# Spaces of continuous functions, box products and almost-$\omega$-resolvable spaces

A. Tamariz-Mascarúa, H. Villegas-Rodríguez

*Abstract.* A dense-in-itself space $X$ is called $C_\square$*-discrete* if the space of real continuous functions on $X$ with its box topology, $C_\square(X)$, is a discrete space. A space $X$ is called *almost-$\omega$-resolvable* provided that $X$ is the union of a countable increasing family of subsets each of them with an empty interior. We analyze these classes of spaces by determining their relations with $\kappa$-resolvable and almost resolvable spaces. We prove that every almost-$\omega$-resolvable space is $C_\square$-discrete, and that these classes coincide in the realm of completely regular spaces. Also, we prove that almost resolvable spaces and almost-$\omega$-resolvable spaces are two different classes of spaces if there exists a measurable cardinal. Finally, we prove that it is consistent with $ZFC$ that every dense-in-itself space is almost-$\omega$-resolvable, and that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost-$\omega$-resolvable.

*Keywords:* box product, $\kappa$-resolvable space, almost resolvable space, almost-$\omega$-resolvable space, Baire irresolvable space, measurable cardinals

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## Introduction

The spaces of continuous functions defined on a topological space $X$ and with values in $\mathbb{R}$, $C(X)$, have been widely studied as a purely algebraic structure ([GJ]), and with a topological (topologico-algebraic) structure ([BNS], [DH]).

One of the natural topologies associated with $C(X)$ is the pointwise convergence topology, which is the topology in $C(X)$ inherited from the Tychonoff topology of $\mathbb{R}^X$. This space is usually denoted by $C_p(X)$. A classical general problem on $C_p$-spaces consists of determining the relations between the topological properties of space $X$ with the topological properties of $C_p(X)$ ([Ar]).

A generalization of the Tychonoff topology for a product of topological spaces, is the box topology (see definition in Section 1) which was introduced by Tietze in [T]. The study of the box product of an infinite family of topological spaces has been useful to construct some interesting topological spaces ([R], [V]).

So, it seems natural to ask about the relations between the topological properties of a space $X$ and those of $C(X)$ with its box topology, which we denote by $C_\square(X)$. The purpose of this article is to analyze some of these relations when

$X$ is dense-in-itself. In particular, we determine for which dense-in-themselves spaces $X$, $C_\square(X)$ is a discrete space. This analysis will lead us to consider the resolvable and irresolvable spaces and the measurable cardinals. Resolvable and irresolvable spaces were studied extensively first by Hewitt [H]. Later, El'kin and Malykhin published a number of papers on this subject and its connection with various topological problems. One of the problems considered by Malykhin [M2] refers to the existence of irresolvable spaces satisfying the Baire Category Theorem. He proved that there is such a space if and only if there is a space $X$ on which every real-valued function is continuous at some point. The question about the existence of a Hausdorff space on which every real-valued function is continuous at some point was posed by M. Katětov in [K]. Bolstein introduced in [B] the spaces $X$ on which it is possible to define a real-valued function $f$ with countable range and such that $f$ is discontinuous at every point of $X$ (he called these spaces *almost resolvable*), and proved that every resolvable space satisfies this condition. Here we introduce the almost-$\omega$-resolvable spaces defined as spaces on which it is possible to define a real-valued function $f$ with countable range, and such that $r\circ f$ is discontinuous in every point of $X$, for every real-valued finite-to-one function $r$. We prove in Section 2 that, for completely regular spaces, $C_\square(X)$ is discrete if and only if $X$ is almost-$\omega$-resolvable. Section 3 is devoted to obtaining the basic properties of $C_\square$-discrete and almost-$\omega$-resolvable spaces, and we determine their relations with spaces having resolvable-like properties. In Section 4 we prove that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost-$\omega$-resolvable, and that, on the contrary, if $V = L$ then every dense-in-itself space is almost-$\omega$-resolvable. Also, we prove that almost-$\omega$-resolvable spaces and almost resolvable spaces are two different classes of spaces if there exists a measurable cardinal. Finally, in the last section we list some unsolved questions concerning these subjects.

## 1. Definitions and generalities

Let $\mathcal{F} = \{X_s : s \in S\}$ be a family of topological spaces. By $\square_{s\in S}X_s$ we denote the cartesian product of the family $\mathcal{F}$ considered with the so called box topology $\tau_\square$, which is that generated by the sets of the form $\prod_{s\in S} A_s$ where, for each $s \in S$, $A_s$ is an open subset of $X_s$. It is obvious that the Tychonoff topology is contained in the box topology, and they coincide when $|S| < \aleph_0$.

It is well known that, for an infinite family $\{X_s : s \in S\}$ of non-trivial topological spaces, $\square_{s\in S}X_s$ is neither first countable nor locally compact, and it never is a topological vector space over $\mathbb{R}$, but it is a topological group if each of the spaces $X_s$ is a topological group. A good survey of the characteristics of the box topology can be found in [Wi].

For a topological space $X$ we denote by $\mathbb{R}^X$ the set of functions from $X$ to $\mathbb{R}$. The subset of $\mathbb{R}^X$ whose elements are the continuous functions is denoted by $C(X)$. The space $\square\mathbb{R}^X$ will be the set $\mathbb{R}^X$ with the box topology, and $C_\square(X)$ is the set $C(X)$ considered as a subspace of $\square\mathbb{R}^X$. Of course, $C_\square(X)$ is a topological subgroup of $\square\mathbb{R}^X$.

For spaces $X$ and $Y$ and for a function $f : X \to Y$, $D(f)$ will denote the set of points in $X$ in which $f$ is not continuous.

A point $x$ in $X$ is an *isolated point* of $X$ if $\{x\}$ is open in $X$, and a space $X$ is *dense-in-itself* if it does not contain isolated points. A space $X$ is *resolvable* if it is the union of two disjoint dense subsets. We say that $X$ is *irresolvable* if it is dense-in-itself and it is not resolvable. For a cardinal number $\kappa > 1$, we say that $X$ is *$\kappa$-resolvable* if $X$ is the union of $\kappa$ pairwise disjoint dense subsets.

The *dispersion character* $\Delta(X)$ of a space $X$ is the minimum of the cardinalities of nonempty open subsets of $X$.

If $X$ is $\Delta(X)$-resolvable, then we say that $X$ is *maximally resolvable*.

A space $X$ is *hereditarily irresolvable* if it is dense-in-itself and every subspace of $X$ is irresolvable. And $X$ is *open-hereditarily irresolvable* if it is dense-in-itself and every open subspace of $X$ is irresolvable.

A space $(X, t)$ is *maximal* if $(X, t)$ is dense-in-itself and $(X, t')$ contains at least an isolated point when $t'$ strictly contains the topology $t$. And a space $X$ is *submaximal* if it is dense-in-itself and every dense subset of $X$ is open.

A space $X$ is called *almost resolvable* if $X$ is the union of a countable collection of subsets each of them with an empty interior.

Of course, every $\kappa$-resolvable space is $\tau$-resolvable if $\tau \leq \kappa$. Besides, every maximal space is submaximal, and these are hereditarily irresolvable spaces, which in turn are open hereditarily irresolvable.

The concept of resolvability and related topics were introduced by Hewitt in [H]; and Bolstein [B] proved that a space $X$ is resolvable if it is the union of a finite collection of subsets each of them with an empty interior. That is, every resolvable space is almost resolvable.

The class of resolvable spaces includes spaces with well known properties:

**Theorem 1.1** ([E2])**.** *If $X$ is dense-in-itself and has a $\pi$-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \Delta(X)$ and each $N \in \mathcal{N}$ satisfies $|N| \geq \Delta(X)$, then $X$ is maximally resolvable.*

**Theorem 1.2.** *All the spaces considered in this theorem are dense-in-themselves.*

(1) *The locally compact Hausdorff spaces are maximally resolvable* [H].
(2) *First countable spaces are maximally resolvable* [E1].
(3) *Hausdorff $k$-spaces are maximally resolvable* [P] (*in particular, metrizable spaces are maximally resolvable* [H]).
(4) *Countably compact regular $T_1$ spaces are $\omega$-resolvable* [CGF].
(5) *Arc connected spaces are $\omega$-resolvable* (*in particular, every topological vector space over $\mathbb{R}$ is $\omega$-resolvable*).
(6) *Every biradial space* (*in particular, every linearly orderable topological space*) *is maximally resolvable* [Vi2].

(7) *Every homogeneous space containing a non-trivial convergent sequence is $\omega$-resolvable* [Vi1].
(8) *If $G$ is a non-countable $T_0$ $\aleph_0$-bounded topological group, then $G$ is $\aleph_1$-resolvable* [Vi2].

The terms not defined in this paper are considered as in [W].

## 2. $C_\Box$-discrete and almost-$\omega$-resolvable spaces

As we have already mentioned, we are interested in knowing when $C_\Box(X)$ is a discrete subspace of $\Box\mathbb{R}^X$; if this happens, we say that $X$ is *$C_\Box$-discrete*. The finite spaces are not $C_\Box$-discrete because if $|X| = n+1$, $n \in \omega$, and $r \in \bigcap_{0\leq i\leq n} A_i$ where $A_i$ is an open set in $\mathbb{R}$ with $0 \in A_i$ for every $0 \leq i \leq n$, then $f \equiv 0$ and $g \equiv r$ are two different continuous functions in $A_0 \times \cdots \times A_n$. *Because of this, from now on we will only consider spaces with cardinality* $\geq \aleph_0$. We also observe that if $X$ is a $T_1$ or a regular space and $a \in X$ is an isolated point, then $X = \{a\} \oplus X \setminus \{a\}$; thus $C_\Box(X) \cong \mathbb{R} \times C_\Box(X \setminus \{a\})$. Therefore, in this case, $C_\Box(X)$ is not discrete.

Furthermore, if $C(X)$ is equal to the set of constant functions, then $C_\Box(X)$ is discrete in $\Box\mathbb{R}^X$. Indeed, let $N = \{x_0, \ldots, x_n, \ldots\} \subset X$ with $x_i \neq x_j$ if $i \neq j$; and let $f(x) = r \in \mathbb{R}$ for every $x \in X$. Then, $\{f\} = C_\Box(X) \cap \prod_{x\in X} A_x$ where $A_{x_n} = (r - \frac{1}{n}, r + \frac{1}{n})$ and $A_x = \mathbb{R}$ for every $x \in X \setminus N$. Thus, we can find a $T_0$ $C_\Box$-discrete space having isolated points: take an infinite set $X$, and let $a \in X$. The collection $\tau = \{A \subset X : a \in A\} \cup \{\emptyset\}$ is a $T_0$ topology in $X$, $\{a\}$ is open and every continuous real-valued function on $X$ is constant.

**Proposition 2.1.** *If $X$ is $C_\Box$-discrete and regular or $T_1$, then $X$ does not have isolated points.*

Observe also that for every space $X$, $C_\Box(X)$ is a closed subspace of $\Box\mathbb{R}^X$. This can be proved as follows: Let $f \in \Box\mathbb{R}^X$, and let $x_0 \in X$ such that $f$ is not continuous in $x_0$. Then, there is an open neighborhood $U$ of $f(x_0)$ such that, for every neighborhood $V$ of $x_0$, there is $x_V \in V$ satisfying $f(x_V) \notin \operatorname{cl} U$. For each $f(x_V)$ we take a neighborhood $W_V$ of $f(x_V)$ with $W_V \cap U = \emptyset$. Let $G = \prod_{x\in X} U_x$ where $U_x = U$ if $x = x_0$, $U_x = W_V$ if $x = x_V$, and $U_x = \mathbb{R}$ in all other cases. It is clear that $f \in G \subset \Box\mathbb{R}^X \setminus C_\Box(X)$.

**Theorem 2.2.** *Let $(X, \tau)$ be a topological space, and let $\tau^* = \tau \setminus \{\emptyset\}$. The following properties are equivalent.*

(1) *There is a partition $\mathcal{X} = \{X_n : n < \omega\}$ of $X$ such that each $A \in \tau^*$ has a nonempty intersection with each of the elements of an infinite subcollection of $\mathcal{X}$.*
(2) *$X = \bigcup_{n<\omega} Y_n$ with $Y_n \subset Y_{n+1}$ and $\operatorname{int}(Y_n) = \emptyset$ for every $n < \omega$.*
(3) *There is a decreasing sequence of dense subsets of $X$ with an empty intersection.*

(4) *There is a function* $f : X \to \omega$ *for which* $D(r \circ f)$ *is equal to* $X$*, for every finite-to-one function* $r : \omega \to \omega$.
(5) *There is a function* $h : X \to \mathbb{R}^+ \setminus \{0\}$ *such that, for each* $A \in \tau^*$*, there is a sequence* $(x_n)_{n<\omega}$ *in* $A$ *such that* $(h(x_n))_{n<\omega}$ *converges to* $0$.
(6) *There is a set* $S \neq \emptyset$ *and, for each* $s \in S$*, there exists* $C_s = \{x_n^s : n < \omega\}$*, such that* (i) $X = \bigcup_{s\in S} C_s$*;* (ii) *for each* $s_0 \in S$ *and each* $n_0 < \omega$*,* $C_{s_0}$ *is not contained in* $\bigcup\{x_k^s : s \in S,\ k \leq n_0\}$*;* (iii) *for each* $A \in \tau^*$*, there is* $s \in S$ *such that* $C_s \subset A$.

PROOF: (1) $\Rightarrow$ (2): The set $Y_n = \bigcup_{0\leq i\leq n} X_n$ has an empty interior for every $n < \omega$.

(2) $\Rightarrow$ (1): If we define $X_0 = Y_0$ and $X_{n+1} = Y_{n+1} \setminus \bigcup_{0\leq i\leq n} Y_n$, then $\{X_n : n < \omega\}$ satisfies that which is required.

(2) $\Leftrightarrow$ (3): Define $D_n = X \setminus Y_n$. Then, $D_{n+1} \subset D_n$ if and only if $Y_n \subset Y_{n+1}$; $D_n$ is dense in $X$ if and only if $\operatorname{int} Y_n = \emptyset$; and $\bigcap_{n\in\omega} D_n = \emptyset$ if and only if $X = \bigcup_{n<\omega} Y_n$.

(2) $\Rightarrow$ (4): The function $f : X \to \omega$ defined by $f(x) = 0$ if $x \in Y_0$, and $f(x) = n + 1$ if $x \in Y_{n+1} \setminus Y_n$, satisfies the required conditions.

(4) $\Rightarrow$ (5): Let $h(x) = \frac{1}{f(x)+1}$ for each $x \in X$, and let $A \in \tau^*$. If for some $n < \omega$, $A \subset \bigcup_{0\leq i\leq n} f^{-1}(i)$ and $x_0 \in A$, then $r \circ f$ is a continuous function in $x_0$, where $r : \omega \to \omega$ is defined by $r(i) = 0$ if $0 \leq i \leq n$, and $r(i) = i$ if $i > n$. But this is not possible, thus, there is a cofinal sequence $(n_k)_{k<\omega}$ in $\omega$ such that $A \cap f^{-1}(n_k) \neq \emptyset$ for every $k < \omega$. If $x_k \in A \cap f^{-1}(n_k)$, then $\lim_{k\to\infty} h(x_k) = 0$.

(5) $\Rightarrow$ (6): Fix a point $x_0 \in X$. For each $A \in \tau^*$, let $(x_n^A)_{n<\omega}$ be a sequence in $A$ such that $(h(x_n^A))_{n<\omega}$ converges to 0. We take $y_n^A \in h^{-1}([\frac{1}{n+1}, \frac{1}{n})) \cap \{x_n^A : n < \omega\} = T_n^A$ if $T_n^A \neq \emptyset$, and $y_n^A = x_0$ if $T_n^A = \emptyset$. Let $C_A = \{y_n^A : n < \omega\}$. Since $h(x) > 0$ for every $x \in X$ and $h(x_n^A) \to 0$, then $|C_A| = \aleph_0$. Besides, because of the election of each $y_n^A$, for each $A_0 \in \tau^*$ and each $n_0 < \omega$, $C_{A_0}$ is not contained in $\bigcup\{y_n^A : A \in \tau^*, k \leq n_0\}$.

If $X \setminus \bigcup_{A\in\tau^*} C_A = K$ is finite, we add the points in $K$ to some $C_A$. In this case, we define $T = \emptyset$. If, on the contrary, $|K| \geq \aleph_0$, let $\{C_t : t \in T\}$ be a partition of $K$ where $|C_t| = \aleph_0$ for each $t \in T$. Then, the collection $\{C_A : A \in \tau\} \cup \{C_t : t \in T\}$ satisfies the conditions in (6).

(6) $\Rightarrow$ (2): For each $s \in S$ we enumerate $C_s$: $\{c_n^s : n < \omega\}$ such that $c_n^s \neq c_m^s$ if $n \neq m$. We define, $Y_0 = \{c_0^s : s \in S\}$, and $Y_{n+1} = \bigcup_{0\leq i\leq n} Y_i \cup \{c_{n+1}^s : s \in S\}$. Now, we have that $X = \bigcup_{n<\omega} Y_n$ and $\operatorname{int} Y_n = \emptyset$ for every $n < \omega$. □

**Definition 2.3.** A space $X$ is *almost-$\omega$-resolvable* if $X$ satisfies one (thus all) of the properties (1)–(6) enlisted in Theorem 2.2. We will call the partition $\{X_n : n < \omega\}$, which satisfies the conditions in (1) of this theorem, an *almost-$\omega$-resolvable partition on* $X$; and the sequence $\{Y_n : n < \omega\}$ in (2) will be called an *almost-$\omega$-resolvable sequence*.

Observe that every almost-$\omega$-resolvable space is an infinite dense-in-itself space. Besides, it is evident that every $\omega$-resolvable space is almost-$\omega$-resolvable, and this class of spaces is contained in the class of almost resolvable spaces.

**Theorem 2.4.** *Let $X$ be an almost-$\omega$-resolvable space. Then $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$.*

Proof: Let $f \in C(X)$, and let $\mathcal{X} = \{X_n : n \in \omega\}$ be a partition on $X$ for which each nonempty open set of $X$ intersects $\aleph_0$ elements in $\mathcal{X}$. We take the following open subset of $\square\mathbb{R}^X$: $G = \prod_{x\in X} G_x$, where $G_x = (f(x) - 1/n, f(x) + 1/n)$ if $x \in X_n$. Of course, $f \in G$. Assume that $g$ is also a continuous function from $X$ to $\mathbb{R}$ which belongs to $G$. Let $x_0$ be a fix point in $X$, and let $n_0$ be a natural number. Since $g$ and $f$ are continuous functions in $x_0$, there is an open set $V$ of $X$ containing $x_0$ such that $f(V) \subset (f(x_0) - \frac{1}{n_0}, f(x_0) + \frac{1}{n_0})$ and $g(V) \subset (g(x_0) - \frac{1}{n_0}, g(x_0) + \frac{1}{n_0})$. Because of the definition of $\mathcal{X}$, there is $k \geq n_0$ such that $V \cap X_k \neq \emptyset$. If $y \in V \cap X_k$, we have that $d(f(x_0), g(x_0)) \leq d(f(x_0), f(y)) + d(f(y), g(y)) + d(g(y), g(x_0)) \leq \frac{1}{n_0} + \frac{1}{k} + \frac{1}{n_0} \leq \frac{3}{n_0}$. Since $x_0$ is fixed and $n_0$ is arbitrary, we conclude that $f = g$. That is, $G \cap C(X) = \{f\}$. □

The converse of the last result is true if we consider completely regular spaces:

**Theorem 2.5.** *Let $X$ be a completely regular space. Then $X$ is an almost-$\omega$-resolvable space if and only if $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$.*

Proof: Assume that $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$. So there is an open set $G = \prod_{x\in X} G_x$ in $\square\mathbb{R}^X$ which only contains the continuous function $f \equiv 0$. Let $d : X \to \omega$ be the function defined by $d(x) =$ the least natural number $n$ such that $(\frac{-1}{n}, \frac{1}{n}) \subset G_x$. Let $Y_n = \{x \in X : d(x) \leq n + 1\}$.

Assume that there exists a nonempty open set $A$ of $X$ satisfying $A \subset Y_{n_0}$ for an $n_0 \in \omega$. We pick a point $x_0$ in $A$. Since $X$ is a completely regular space, there is a continuous function $h : X \to [0, \frac{1}{2(n_0)}]$ such that $h(x_0) = \frac{1}{2(n_0)}$ and $h(y) = 0$ if $y \in X \setminus A$. It happens that $h \in G \cap C(X)$ and, of course, $h \neq f$. This contradiction implies that $\operatorname{int} Y_n = \emptyset$ for every $n < \omega$. So, $\{Y_n : n < \omega\}$ is an almost-$\omega$-resolvable sequence on $X$. □

We are going to see that almost resolvable spaces, almost $\omega$-resolvable spaces and $C_\square$-discrete spaces are very similar classes of spaces, and, in fact, we will prove that the proposition "every dense-in-itself topological space is almost-$\omega$-resolvable" is consistent with *ZFC*.

## 3. Basic properties of $C_\square$-discrete and almost $\omega$-resolvable spaces

Since a $T_0$ topological space is dense-in-itself if and only if its dispersion character is infinite, we obtain:

**Proposition 3.1.** *Every $T_0$ dense-in-itself space with cardinality $\aleph_0$ is an almost-$\omega$-resolvable space.*

**Proposition 3.2.** *If $X$ contains an almost-$\omega$-resolvable (resp., $C_\square$-discrete) dense subspace, then $X$ is an almost-$\omega$-resolvable (resp., $C_\square$-discrete) space.*

PROOF: Assume that $D$ is a dense subspace of $X$, and let $\mathcal{D} = \{D_n : n < \omega\}$ be an almost-$\omega$-resolvable partition of $D$. The partition of $X$ determined by $\mathcal{X} = \{X \setminus D\} \cup \mathcal{D}$ witnesses that $X$ is almost-$\omega$-resolvable.

Now, assume that $D$ is a dense subset of $X$ and $D$ is $C_\square$-discrete. Let $f \in C(X)$. There exists an open set $B = \prod_{d\in D} B_d$ of $\square\mathbb{R}^D$ such that $f|_D$ is the only continuous function contained in $B$. Then, $\{f\} = G \cap C(X)$ where $G = \prod_{x\in X} G_x$ and $G_x = B_x$ if $x \in D$ and $G_x = \mathbb{R}$ if $x \notin D$. $\square$

**Corollary 3.3.** *Every $T_0$ dense-in-itself space $X$ such that* $\min\{|D| : D \subset X,$ *$D$ is dense in $X$ and dense-in-itself* $\} = \aleph_0$, *is an almost-$\omega$-resolvable space. In particular, every $T_1$ dense-in-itself separable space is almost-$\omega$-resolvable.*

Observe that for every topological space $X$ there is a dense subset of $X$, $D_X = F_X \cup G_X$, such that every point in $F_X$ (if $F_X$ is not empty) is an isolated point of $D_X$ and $G_X$ is dense-in-itself ($G_X$ can be an empty set). It is possible to prove that every $T_0$ space $X$ without isolated points for which $|G_X| = \aleph_0$ and $\mathrm{cl}_X F_X \cap \mathrm{cl}_X G_X = \emptyset$, is a $C_\square$-discrete space. A similar result does not hold for almost-$\omega$-resolvable spaces as the examples in Section 4 testify.

We have already mentioned that every $\omega$-resolvable space is almost-$\omega$-resolvable, thus by Theorems 1.1 and 1.2 we obtain:

**Theorem 3.4.** *Dense-in-themselves spaces with the following properties are almost-$\omega$-resolvable: locally compact Hausdorff spaces, first countable spaces, Hausdorff $k$-spaces (in particular, metrizable spaces), countably compact regular $T_1$ spaces, arc connected spaces, biradial spaces (in particular, linearly orderable topological spaces), homogeneous spaces with a non-trivial convergent sequence, non-countable $T_0$ $\aleph_0$-bounded groups, and spaces $X$ with a $\pi$-network $\mathcal{N}$ such that $|\mathcal{N}| \le \Delta(X)$ and $|N| \ge \Delta(X)$ for every $N \in \mathcal{N}$.*

We present now some subclasses of almost-$\omega$-resolvable spaces which are not contained in the class of $\omega$-resolvable spaces.

**Theorem 3.5.** *Let $X$ be a dense-in-itself space. If $X$ is*

(i) *of the first category, or*
(ii) *$T_0$ and $\sigma$-discrete, or*
(iii) *a $T_0$ $\aleph_0$-bounded topological group,*

*then $X$ is an almost-$\omega$-resolvable space.*

PROOF: Let $X$ be of the first category. Since $|X| \ge \aleph_0$, we can split $X$ into a sequence $\{X_n : n < \omega\}$ such that $X_n \ne \emptyset$ and $\mathrm{int}_X(\mathrm{cl}_X X_n) = \emptyset$ for each $n < \omega$. Then, $\{X_n : n < \omega\}$ is an almost-$\omega$-resolvable partition on $X$.

Assume now that $X = \bigcup_{n<\omega} X_n$, where each $X_n$ is a discrete subspace of $X$. Since $X$ is dense-in-itself and $T_0$, $X$ must be the union of an infinite countable collection of different discrete subspaces. So, we can assume, without loss of generality, that $X_n \neq X_m$ if $n \neq m$. Let $Z_0 = X_0$, $Z_1 = X_1 \setminus X_0$, ... ,$Z_{n+1} = X_{n+1} \setminus \bigcup_{0\leq i\leq n} X_n, \dots$ . Each $Z_n$ is a discrete subspace of $X$, and if $Z_n = \emptyset$ for every $n > n_0$, then $X$ is the union of a finite collection of discrete subspaces. But this implies that $X$ has an isolated point, which is not possible. Therefore, there exists an increasing sequence $\{n_k : k < \omega\}$ of natural numbers, such that $Z_n \neq \emptyset$ if and only if $n \in \{n_k : k < \omega\}$. We now have that the collection $\{Z_{n_k} : k < \omega\}$ is an almost-$\omega$-resolvable partition on $X$.

The last assertion of the theorem is a consequence of Proposition 3.1 and Theorem 3.4. $\square$

For each infinite cardinal number $\kappa$, there are dense-in-themselves first category and $\sigma$-discrete spaces with cardinality $\kappa$, which are not resolvable. Indeed, van Douwen constructed a countable irresolvable Hausdorff zero-dimensional space $X$ in [vD] using a maximal independent family of subsets of $\omega$. The free topological sum of $\kappa$ copies of $X$ is a first category $\sigma$-discrete space with cardinality $\kappa$ and it is not resolvable. Besides, Malykhin in [M1] constructed an extremally disconnected group topology on the countable Boolean group $G = \bigoplus_{n<\omega}\{0, 1\}$, in the system $[ZFC + P(c)]$, which is maximal among regular Hausdorff topologies without isolated points (so, $G$ is an irresolvable $\aleph_0$-bounded topological group).

The following two propositions are easy to prove.

**Proposition 3.6.** *If $\{X_s : s \in S\}$ is a family of almost-$\omega$-resolvable (resp., $C_\square$-discrete) spaces, then the free topological sum of the elements of this family, $\bigoplus_{s\in S} X_s$, is almost $\omega$-resolvable (resp., $C_\square$-discrete).*

**Proposition 3.7.** *Every open subspace of an almost-$\omega$-resolvable space is almost-$\omega$-resolvable.*

It is not possible to get, in *ZFC*, a proposition like Proposition 3.7 for dense or $G_\delta$ closed subsets. Indeed, assume that $(T, \tau)$ is a dense-in-itself Tychonoff non almost-$\omega$-resolvable space $T$ (see Theorem 4.16.1 below), then $\beta(T)$, the Stone-Čech compactification of $T$, is maximally resolvable (Theorem 1.2) (so, it is almost-$\omega$-resolvable) and $T$ is dense in $\beta(T)$. Besides, $Y = T \times \prod_{n<\omega} X_n$, where each $X_n$ is a non-trivial $T_1$ space with countable pseudo-character, is an almost-$\omega$-resolvable space (see Theorem 3.9 below), and $T$ is homeomorphic to a $G_\delta$ closed subset of $Y$. On the other hand, it is not possible to prove in *ZFC* that $C_\square$-discreteness is inherited by open subspaces: Consider the space $Z = T \cup \{p\}$ with $p \notin T$, and such that $A \subset Z$ is open if and only if $A \in \tau$ or $p \in A$ and $|X \setminus A| < \aleph_0$. We have that $C(Z)$ is equal to the set of constant functions, and, hence, $C_\square(Z)$ is discrete. Moreover, $T$ is open in $Z$ and it is not $C_\square$-discrete.

The proof of the following result follows the pattern of the proof of the Theorem in [CLF] (see also [CGF], Theorem 2.2).

**Theorem 3.8.** *If $X$ is the union of almost-$\omega$-resolvable (resp., almost resolvable, $\kappa$-resolvable) subspaces, then $X$ is almost-$\omega$-resolvable (resp., almost resolvable, $\kappa$-resolvable).*

PROOF: We prove the theorem for almost-$\omega$-resolvability. For almost resolvable or $\kappa$-resolvable spaces the proof is analogous. Assume that $X = \bigcup_{j\in J} X_j$ where each $X_j$ is almost-$\omega$-resolvable. Let $\mathcal{A} = \{A_s : s \in S\}$ be a maximal family of pairwise disjoint, nonempty almost-$\omega$-resolvable subspaces of $X$. For $s \in S$, let $\mathcal{C}_s = \{A(s,n) : 0 < n < \omega\}$ be an almost-$\omega$-resolvable partition of $A_s$. If there is $j \in J$ for which $B = X_j \setminus \mathrm{cl}_X \bigcup \mathcal{A} \neq \emptyset$, then, by Proposition 3.7, $B$ is almost-$\omega$-resolvable and its intersection with each element $A \in \mathcal{A}$ is equal to the empty set. This contradicts the maximality of $\mathcal{A}$; so, $\bigcup \mathcal{A}$ must be dense in $X$. Now, the family $\{X_n : n < \omega\}$, where $X_0 = X \setminus \bigcup \mathcal{A}$ and $X_n = \bigcup_{s\in S} A(s,n)$ for every $0 < n < \omega$, is an almost-$\omega$-resolvable partition of $X$. $\square$

If $X$ is a Tychonoff product with an infinite number of non-trivial $T_1$ factors, then a $\sigma$-product in $X$ is dense, has no isolated points and is a first category set. Then, by Proposition 3.2 and Theorem 3.5, $X$ is almost-$\omega$-resolvable. The following theorem and corollary seem not to have been noted before. The maximal resolvability of a product of maximal resolvable spaces is treated in [C] and [CP].

**Theorem 3.9.** *The Tychonoff product $X$ of an infinite collection $\{X_\lambda : \lambda < \kappa\}$ of non-trivial $T_1$ topological spaces is, at least, $2^\kappa$-resolvable. In particular, if $\sup_{\lambda<\kappa} |X_\lambda| \leq 2^\kappa$, then $X$ is maximally resolvable.*

PROOF: The space $X$ can be covered by copies of the Cantor cube of weight $\kappa$, $C = \{0,1\}^\kappa$, and $C$ is maximally resolvable; that is, $C$ is $2^\kappa$-resolvable. We finish the proof by applying Theorem 3.8.

The last part of the Theorem is obtained because $\Delta(X) \leq (\sup_{\lambda<\kappa} |X_\lambda|)^\kappa$. $\square$

**Corollary 3.10.** *Every $T_2$ locally path connected space (in particular, every topological vector space over $\mathbb{R}$) is, at least, $2^\omega$-resolvable.*

PROOF: If $X$ is a $T_2$ locally path connected space, then $X$ can be covered by copies of the real line, and $\mathbb{R}$ is $2^\omega$-resolvable. It remains to refer to Theorem 3.8. $\square$

**Corollary 3.11.** *Let $X$ be a homogeneous space. If $X$ contains an almost-$\omega$-resolvable subspace, then $X$ is almost-$\omega$-resolvable.*

**Proposition 3.12.** *Let $X$ and $Y$ be two spaces, and $f : X \to Y$ be an onto function.*

(1) *If $\mathrm{int}\, f(A) \neq \emptyset$ for every nonempty open subset $A$ of $X$ (in particular, if $f$ is open) and $Y$ is almost-$\omega$-resolvable, then $X$ is almost-$\omega$-resolvable.*

(2) *If $f$ is a one-to-one continuous function and $X$ is almost-$\omega$-resolvable (resp., $C_\Box$-discrete), then $Y$ is almost-$\omega$-resolvable (resp., $C_\Box$-discrete).*

Proof: (1) Let $\{Y_n : n < \omega\}$ be an almost-$\omega$-resolvable partition of $Y$. Then, $\{f^{-1}(Y_n) : n < \omega\}$ is an almost-$\omega$-resolvable partition of $X$.

(2) If $\{X_n : n < \omega\}$ is an almost-$\omega$-resolvable partition of $X$. Then, $\{f(X_n) : n < \omega\}$ is an almost-$\omega$-resolvable partition of $Y$.

Now assume that $X$ is $C_\Box$-discrete. So, $C_\Box(X)$ is a discrete subspace of $\Box\mathbb{R}^X = \Box\mathbb{R}^Y$. Moreover, the function $f^\# : C_\Box(Y) \to C_\Box(X)$ defined by $f^\#(g) = g \circ f$ is an embedding. Then, $C_\Box(Y)$ is discrete in $\Box\mathbb{R}^Y$. $\Box$

The following result is a consequence of Proposition 3.12.1.

**Corollary 3.13.** *Let $\{X_s : s \in S\}$ be a family of topological spaces, and let $X = \prod_{s\in S} X_s$. If for an $s_0 \in S$, $X_{s_0}$ is almost-$\omega$-resolvable, and $t$ is a topology on $X$ contained in the box topology of $X$, then $(X,t)$ is almost-$\omega$-resolvable.*

**Remarks 3.14.**

(1) With regard to Proposition 3.12, we give the following counterexamples which prove that this proposition cannot be ameliorated in various senses: Let $Z = [0,1] \oplus X$ where $X$ is the non almost-$\omega$-resolvable space in Example 4.2 below. Let $Y$ be the quotient space obtained from $Z$ by identifying 0 in $[0,1]$ with $p$ in $X$; and let $q : Z \to Y$ be the quotient map. We have that $q$ is a continuous finite-to-one quotient map, $Z$ is not almost-$\omega$-resolvable and $Y$ is almost-$\omega$-resolvable. In addition, the function $g : Z \to [0,1]$ defined as $g|_{[0,1]} = \mathrm{id}$ and $g|_X \equiv 0$ is continuous and closed. On the other hand, the projection $\pi : X \times \mathbb{Q} \to X$, where $\mathbb{Q}$ is the space of rational numbers with its usual topology, is a continuous open map with countable fibers from an almost-$\omega$-resolvable space (Corollary 3.13) and with values in a space which does not satisfy this property. Finally, let $W$ be a space which is not $C_\Box$-discrete, $V = W \oplus X$ and $f : V \to X$ which sends every point in $W$ to $p$, and coincides with the identity function in $X$; then, $f$ is a continuous function from a space which is not $C_\Box$-discrete onto a $C_\Box$-discrete space.

(2) In a similar way as in Theorem 3.8, several results presented in this section can be formulated by almost resolvable spaces and for $\kappa$-resolvable spaces. Indeed, observe that Propositions 3.2, 3.6, 3.7, and 3.12, and Corollaries 3.11 and 3.13 remain true if we put "almost resolvable" or "$\kappa$-resolvable" instead of "almost-$\omega$-resolvable".

## 4. Measurable cardinals, almost resolvable spaces and almost-$\omega$-resolvable spaces

As we have already mentioned, every resolvable space $X$ and every almost-$\omega$-resolvable space is almost resolvable. The spaces mentioned in the paragraph following Theorem 3.5 are examples of irresolvable almost-$\omega$-resolvable spaces of cardinality $\kappa$, where $\kappa$ is an arbitrary infinite cardinal number. Now we are going to give examples of almost-$\omega$-resolvable hereditarily irresolvable Tychonoff spaces with an arbitrarily large dispersion character. Also, we are going to see that the existence of a measurable cardinal number provides us with an example of a resolvable $C_\Box$-discrete space which is not almost-$\omega$-resolvable. A simple modification of this space produces a $C_\Box$-discrete space which is not even almost resolvable:

**Example 4.1.** *For each cardinal number $\kappa$, there are Tychonoff spaces $X$ which are almost-$\omega$-resolvable, hereditarily irresolvable and $\Delta(X) \geq \kappa$.*

PROOF: Let $\alpha$ be a cardinal number such that $\kappa \leq \alpha$ and $\mathrm{cof}(\alpha) = \aleph_0$. Let $H$, $G$ and $\tau$ be the topological groups and the topology in $G$, respectively, defined in [LF, pages 33 and 34], with $|H| = \alpha$. L. Feng proved there that $(H, \tau|_H)$ is an irresolvable card-homogeneous (every open subset of $H$ has the same cardinality as $H$) Tychonoff space, and each subset $S \subset H$ with cardinality $< \alpha$ is a nowhere dense subset of $H$.

Let $(\alpha_n)_{n<\omega}$ be a sequence of cardinal numbers such that $\alpha_n < \alpha_{n+1}$ for every $n < \omega$ and $\sup\{\alpha_n : n < \omega\} = \alpha$. We take subsets $H_n$ of $H$ with the properties $H_n \subset H_{n+1}$ and $|H_n| = \alpha_n$ for each $n < \omega$, and $H = \bigcup_{n<\omega} H_n$. We have that each $H_n$ is nowhere dense in $H$; so $\{H_n : n < \omega\}$ is an almost-$\omega$-resolvable sequence on $H$. That is, $H$ is almost-$\omega$-resolvable.

By the Hewitt Decomposition Theorem (see Lemma 4.7 below), there exists a nonempty open subset $U$ of $H$ which is hereditarily irresolvable. Besides, $\Delta(U) = \Delta(H) \geq \kappa$ and it is almost-$\omega$-resolvable (Proposition 3.7). $\Box$

**Example 4.2.** *If there is a measurable cardinal $\alpha$, then there is a $T_0$ resolvable $C_\Box$-discrete Baire space $X$ which is not almost-$\omega$-resolvable and $\Delta(X) = \alpha$.*

PROOF: Let $\alpha$ be a non-countable Ulam-measurable cardinal, and let $p$ be a free ultrafilter on $\alpha$ $w^+$-complete. Let $X = \alpha \cup \{p\}$. We define a topology $t$ for $X$ as follows: $A \in t \setminus \{\emptyset\}$ if and only if $p \in A$ and $A \cap \alpha \in p$. This space is resolvable because $\alpha$ and $\{p\}$ are two disjoint dense subsets, and $\Delta(X) = \alpha$ because the cardinality of each element of $p$ is $\alpha$. Assume now that $X = \bigcup_{n<\omega} Y_n$, and $Y_n \subset Y_{n+1}$ for every $n < \omega$. Let $n_0$ be a natural number such that $p \in Y_{n_0}$, and let $n_1 < \omega$ such that $Y_{n_1} \cap \alpha \in p$ (this is because $p$ is $\omega^+$-complete). Thus, if $k \geq \max\{n_0, n_1\}$, then $\mathrm{int}\, Y_k \neq \emptyset$. This means that $(X, t)$ is not almost-$\omega$-resolvable. Furthermore, $X$ is $C_\Box$-discrete because the only continuous functions from $X$ to $\mathbb{R}$ are constant functions. Finally, $X$ is a Baire space because if $\{U_n : n < \omega\}$ is a sequence of open dense subsets of $X$, then $U_n \cap \alpha \in p$ for each $n < \omega$. Since $p$ is $\omega^+$-complete, $D = \bigcap_{n<\omega}(U_n \cap \alpha) \in p$. Therefore, $D$ is a dense subset of $X$. $\Box$

**Example 4.3.** *If there is a measurable cardinal $\alpha$, then there is a $T_1$ submaximal $C_\square$-discrete Baire space $X$ which is not almost resolvable and $\Delta(X) = \alpha$.*

Proof: Let $\alpha$ be a non-countable Ulam-measurable cardinal, and let $p$ be a free ultrafilter on $\alpha$ $w^+$-complete. Let $X = \alpha \cup \{p\}$. We define a topology $\sigma$ for $X$ as follows: $A \in \sigma \setminus \{\emptyset\}$ if and only if $A \cap \alpha \in p$. This space is not almost resolvable because if $X = \bigcup_{n<\omega} X_n$, at least for a $n_0 \in \omega$, $X_{n_0} \cap \alpha \in p$; that is $\operatorname{int} X_{n_0} \neq \emptyset$. Because of the same argument of the former example, $(X, \sigma)$ is $C_\square$-discrete, Baire and $\Delta(X) = \alpha$.

Also, observe that a subset $D$ of $X$ is dense in $(X, \sigma)$ if and only if $D \cap \alpha \in p$. So, $D$ is open. This means that $(X, \sigma)$ is submaximal. □

**Example 4.4.** *If there is a measurable cardinal $\alpha$, then there is a $T_1$ compact $C_\square$-discrete Baire space $Y$ which is not almost resolvable* (*in particular, it is irresolvable*), *and $\Delta(Y) = \alpha$.*

Proof: Let $(X, \sigma)$ be as in Example 4.3. Now, let $q \notin X$ and let $\mathcal{F} = \{\{q\} \cup A : A \subset \alpha \text{ and } |\alpha \setminus A| < \aleph_0\}$. Let $\tau$ be the topology in $Y = X \cup \{q\}$ generated by the base $\sigma \cup \mathcal{F}$. Then, $(Y, \tau)$ is compact, $T_1$ and non almost resolvable. It is not difficult to prove that $(Y, \tau)$ is a Baire space and $C_\square(Y)$ coincides with the set of constant functions. □

It is not possible to get a $T_2$ irresolvable Baire space from measurable cardinals. Indeed, Theorem 3.8 in [KST] states that there are models of *ZFC* with measurable cardinals in which all Baire $T_2$ spaces are resolvable.

We will prove that the existence of a measurable cardinal number is equiconsistent with the existence of a Tychonoff space which is not $C_\square$-discrete. In order to achieve this goal we need to establish some previous statements.

The following lemma is Fact 1.13 in [vD].

**Lemma 4.5.** *A dense-in-itself space $X$ is open-hereditarily irresolvable if and only if for every $A \subset X$, $\operatorname{int} A = \emptyset$ implies $\operatorname{int}(\operatorname{cl} A) = \emptyset$.*

Then we conclude:

**Proposition 4.6.** *In the class of open-hereditarily irresolvable* (*completely regular*) *spaces, almost resolvability and almost-$\omega$-resolvability* (*and $C_\square$-discreteness*) *coincide.*

Proof: These are consequences of Lemma 4.5 and Theorems 2.5 and 3.5. □

We are going to prove that every Baire irresolvable space is not almost-$\omega$-resolvable by using the following two lemmas. (The referee pointed out to the authors that the proof of Corollary 4.9 below, was already given in [Ma].)

**Lemma 4.7** ([H]). *Every dense-in-itself topological space $X$ is the union of a resolvable subspace $X_1$, and a hereditarily irresolvable open subspace $X_2$, and $X_1 \cap X_2 = \emptyset$.*

**Lemma 4.8** ([FL]). *For a dense-in-itself space $X$ the following statements are equivalent.*

(1) *$X$ is an almost resolvable space.*
(2) *$X = X_1 \cup X_2$ where $X_1$ is closed (with a nonempty interior if $X_1 \neq \emptyset$) and it is resolvable, and $X_2$ is a first category open set.*
(3) *There is a function $f : X \to \omega$ such that $D(f) = X$.*

**Corollary 4.9.** *If $X$ is a Baire irresolvable space, then $X$ is not almost resolvable; in particular, $X$ is not almost-$\omega$-resolvable. Besides, if $X$ is completely regular, $C_\square(X)$ is not discrete in $\square\mathbb{R}^X$.*

PROOF: If $X$ is a Baire irresolvable space, then, by Lemma 4.7, it contains a nonempty open hereditarily irresolvable subspace $Y$, and $Y$ is a Baire space. If $X$ is almost resolvable, then $Y$ is almost resolvable. But this means that $Y$ is a set of the first category (Lemma 4.8). This is a contradiction. $\square$

Even more, it is consistent with *ZFC* that every Baire dense-in-itself space is resolvable (see Theorem 4.11 below). Indeed:

**Proposition 4.10.** *Assume $V = L$. Then every space without isolated points is almost resolvable.*

PROOF: Proposition 3.9 in [KST] says that $V = L$ implies that every space of regular cardinality without isolated points is almost resolvable.

Now, let $X$ be a space without isolated points such that $|X| = \kappa$ is a singular cardinal. $\kappa^+$ is regular and $Z = \bigoplus_{\lambda<\kappa^+} X_\lambda$, with $X_\lambda = X$ for every $\lambda < \kappa^+$, is a topological space with regular cardinality and without isolated points. So, since we are assuming that $V = L$, $Z$ is almost resolvable; that is $Z$ can be written as $\bigcup_{n<\omega} J_n$ where $\operatorname{int}_Z J_n = \emptyset$. Then $X_0 = \bigcup_{n<\omega} L_n$ where $L_n = X_0 \cap J_n$ for every $n < \omega$. We have that $\operatorname{int}_{X_0} L_n = \emptyset$ and $X_0$ is homeomorphic to $X$, so $X$ is almost resolvable. $\square$

Then, by Corollary 4.9 and Proposition 4.10, we obtain:

**Theorem 4.11.** *Assume $V = L$. Then every Baire space (in particular, every Tychonoff pseudocompact space) without isolated points is resolvable.*

The authors of [A] prove their Theorem 3.1 assuming that every space is Hausdorff. By a slight modification of that proof we obtain the same result without assuming any separation axiom:

**Lemma 4.12.** *The following assertions are equivalent in ZFC.*

(1) *There is a Baire irresolvable space.*
(2) *There is a maximal space which is not $\sigma$-discrete.*
(3) *There is a submaximal space which is not $\sigma$-discrete.*

We remind the reader that a dense-in-itself Tychonoff space $(X,t)$ is *maximal Tychonoff* if for every Tychonoff topology $t'$ on $X$ with $t \subset t'$ and $t \neq t'$, $(X,t')$ has an isolated point.

**Remarks 4.13.**

(1) If $\mathcal{C}$ is a chain of Tychonoff topologies on a set $X$, then the topology $\tau$ generated by $\bigcup \mathcal{C}$ is a Tychonoff topology in $X$; if in addition, $(X,t)$ is dense-in-itself for every $t \in \mathcal{C}$, so is $(X,\tau)$.

(2) Every submaximal space is $T_1$. Moreover, every $\sigma$-discrete $T_0$ space is almost-$\omega$-resolvable (Theorem 3.5), and every maximal Tychonoff space is hereditarily irresolvable ([vD, Theorem 1.8 and Fact 1.11]).

**Theorem 4.14.** *The following assertions are equivalent in ZFC.*

(1) *There is a Baire irresolvable* (*Tychonoff*) *space.*

(2) *There is a maximal* (*maximal Tychonoff*) *space which is not almost resolvable.*

(3) *There is a submaximal space which is not almost resolvable.*

(4) *There is a maximal* (*maximal Tychonoff*) *space which is not almost-$\omega$-resolvable.*

(5) *There is a submaximal space which is not almost-$\omega$-resolvable.*

Proof: (1) $\Rightarrow$ (2): Let $(X,t)$ be a Baire irresolvable (Tychonoff) space. Because of Corollary 4.9, $X$ is not almost resolvable. Let $(X,t')$ be a maximal (maximal Tychonoff) topology containing $t$ (The existence of $t'$ is guaranteed by Remark 4.13.1 and Zorn Lemma). Assume that $(X,t')$ is almost resolvable. It happens that the direct image of an almost resolvable space under a continuous bijective function is again almost resolvable; so, $(X,t)$ must be almost resolvable, which is a contradiction by Corollary 4.9. So, $(X,t')$ is not almost resolvable.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5): Every maximal space is submaximal.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5): Every almost-$\omega$-resolvable space is almost resolvable.

(5) $\Rightarrow$ (1): The statement in (5) implies that there is a submaximal space which is not $\sigma$-discrete (Remark 4.13.2), and this assertion implies that there is a Baire irresolvable space (Lemma 4.12).

Tychonoff case of (4) $\Rightarrow$ Tychonoff case of (1): Assume that $(X,t)$ is a maximal Tychonoff space which is not almost-$\omega$-resolvable. Let $\mathcal{S} = \{U \in t : U \neq \emptyset$ and $U$ is almost-$\omega$-resolvable $\}$. Using Theorem 3.8 and Proposition 3.2, we obtain that $Y = \mathrm{cl} \bigcup \mathcal{S}$ is almost-$\omega$-resolvable. Therefore, $Z = X \setminus Y$ is an open dense-in-itself nonempty subspace of $X$ which has no open almost-$\omega$-resolvable subsets. Since $X$ is a maximal Tychonoff space, it is hereditarily irresolvable (Remark 4.13.2), hence $Z$ is irresolvable. Besides, $Z$ is a Baire space: indeed, if $U_n$ is an open dense subset of $Z$ for every $n \in \omega$, then $T = \bigcup_{n<\omega}(Z \setminus U_n)$ is of the first category, hence, it is almost-$\omega$-resolvable; so $T$ cannot contain a nonempty open subset of $Z$. This means that $\bigcap\{U_n : n < \omega\}$ is dense in $Z$, and hence $Z$ is a Baire space. $\square$

In the article of Kunen, Szymanski and Tall [KST], Theorem 3.3, they established the following result which contrasts with Theorem 4.11.

**Theorem 4.15.** *ZFC is consistent with the existence of a measurable cardinal if and only if ZFC is consistent with the existence of an irresolvable (zero-dimensional, Tychonoff) Baire space.*

As a consequence of Theorems 4.14 and 4.15 we obtain:

**Theorem 4.16.**

(1) *ZFC is consistent with the existence of a measurable cardinal if and only if ZFC is consistent with the existence of a Tychonoff dense-in-itself space $X$ for which $C_\square(X)$ is not discrete in $\square\mathbb{R}^X$.*
(2) *ZFC is consistent with "every space $X$ without isolated points is almost-$\omega$-resolvable and $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$".*
(3) *Con (ZFC) $\to$ Con (ZFC + for every $T_1$ or regular space $X$, $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$ if and only if $X$ does not have isolated points).*

Proof: (1) This is a consequence of Theorems 4.14 and 4.15.

(2) In the system $[ZFC + V = L]$ every maximal space $X$ is almost-$\omega$-resolvable because of Theorems 4.11 and 4.14. Let $(X, t)$ be a topological space (dense-in-itself). Then, we can consider a maximal topology $t'$ of $X$ such that $t \subset t'$. So, $(X, t')$ is almost-$\omega$-resolvable; then $(X, t)$ is also almost-$\omega$-resolvable, because $\mathrm{id} : (X, t') \to (X, t)$ is bijective and continuous (Proposition 3.12). Moreover, every almost $\omega$-resolvable space is $C_\square$-discrete.

(3) Here we need to apply (2) and Proposition 2.1. $\square$

Examples 4.1, 4.2, 4.3 and 4.4 and this last theorem gives us a good picture of what we can expect about almost-$\omega$-resolvable spaces when we assume the existence of measurable cardinals or $V = L$. We finish this section by giving, in Theorem 4.19, one more detail of this panorama.

The following lemma was proved in [FL].

**Lemma 4.17.** *Suppose $X$ is a dense-in-itself topological space. Then $X$ can be written as the union of mutually disjoint subsets $F$, $B$ and $N$ (resp., $R$, $I$, and $M$) such that $F$, $B$, $R$ and $I$ are open, $N$ and $M$ are nowhere dense, $F$ is of the first category, $R$ is resolvable, $B$ is Baire or $B = \emptyset$, and $I$ is irresolvable or $I = \emptyset$.*

**Corollary 4.18.** *A dense-in-itself topological space $X$ is almost resolvable if and only if $X = X_0 \cup X_1 \cup X_2 = Y_0 \cup Y_1$, where $X_0$ is open and of the first category, $X_1$ is open, Baire and resolvable or $X_1 = \emptyset$, $X_2$ is a nowhere dense set, $Y_0$ is of the first category and $Y_1$ is open Baire and resolvable or $Y_1 = \emptyset$. Besides, $X_0$, $X_1$ and $X_2$ (resp., $Y_0$ and $Y_1$) are mutually disjoint.*

Proof: Assume that $X$ is almost resolvable and put $X = F \cup B \cup N = R \cup I \cup M$ as in Lemma 4.16.

We have that $B \cap I$ is empty or is an open Baire almost resolvable space. By Corollary 4.9, if $B \cap I \neq \emptyset$, then $B \cap I$ must be resolvable.

Therefore $X = F \cup (B \cap R) \cup (B \cap I) \cup (B \cap M) \cup N$. The set $(B \cap R) \cup (B \cap I)$ is empty or is open, Baire and resolvable; and the set $(B \cap M) \cup N$ is nowhere dense, so $F \cup (B \cap M) \cup N$ is of the first category. Take $X_0 = F$, $X_1 = (B \cap R) \cup (B \cap I)$, $X_2 = (B \cap M) \cup N$, $Y_0 = X_0 \cup X_2$ and $Y_1 = X_1$.

The converse is a consequence of Lemma 4.8. □

**Theorem 4.19.** *The following properties are equivalent.*

(1) *There exists a resolvable Baire* (*Tychonoff*) *space which is not almost-$\omega$-resolvable.*
(2) *There exists a resolvable* (*Tychonoff*) *space which is not almost-$\omega$-resolvable.*
(3) *There exists an irresolvable almost resolvable* (*Tychonoff*) *space which is not almost-$\omega$-resolvable.*
(4) *There exists an almost resolvable* (*Tychonoff*) *space which is not almost-$\omega$-resolvable.*

Proof: (1) ⇒ (2) and (3) ⇒ (4) are trivial.

(2) ⇒ (3): Assume that $X$ is resolvable (Tychonoff) and it is not almost-$\omega$-resolvable. Let $Y$ be a countable irresolvable Tychonoff space (see for example [vD]), then $X \oplus Y$ is irresolvable (Tychonoff) and it is not almost-$\omega$-resolvable. Besides, since $X$ and $Y$ are almost resolvable ($X$ is resolvable and $Y$ is countable), then $X \oplus Y$ is almost resolvable.

(4) ⇒ (2): In fact, let $X$ be an almost resolvable (Tychonoff) space which is not almost-$\omega$-resolvable. By Lemma 4.8, $X = X_1 \cup X_2$, where $X_1$ is resolvable and $X_2$ is of the first category. If $X_1$ were empty or almost-$\omega$-resolvable, then $X$ would be almost-$\omega$-resolvable. Therefore, $X_1$ is a resolvable non almost-$\omega$-resolvable (Tychonoff) space.

(3) ⇒ (1): Because of Corollary 4.18, the irresolvable almost resolvable and non almost-$\omega$-resolvable (Tychonoff) space $X$ can be written as the union of two disjoint subspaces $Y_0$ and $Y_1$, where $Y_0$ is of the first category and $Y_1$ is open, Baire and resolvable. Since $X$ is not almost-$\omega$-resolvable, then $Y_1$ is a nonempty Baire resolvable (Tychonoff) space which is not almost-$\omega$-resolvable. □

By Example 4.2, if there is a measurable cardinal, then each of the equivalent conditions in Theorem 4.19 holds.

## 5. Questions

The results obtained in Section 4 produce several questions:

**Questions 5.1.**

(1) Is the existence of a measurable cardinal a consequence of the existence of a Tychonoff space without isolated points which is not almost-$\omega$-resolvable?

(2) Is it possible to construct, assuming the existence of a measurable cardinal, a regular (or a completely regular) space without isolated points which is not almost-$\omega$-resolvable?

**Questions 5.2.**

(1) Are almost resolvability and almost-$\omega$-resolvability equivalent in the class of irresolvable spaces?
(2) Is every Baire irresolvable space $C_\Box$-discrete?
(3) Is the existence of an irresolvable Baire space equivalent to the existence of a maximal non $C_\Box$-discrete space?
(4) Does *Con* (*ZFC*) imply *Con* (*ZFC* + every dense-in-itself space is $C_\Box$-discrete + there exists a dense-in-itself non almost-$\omega$-resolvable space)?
(5) Is every non almost resolvable irresolvable space a Baire space?
(6) Is every maximal space $C_\Box$-discrete?

W.W. Comfort and S. García-Ferreira asked in [CGF] whether every Tychonoff pseudocompact space without isolated points is resolvable. Theorem 4.11 implies that there is a model of *ZFC* in which the answer to this question is in the affirmative. Besides, Malykhin asked if every Lindelöf Tychonoff space without isolated points with $\Delta(X) > \omega$ was resolvable [CGF]. This leads us to the following problems:

**Questions 5.3.**

(1) Is every pseudocompact Tychonoff dense-in-itself space almost-$\omega$-resolvable?
(2) Is every Lindelöf Tychonoff dense-in-itself space almost-$\omega$-resolvable?

The following problem is motivated by Corollary 3.10, Examples 4.2, 4.3 and 4.4, and Question 8.13 in [CGF].

**Question 5.4.** Is every connected Tychonoff dense-in-itself space almost-$\omega$-resolvable?

Of course, the more general question regarding topological groups is:

**Question 5.5.** Is every dense-in-itself topological group almost-$\omega$-resolvable?

In [CM] it is proved that every Baire topological group is resolvable, thus a more restricted but still reasonable question is:

**Question 5.6.** Is every dense-in-itself Baire topological group almost-$\omega$-resolvable?

Theorem 3.9 and Corollary 3.10 produce the following:

**Questions 5.7.**

(1) Is every Tychonoff product of an infinite collection of non-trivial $T_1$ topological spaces a maximally resolvable space?
(2) Is every arc connected $T_2$ space maximally resolvable?

We finish this list of problems with the following natural questions:

**Questions 5.8.**

(1) (O. Okunev) Let $\{\tau_s : s \in S\}$ be a chain of almost-$\omega$-resolvable topologies on $X$. Is, then, the topology generated by $\bigcup_{s\in S} \tau_s$ almost-$\omega$-resolvable?
(2) Is $X$ almost-$\omega$-resolvable if $X^2$ satisfies this property?

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Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, Mexico D.F., 04510, México

*E-mail*: atamariz@servidor.unam.mx

Escuela de Ciencias Físico Matemáticas, Universidad Autónoma de Sinaloa, Ciudad Universitaria, Ave. de las Américas Culiacan, Sinaloa, 80010 México

*E-mail*: vrhi@uas.uasnet.mx

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