chapter III., Theorem 3.11] there exists $\zeta_1 \in (0, \delta^2)$ such that

$$\mathbf{w} \in L^\infty(t_0, t_0 + \zeta_1, W^{1,2}(B_{\delta_3}(\mathbf{x}_0))). \tag{87}$$

It gives together with (83) and the definition of $\mathbf{w}$ that

$$\mathbf{v} \in L^\infty(t_0 - \delta^2, t_0 + \zeta_1, W^{1,2}(B_{\delta_1}(\mathbf{x}_0))). \tag{88}$$

The regularity of $(\mathbf{x}_0, t_0)$ now follows from the standard Prodi-Serrin's conditions. The proof of Theorem 6 is complete. □

Remark 2. *Theorem* 5 *can also be proved if we consider* $b = 3$ *or* $\beta = 3/2$ *in* (12) *and* (13) *and the appropriate norms are sufficiently small. Similarly, Theorem* 6 *can also be proved if* $\tilde{b} = 3$ *or* $b = 3$ *or* $\beta = 3/2$ *in* (14)–(16) *and the appropriate norms are sufficiently small.*

References

[1] Caffarelli L., Kohn R., Nirenberg L., *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Apl. Math. **35** (1982), 771–831.

[2] Kučera P., Skalák Z., *Smoothness of the velocity time derivative in the vicinity of regular points of the Navier-Stokes equations*, Proceedings of the 4^{th} Seminar "Euler and Navier-Stokes Equations (Theory, Numerical Solution, Applications)", Institute of Thermomechanics AS CR, Editors: K. Kozel, J. Příhoda, M. Feistauer, Prague, 2001, pp. 83–86.

[3] Ladyzhenskaya O.A., Seregin G.A., *On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations*, J. Math. Fluid Mech. **1** (1999), 356–387.

[4] Lin F., *A new proof of the Caffarelli-Kohn-Nirenberg Theorem*, Comm. Pure Appl. Math. **51** (1998), 241–257.

[5] Neustupa J., *Partial regularity of weak solutions to the Navier-Stokes equations in the class* $L^\infty(0, T, L^3(\Omega)^3)$, J. Math. Fluid Mech. **1** (1999), 309–325.

[6] Neustupa J., *A removable singularity of a suitable weak solution to the Navier-Stokes equations*, preprint.

[7] Nečas J., Neustupa J., *New conditions for local regularity of a suitable weak solution to the Navier-Stokes equations*, preprint.

[8] Skalák Z., *Removable Singularities of Weak Solutions of the Navier-Stokes Equations*, Proceedings of the 4^{th} Seminar "Euler and Navier-Stokes Equations (Theory, Numerical Solution, Applications)", Institute of Thermomechanics AS CR, Editors: K. Kozel, J. Příhoda, M. Feistauer, Prague, 2001, pp. 121–124.

[9] Temam R., *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland Publishing Company, Amsterdam, New York, Oxford, revised edition, 1979.

CTU in Prague, Faculty of Civil Engineering, Department of Mathematics, Prague, Czech Republic

(*Received* February 15, 2002, *revised* June 5, 2002)