On the subsets of non locally compact points of ultracomplete spaces

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Abstract. In 1998, S. Romaguera [13] introduced the notion of cofinally Čech-complete spaces equivalent to spaces which we later called ultracomplete spaces. We define the subset of points of a space X at which X is not locally compact and call it an nlc set. In 1999, García-Máynez and S. Romaguera [6] proved that every cofinally Čech-complete space has a bounded nlc set. In 2001, D. Buhagiar [1] proved that every ultracomplete GO-space has a compact nlc set. In this paper, ultracomplete spaces which have compact nlc sets are studied. Such spaces contain dense locally compact subspaces and coincide with ultracomplete spaces in the realms of normal γ-spaces or ks-spaces.

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1. Preliminaries

In this paper, *all spaces are Tychonoff and all maps are continuous and onto*. For a subset A of Y, where Y is also a subset of a space X, by $\overline{A}^Y$ $(\partial_Y A)$ we denote the closure (the boundary) of A in Y. For a collection $\mathcal{A}$ of subsets of a space X, by $\bigcup\mathcal{A}$, $\bigcap\mathcal{A}$ and $\bigcap\overline{\mathcal{A}}$ we denote the union $\bigcup\{A \mid A \in \mathcal{A}\}$, the intersection $\bigcap\{A \mid A \in \mathcal{A}\}$ and $\bigcap\{\overline{A} \mid A \in \mathcal{A}\}$ respectively. Also, by $\mathcal{A}^F$ we denote the collection of all unions of finite subcollections from $\mathcal{A}$. Moreover, let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X, then we say that $\mathcal{A}$ meshes with $\mathcal{B}$ if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. When $\mathcal{A}$ and $\mathcal{B}$ are both coverings of X, we say that $\mathcal{A}$ refines $\mathcal{B}$ if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$ and we express this by $\mathcal{A} < \mathcal{B}$.

By $\mathbb{R}$ and $\mathbb{N}$, we denote the real line and the set of all natural numbers, respectively.

We refer the reader to [5], [12] for undefined terms.

For a space X, by a *compactification* of X we understand a Hausdorff compactification, in particular, by βX we denote the Stone-Čech compactification of X.

Recall that a collection $\mathcal{B}$ of open subsets of a space X is called a *base* for a subset A in X if all the elements of $\mathcal{B}$ contain A and for any open subset V

containing A there exists a $U \in \mathcal{B}$ such that $A \subset U \subset V$. The *character* of A in X is defined to be the smallest cardinal number of the form $|\mathcal{B}(A)|$, where $\mathcal{B}(A)$ is a base for A in X, and is denoted by $\chi(A, X)$. A space X is said to be of *countable type* provided every compact subset is contained in a compact subset of countable character ([14]).

In [2], [13], the class of ultracomplete spaces, which were called cofinally Čech-complete by S. Romaguera, was studied and this class lies between the class of locally compact spaces and the class of Čech-complete spaces.

We recall the definition of ultracomplete spaces.

Definition 1.1 ([2]). A space X is said to be *ultracomplete* if X has countable character in βX, i.e. $\chi(X, \beta X) \leq \omega_0$, equivalently, the remainder $\beta X \setminus X$ is hemicompact.

The well known equivalent conditions of ultracompleteness are as follows.

Theorem 1.2 ([2], [13]). *For a space X, the following conditions are equivalent:*

(1) *X is ultracomplete;*
(2) *$\chi(X, cX) \leq \omega_0$ for every compactification cX of X;*
(3) *$\chi(X, cX) \leq \omega_0$ for some compactification cX of X;*
(4) *there exists a sequence $\{\mathcal{U}_n\}_{n\geq 1}$ of open coverings of X such that, if $\mathcal{F}$ is a filter base of X which meshes with some sequence $\{U_n\}_{n\geq 1}$, where $U_n \in \mathcal{U}_n$, then $\mathcal{F}$ has a cluster point;*
(5) *there exists a sequence $\{\mathcal{U}_n\}_{n\geq 1}$ of open coverings of X such that for every open covering $\mathcal{W}$ of X, there exists an $m \in \mathbb{N}$ such that $\mathcal{U}_m$ refines $\mathcal{W}^F$.*

Let us recall that a nonempty subset A of a space X is *bounded* in X if every real valued continuous function on X is bounded on A.

The next lemma can be easily proved.

Lemma 1.3. *A subset A of a space X is bounded in X if and only if the collection $\{U \in \mathcal{B} \mid U \cap A \neq \emptyset\}$ is finite for every locally finite collection $\mathcal{B}$ of nonempty open subsets.*

We denote by X_C [6] the subset of points of a space X at which X is not locally compact and call X_C its *nlc set*. Note that X_C is closed in X.

A. García-Máynez and S. Romaguera [6] proved the following theorem.

Theorem 1.4. *If X is an ultracomplete space, then X_C is a bounded subset in X.*

Throughout this paper, we are primarily concerned with ultracomplete spaces whose nlc sets are compact.

We can slightly generalize the result of M. Henriksen and J.R. Isbell [7] that $\overline{cX \setminus X} = (cX \setminus X) \cup X_C$ for any compactification cX of a space X.

Lemma 1.5. *If X is a dense subset of a locally compact space Z, then* $\overline{Z\backslash X} = (Z\backslash X) \cup X_C$.

Proposition 1.6 ([2], [13], [14]). *For a space X, the following implications hold:*

X is locally compact $\Longrightarrow$ X is ultracomplete $\Longrightarrow$ X is Čech-complete $\Longrightarrow$ X is of countable type $\Longrightarrow$ X is of point countable type $\Longrightarrow$ X is a k-space.

Therefore, Theorem 4 of [6] can be rewritten as follows.

Theorem 1.7. *For a space X, the following conditions are equivalent:*

(1) *X is ultracomplete and X_C is compact;*
(2) *the nlc set X_C is contained in a compact subset of countable character;*
(3) *X is the union of a compact subset of countable character and of an open locally compact subset.*

We now give other conditions for the nlc sets of ultracomplete spaces to be compact.

Theorem 1.8. *Let cX be any compactification of a space X. Then the following conditions are equivalent:*

(1) *X is ultracomplete and X_C is compact;*
(2) *X is of countable type and X_C is compact;*
(3) *$cX \setminus X$ is hemicompact and locally compact;*
(4) *the nlc set X_C is compact and $\chi(X_C, \overline{cX \setminus X}) \leq \omega_0$;*
(5) *$cX \setminus X$ is the countable union of an increasing sequence $\{K_n\}_{n\geq 1}$ of compact subsets such that $\bigcap_{k\leq n} \partial_{cX\setminus X} K_n = \emptyset$ for every $k \in \mathbb{N}$.*

PROOF: The equivalence of (1) and (2) follows from Proposition 1.6 and Theorem 1.7, while the equivalence of (1) and (3) follows from Definition 1.1 and Lemma 1.5. Also, the equivalence of (3) and (4) is evident in light of Lemma 1.5. Since (3) $\Longrightarrow$ (5) follows from [5, p. 250], we prove the implication (5) $\Longrightarrow$ (3). Suppose that $cX \setminus X$ has no compact neighborhood at some point p of $cX \setminus X$. Then, we have that $p \in K_k \subset K_{k+1} \subset \dots$ for some $k \in \mathbb{N}$ and $p \in \bigcap_{k\leq n} \partial_{cX\setminus X} K_n$, which is a contradiction. Hence $cX \setminus X$ is locally compact and so, there exists a sequence $\{H_n\}_{n\geq 1}$ of compact subsets of $cX \setminus X$ such that $K_n \cup H_{n-1} \subset \operatorname{int} H_n$ for $n \geq 2$. If there exist $x_n \in L \setminus H_n$ for some compact subset L of $cX \setminus X$ and for every $n \in \mathbb{N}$, then for some $m \in \mathbb{N}$, $L \cap (\operatorname{int} H_m)$ contains a cluster point x_0 of a sequence $\{x_n\}_{n\geq 1}$. This contradiction implies that $cX \setminus X$ is hemicompact. □

We now give the definition of c-ultracomplete spaces.

Definition 1.9. A space X is said to be *c-ultracomplete* if it satisfies condition (1), and hence all the conditions, in Theorem 1.8.

From the definition, we have following implications:

$$\text{locally compact} \Longrightarrow c\text{-ultracomplete} \Longrightarrow \text{ultracomplete.}$$

In the realm of paracompact spaces, c-ultracompleteness and ultracompleteness are equivalent by Theorem 1.4 and Lemma 1.3. Also, D. Buhagiar [1] proved the following result for GO-spaces.

Theorem 1.10. *Every ultracomplete GO-space is c-ultracomplete.*

But in general none of the above implications are reversible.

Example 1.11 ([1]). Let $Y = [0, \omega_1)$ be the set of all ordinals less than the first uncountable ordinal with the order topology and let X be the countable product $\prod_{n\geq 1} Y_n$, where $Y_n = Y$ for every $n \in \mathbb{N}$. Then, X is ultracomplete, countably compact but not c-ultracomplete.

Example 1.12 ([2]). Let X be the subspace $[0,1] \setminus \{1/n \mid n \geq 2\}$ of $\mathbb{R}$. Then, X is a c-ultracomplete, σ-compact, separable metric space. But, it is neither locally compact nor hemicompact.

I do not know whether a Čech-complete, countably compact space is ultracomplete or not [3, Problem 5.1]. But, we get the following result with the special case.

Theorem 1.13. *Let D be an infinite discrete space and let X be a subspace of βD such that $D \subset X$. Then, the following assertions hold:*

(1) *X is not sequentially compact;*
(2) *if X is Čech-complete and $|\beta D \setminus X| < 2^{\mathfrak{c}}$, then X is ultracomplete, countably compact;*
(3) *if $\omega_0 \leq |\beta D \setminus X| < 2^{\mathfrak{c}}$, then X is not locally compact.*

Proof: (1) is well known [5, p. 228], so we prove (2) and (3).

(2): First, we show that X is ultracomplete. There exists a countable decreasing sequence $\{U_n\}_{n\geq 1}$ of open subsets in βD such that $X = \bigcap_{n\geq 1} U_n$. Suppose that there exists an open subset W in βD such that $X \subset W$ and $U_n \setminus W \neq \emptyset$ for every $n \in \mathbb{N}$. Take arbitrary points $x_n \in U_n \setminus W$ for every $n \in \mathbb{N}$. If for some $x_0 \in \beta D$, $x_0 = x_{n(i)}$ for $n(1) < n(2) < \dots$, then $x_0 \in X$ and this is a contradiction. Therefore, there exists an infinite subset K of $\{x_n\}_{n\geq 1}$. On the other hand, $\overline{K} \subset \beta D \setminus W \subset \beta D \setminus X$ and $|\overline{K}| \geq 2^{\mathfrak{c}}$. This contradiction asserts that $\{U_n\}_{n\geq 1}$ is a base of X in βD.

Secondly, we prove that X is countably compact. Let F be a countably infinite subset of X. Then we have that $|\overline{F} \cap X| \geq 2^{\mathfrak{c}}$.

Indeed, if $|\overline{F} \cap X| < 2^{\mathfrak{c}}$, then

$$2^{\mathfrak{c}} \leq |\overline{F}| = |(\overline{F} \cap X) \cup (\overline{F} \cap (\beta D \setminus X))| \leq |\overline{F} \cap X| + |\beta D \setminus X| < 2^{\mathfrak{c}},$$

which is a contradiction. Hence, $\overline{F}^X \setminus F \neq \emptyset$.

(3): If X is locally compact, then $\beta D \setminus X$ is an infinite closed subset in βD. Therefore $|\beta D \setminus X| \geq 2^{\mathfrak{c}}$, which is a contradiction. □

Corollary 1.14. *Let D be an infinite discrete space. If a Čech-complete space X is a continuous image of a subspace Z of βD such that $D \subset Z$ and $|\beta D \setminus Z| < 2^{\mathfrak{c}}$, then X is ultracomplete, countably compact.*

Proof: Let $h : \beta D \longrightarrow \beta Z$ be a homeomorphism such that $h(z) = z$ for each $z \in Z$ and let $f : Z \longrightarrow X$ be a surjective map. Then, f has a continuous extension $g : \beta Z \longrightarrow \beta X$. We put $W = (g \circ h)^{-1}(X)$. Then, the restriction $k : W \longrightarrow X$ of $g \circ h$ is perfect and $Z \subset W \subset \beta D$. Therefore, W is Čech-complete and $|\beta D \setminus W| < 2^{\mathfrak{c}}$, which implies that W is ultracomplete, countably compact. Therefore, X is also ultracomplete, countably compact. □

Now, we construct a c-ultracomplete, countably compact space which is not locally compact.

Example 1.15. Let D be an infinite discrete space. Then, there exist a sequence $A = \{a_n\}_{n\geq 1} \subset \beta D \setminus D$ and a sequence $\{V_n\}_{n\geq 1}$ of clopen subsets in βD such that $a_n \in V_n$ and $V_n \cap V_m = \emptyset$ whenever $n \neq m$. Then, $X = \beta D \setminus A$ is Čech-complete and $|\beta D \setminus X| = \omega_0 < 2^{\mathfrak{c}}$. Therefore, by Theorem 1.13, X is ultracomplete, countably compact but not locally compact. Next, we show that X is c-ultracomplete. Indeed, since $\beta X \setminus X = \beta D \setminus X = A$ is a countable discrete space, $\beta X \setminus X$ is locally compact and hemicompact. Hence, X is c-ultracomplete by Theorem 1.8.

Theorem 1.16. *For a space X, the following conditions are equivalent:*

(1) *X is c-ultracomplete;*

(2) *there exists a sequence $\{\mathcal{U}_n\}_{n\geq 1}$ of open coverings of X satisfying the following two conditions*
   (a) *for every $n \in \mathbb{N}$, X_C is contained in the union of some finite subcollection of $\mathcal{U}_n$, and*
   (b) *if $\mathcal{F}$ is a filter base of X which meshes with some $\{U_n\}_{n\geq 1}$, where $U_n \in \mathcal{U}_n$, then $\mathcal{F}$ has a cluster point;*

(3) *there exists a sequence $\{\mathcal{U}_n\}_{n\geq 1}$ of open coverings of X satisfying condition* (a) *of* (2) *and the following condition*
   (c) *if $\mathcal{U}$ is any open covering of X, then there exists an $m \in \mathbb{N}$ such that $\mathcal{U}_m$ refines $\mathcal{U}^F$ (i.e. $\mathcal{U}_m < \mathcal{U}^F$).*

Proof: The implications (1) ⇒ (2) and (2) ⇒ (3) are proved in (1) ⇒ (3) and (3) ⇒ (4) of Theorem 2.2 in [2]. Here, we note that the same sequence satisfying (b) of (2) also satisfies (c) of (3).

(3) ⇒ (1): Ultracompleteness of X is also proved in (4) ⇒ (1) of Theorem 2.2 in [2]. We show that the nlc set X_C is compact. Suppose that X_C is not compact.

Then, there exists a closed filter base $\mathcal{F}$ of X_C such that $\bigcap \mathcal{F} = \emptyset$. Therefore, $\mathcal{U} = \{X \backslash F \mid F \in \mathcal{F}\}$ is open covering of X and hence, some $\mathcal{U}_m$ refines $\mathcal{U}^F$. On the other hand, there exists a finite subcollection $\{U_1, \dots, U_p\}$ of $\mathcal{U}_m$ such that $X_C \subset \bigcup_{i=1}^{p} U_i$. Also for each i $(1 \leq i \leq p)$, we have that $U_i \subset \bigcup_{k=1}^{l(i)} (X \setminus F_{i,k})$ for some finite subcollection $\{X \setminus F_{i,k} \mid k = 1, \dots, l(i)\}$ of $\mathcal{U}$. Therefore, $X_C \subset X \setminus \{\bigcap_{i=1}^{p} (\bigcap_{k=1}^{l(i)} F_{i,k})\}$ and $\bigcap_{i=1}^{p} (\bigcap_{k=1}^{l(i)} F_{i,k}) \neq \emptyset$, which is a contradiction. This fact asserts that X is c-ultracomplete. $\square$

One can easily prove the following two results.

Proposition 1.17. *The product $X \times Y$ of a c-ultracomplete space X and a compact space Y is c-ultracomplete.*

Proposition 1.18. *The closed subspace of a c-ultracomplete space is c-ultracomplete.*

Example 1.19. An open dense subspace of a c-ultracomplete space is not necessarily ultracomplete.

Indeed, the space X given in Example 1.15 is c-ultracomplete, non locally compact and $X \times D$ is an open dense subspace of the c-ultracomplete space $X \times \beta D$. But, $X \times D$ is homeomorphic to the topological sum $\bigoplus\{X_d \mid d \in D\}$, where $X_d = X$, which is not ultracomplete [3, Theorem 3.3].

We consider ultracomplete spaces which contain dense locally compact subsets.

Example 1.20. There exists an ultracomplete space in which no dense subspace is locally compact.

Indeed, let X be the ultracomplete space given in Example 1.11. If X contains a locally compact dense subspace P, then P is open in X, which is a contradiction.

Theorem 1.21. *Every c-ultracomplete space contains an open dense locally compact subspace.*

Proof: $X_C = \partial X_C$ since the nlc set X_C is compact. Therefore, $X \backslash X_C$ is an open dense locally compact subspace of X. $\square$

Remark. The reverse of the above theorem is not true. For example, the subspace $X = \mathbb{N} \cup \{p\}$, where $p \in \beta\mathbb{N} \setminus \mathbb{N}$, of $\beta\mathbb{N}$ contains an open dense discrete subspace $\mathbb{N}$, but X is not a k-space.

2. Maps and c-ultracomplete spaces

We proved ([2], [6]) that ultracompleteness is an invariant of open maps. One can strengthen this result by using bi-quotient maps.

Definition 2.1 ([11]). A map $f : X \longrightarrow Y$ is *bi-quotient* if, whenever $y \in Y$ and $\mathcal{U}$ is a covering of $f^{-1}(y)$ by open subsets in X, then there exists a finite subcollection $\{U_1, \ldots, U_n\}$ of $\mathcal{U}$ such that $\bigcup_{i=1}^n f(U_i)$ is a neighborhood of y in Y. Evidently every open map is bi-quotient.

Theorem 2.2. *Let $f : X \longrightarrow Y$ be a bi-quotient map. If X is an ultracomplete (a c-ultracomplete) space, then Y is ultracomplete (c-ultracomplete, respectively).*

Proof: First, let X be an ultracomplete space. We show that Y is also ultracomplete. By Theorem 1.2, there exists a sequence $\{\mathcal{U}_n\}_{n\geq 1}$ of open coverings of X such that for every open covering $\mathcal{W}$ of X, there exists $\mathcal{U}_k$ which refines $\mathcal{W}^F$. Let $\mathcal{V}_n = \{\operatorname{int} f(V) \mid V \in \mathcal{U}_n^F\}$ for every $n \in \mathbb{N}$. Then, $\{\mathcal{V}_n\}_{n\geq 1}$ is a sequence of open coverings of Y. Now, let $\mathcal{H}$ be an open covering of Y and denote by $f^{-1}(\mathcal{H})$ the covering $\{f^{-1}(H) \mid H \in \mathcal{H}\}$. Then, the fact that some $\mathcal{U}_m$ refines $(f^{-1}(\mathcal{H}))^F$ asserts that $\mathcal{V}_m$ refines $\mathcal{H}^F$. Therefore, Y is ultracomplete by Theorem 1.2.

Next, let X be a c-ultracomplete space. By the above proof, we only need to show the compactness of Y_C. Let $y \in Y_C \setminus f(X_C)$. Then for any $x \in f^{-1}(y)$, there exists an open neighborhood $U(x)$ of x such that $\overline{U(x)}$ is compact and $\overline{U(x)} \cap X_C = \emptyset$. By the bi-quotientness of f, there exists a finite subset $\{x_1, \ldots, x_n\} \subset f^{-1}(y)$ such that

$$y \in \operatorname{int}\{f(U(x_1)) \cup \cdots \cup f(U(x_n))\} \subset f\{\overline{U(x_1) \cup \cdots \cup U(x_n)}\},$$

where the latter set is compact. This contradiction implies that $Y_C \subset f(X_C)$ and hence, Y_C is compact. $\square$

We note that ultracompleteness is not preserved by closed maps ([2]).

Now, we have that $f(X_C) = Y_C$ [7] for a perfect map $f : X \longrightarrow Y$. So, we can easily see the following result by Theorem 1 in [6] (or [2]).

Theorem 2.3. *Let $f : X \longrightarrow Y$ be a perfect map. Then, X is c-ultracomplete if and only if Y is c-ultracomplete.*

It is not difficult to see that the topological sum $X \oplus Y$ of c-ultracomplete spaces X and Y is also c-ultracomplete ([3]). Therefore, we have the following result.

Proposition 2.4. *Let a space X be the union of c-ultracomplete subsets A and B. If both A and B are open or closed in X, then X is c-ultracomplete.*

But, Remark of Theorem 1.21 asserts that the union of a discrete open subset and of a single point is not necessarily ultracomplete.

For infinite sums, we have the following theorem by Proposition 2.4 and [3, Theorem 3.3].

Theorem 2.5. *Let A be some indexing set. Then the topological sum $X = \bigoplus_{\alpha\in A} X_\alpha$ is c-ultracomplete if and only if there exists a finite subset $A_0 \subset A$ such that X_α is locally compact for every $\alpha \in A \setminus A_0$ and X_α is c-ultracomplete for every $\alpha \in A_0$.*

Corollary 2.6. *Let A be some indexing set and let $\{X_\alpha \mid \alpha \in A\}$ be a locally finite covering of a space X, where each X_α is a c-ultracomplete (ultracomplete) closed subsets. Then X is c-ultracomplete (ultracomplete) if and only if X_α is locally compact for every $\alpha \in A \setminus A_0$ for some finite subset $A_0 \subset A$.*

Proof: First, we define a map $p : \bigoplus_{\alpha\in A} X_\alpha \longrightarrow X$ from the topological sum of $\{X_\alpha \mid \alpha \in A\}$ onto X. Let $p_\alpha : X_\alpha \longrightarrow X$ be an embedding such that $p_\alpha(x) = x$ for every $\alpha \in A$ and $p(x) = p_\alpha(x)$ if $x \in X_\alpha$. Then, the map p is perfect by local finiteness of the closed covering $\{X_\alpha \mid \alpha \in A\}$. Therefore, the proof of this corollary follows from Theorem 2.3 and 2.5. □

3. Subsets of countable character

In this section, we discuss in varying detail the contents given in Theorem 1.7 and also give some conditions under which ultracomplete spaces are c-ultracomplete. We begin with a definition.

Definition 3.1. A sequence $\{W_n\}_{n\geq 1}$ of (open) subsets of a space X is said to be (*open*) *complete* if, whenever $\mathcal{F}$ is any filter base of X such that for every $n \in \mathbb{N}$, $F_n \subset W_n$ for some $F_n \in \mathcal{F}$, then $\mathcal{F}$ has a cluster point. And, a subset A is said to be *contained* in $\{W_n\}_{n\geq 1}$ if $A \subset \bigcap_{n\geq 1} W_n$.

We note that if a closed subset K is contained in a complete sequence, then K is compact.

Throughout this section, the following theorem is fundamental.

Theorem 3.2. *For a space X, the following conditions are equivalent:*

(1) *X is ultracomplete (c-ultracomplete);*
(2) *X is the union of two ultracomplete (c-ultracomplete, respectively) subsets P and Q and, ∂Q is contained in an open complete sequence;*
(3) *X is the union of two ultracomplete (c-ultracomplete, respectively) subsets P and Q and, Q satisfies the following condition*
    (∗) *there exist a compact subset K and a subset L of countable character satisfying $\partial Q \subset L \subset K$.*

Proof: First, we prove the ultracomplete case. Since the implication (1) ⇒ (2) is evident from $X = X \cup \emptyset$, we prove the implications (2) ⇒ (3) and (3) ⇒ (1).

(2) ⇒ (3): We can assume that there exists a decreasing open complete sequence $\{W_n\}_{n\geq 1}$ containing ∂Q. Since ∂Q is compact, we can construct an open complete sequence $\{H_n\}_{n\geq 1}$ such that $\overline{H_{n+1}} \subset H_n \subset W_n$ for every $n \in \mathbb{N}$ and

$\partial Q \subset L = \bigcap_{n\geq 1} H_n = \bigcap_{n\geq 1} \overline{H_n}$. This implies that $K = L$ is compact with the base $\{H_n \mid n \geq 1\}$.

(3) $\Rightarrow$ (1): Let $\{W_n\}_{n\geq 1}$ be a decreasing open base of L in X. And, let $\{\mathcal{U}_n\}_{n\geq 1}$ and $\{\mathcal{V}_n\}_{n\geq 1}$ be a sequence of open coverings of P and Q respectively, satisfying condition (4) of Theorem 1.2 and, let $\mathcal{U}_{n+1} < \mathcal{U}_n$ and $\mathcal{V}_{n+1} < \mathcal{V}_n$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we put

$$\mathcal{G}_n = \{U \setminus \overline{Q} \mid U \in \mathcal{U}_n\} \cup \{W_n\} \cup \{V \cap \operatorname{int} Q \mid V \in \mathcal{V}_n\}.$$

We show that the sequence $\{\mathcal{G}_n\}_{n\geq 1}$ of open coverings of X satisfies condition (4) of Theorem 1.2. Consider a filter base $\mathcal{F}$ of X which meshes with some sequence $\{G_n \mid G_n \in \mathcal{G}_n\}_{n\geq 1}$. Let $K \cap \overline{F} \neq \emptyset$ for any $F \in \mathcal{F}$. Then, we have that $\bigcap \overline{\mathcal{F}} \neq \emptyset$ by compactness of K. Therefore, one can assume that $K \cap \overline{H} = \emptyset$ for some $H \in \mathcal{F}$. Then, there exists an $m \in \mathbb{N}$ such that $L \subset W_m \subset X \setminus \overline{H}$ and hence, $G_k \neq W_k$ for every $k \geq m$. This implies that for every $k \geq m$, $G_k = U_k \setminus \overline{Q}$ for some $U_k \in \mathcal{U}_k$ or $G_k = V_k \cap \operatorname{int} Q$ for some $V_k \in \mathcal{V}_k$. Therefore, let us assume that $G_{k(n)} = U_{k(n)} \setminus \overline{Q}$ for some infinite sequence $\{k(1) < k(2) < \dots\}$. Then, for every $n \in \mathbb{N}$, we have $\emptyset \neq F \cap G_{k(n)} \subset (F \cap P) \cap U_{k(n)}$ for every $F \in \mathcal{F}$. Therefore, $\mathcal{F}(P) = \{F \cap P \mid F \in \mathcal{F}\}$ is a filter base of P which meshes with the subsequence $\{U_{k(n)} \mid n \in \mathbb{N}\}$. Since $\{\mathcal{U}_{k(n)}\}_{n\geq 1}$ satisfies condition (3) of Theorem 1.2, we have that $\emptyset \neq \bigcap \overline{\mathcal{F}(P)}^P \subset \bigcap \overline{\mathcal{F}}$, which implies that X is ultracomplete.

The case when $G_{l(n)} = V_{l(n)} \cap \operatorname{int} Q$ for some infinite sequence $\{l(1) < l(2) < \dots\}$ is the same.

Secondly, for the c-ultracomplete case, we only prove the implication (3) $\Rightarrow$ (1). Since X is ultracomplete by the ultracomplete case, we show that the nlc set X_C is compact. We have that $X_C \cap (P \setminus \overline{Q}) \subset P_C$ and $X_C \cap \overline{Q} \subset \partial Q \cup Q_C$. Therefore, the subset $X_C = (X_C \cap \overline{Q}) \cup (X_C \cap (P \setminus \overline{Q}))$ is a closed subset of a compact subset $P_C \cup \partial Q \cup Q_C$ and hence, X_C is compact. □

Remark. The space $X = \mathbb{N} \cup \{p\}$ given in the remark after Theorem 1.21 is the union of a locally compact open subset $\mathbb{N}$ and a compact subset $\{p\}$, but X is not k-space. This fact shows that condition $(*)$ of the above theorem cannot be omitted.

Theorem 3.3. *For a space X, the following conditions are equivalent:*

(1) *X is ultracomplete (c-ultracomplete, Čech-complete);*
(2) *X is the union of subsets P, Q such that $\operatorname{int} P$, $\operatorname{int} Q$ are both ultracomplete (c-ultracomplete, Čech-complete, respectively) and, ∂Q is contained in an open complete sequence;*
(3) *X is the union of subsets P, Q such that $\operatorname{int} P$, $\operatorname{int} Q$ are both ultracomplete (c-ultracomplete, Čech-complete, respectively) and, Q satisfies condition $(*)$ of Theorem 3.2.*

Proof: First, we prove the ultracomplete case. Since the proof of (2) $\Rightarrow$ (3) is analogous to the proof of (2) $\Rightarrow$ (3) of Theorem 3.2, we only prove the implication (3) $\Rightarrow$ (1). Let $\{W_n\}_{n\geq 1}$ be a decreasing open base of L in X. And, let $\{\mathcal{U}_n\}_{n\geq 1}$ and $\{\mathcal{V}_n\}_{n\geq 1}$ be a sequence of open coverings of $\operatorname{int} P$ and $\operatorname{int} Q$ respectively, satisfying condition (4) of Theorem 1.2 and, let $\mathcal{U}_{n+1} < \mathcal{U}_n$ and $\mathcal{V}_{n+1} < \mathcal{V}_n$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we put $\mathcal{G}_n = \mathcal{U}_n \cup \{W_n\} \cup \mathcal{V}_n$. Then, we can see that the sequence $\{\mathcal{G}_n\}_{n\geq 1}$ of open coverings of X satisfies condition (4) of Theorem 1.2. Therefore, the rest of the proof should be completed in the same manner as (3) $\Rightarrow$ (1) in Theorem 3.2.

Secondly, for the c-ultracomplete case, we only prove the implication (3) $\Rightarrow$ (1). Ultracompleteness of X is evident. Next, we put $\operatorname{int} P = R$ and $\operatorname{int} Q = S$. Then, we have that $X_C \cap R \subset R_C$ and $X_C \cap \overline{Q} \subset \partial Q \cup S_C$. Since $X = R \cup \overline{Q}$, X_C is compact, which implies that X is c-ultracomplete.

Last, for the Čech-complete case, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are evident and (3) $\Rightarrow$ (1) can be proved in the same manner as the ultracomplete case. □

We can easily see a similar result for locally compact spaces.

Proposition 3.4. *A space X is locally compact if and only if X is the union of subsets P and Q such that $\operatorname{int} P$ and $\operatorname{int} Q$ are both locally compact and, ∂Q is contained in an interior of some compact subset K.*

Proof: The *only if* part is evident and the *if* part follows from the fact that $X = \operatorname{int} P \cup \operatorname{int} K \cup \operatorname{int} Q$ is locally compact. □

Example 3.5. Let X be the subspace $\mathbb{R} \setminus \{1/k \mid k = \pm 1, \pm 2, \dots\}$, and $P = (-\infty, 0) \setminus \{1/k \mid k = -1, -2, \dots\}$ and $Q = [0, \infty) \setminus \{1/k \mid k = 1, 2, \dots\}$. Then, $\operatorname{int} P$ and $\operatorname{int} Q$ are locally compact and $\partial Q = \{0\}$ is compact subset of countable character, but X is not locally compact.

We remark that by Theorem 3.3, X is c-ultracomplete.

Definition 3.6. For a space X, a structure $(\{g_n(x)\}_{n\geq 1} \mid x \in X)$ is called a *g-structure* if $g_n(x)$ is an open neighborhood of x and $g_{n+1}(x) \subset g_n(x)$ for any $x \in X$ and every $n \in \mathbb{N}$. We now consider the following conditions on a g-structure $(\{g_n(x)\}_{n\geq 1} \mid x \in X)$.

(A) If $y_n \in g_n(p)$, $x_n \in g_n(y_n)$ for every $n \in \mathbb{N}$, then p is a cluster point of $\{x_n\}_{n\geq 1}$, equivalently, if $y_n \in g_n(x_n)$ for every $n \in \mathbb{N}$ and $\lim x_n = p$ (p is a cluster point of $\{x_n\}_{n\geq 1}$), then $\lim y_n = p$ (p is a cluster point of $\{y_n\}_{n\geq 1}$, respectively).

(B) If $y_n \in g_n(x_n)$ for every $n \in \mathbb{N}$ and $\lim y_n = p$, then $\lim x_n = p$.

A space X is said to be a *γ-space* [8] (*ks-space* [15]) if there exists a g-structure satisfying condition (A) (condition (B), respectively).

Remark. The space X given in the remark after Theorem 1.21 is a paracompact ks-space which is not a γ-space. Also, there exists a ks-space which is not paracompact ([10]).

The Sorgenfrey line is a paracompact γ-space which is not a ks-space.

The next lemma can be easily proved ([4], [15]).

Lemma 3.7. (1) *If X is a γ-space, then any compact subset of X is of countable character in X.*

(2) *If A is a countably compact subset of a γ-space or a ks-space X, then A is compact metrizable.*

(3) *If A is a closed bounded subset of a normal γ-space X, then A is a compact metrizable subset of countable character.*

Theorem 3.8. *For a normal γ-space X, the following conditions are equivalent:*

(1) *X is c-ultracomplete;*
(2) *X is ultracomplete;*
(3) *the nlc set X_C is bounded in X;*
(4) *$\chi(X_C, cX) \leq \omega_0$ for some compactification cX of X;*
(5) *X is the union of a bounded subset A in X and of an ultracomplete subset B.*

Proof: The equivalence of (1) and (2) follows Theorem 1.4 and Lemma 3.7. The implication (2) $\Rightarrow$ (3) follows from Theorem 1.4.

(3) $\Rightarrow$ (4): X_C is compact and $\chi(X_C, X) \leq \omega_0$ by Lemma 3.7. Therefore, for the Stone-Čech compactification βX, we have that $\chi(X_C, \beta X) = \chi(X_C, X) \leq \omega_0$ ([7, Lemma 3.5]).

(4) $\Rightarrow$ (2): By Theorem 1.2, it is sufficient to prove that $\chi(X, cX) \leq \omega_0$. Let $\{W_n\}_{n\geq 1}$ be a decreasing open base of $X_C = X \cap \overline{cX \setminus X}$ in cX. If U is any open subset in cX containing X, then there exists an $m \in \mathbb{N}$ such that $X \cap \overline{cX \setminus X} \subset W_m \subset U$ and hence, $X \subset W_m \cup X \subset U$. Now, $W_n \cup X$ is open in cX for every $n \in \mathbb{N}$. Indeed, if $z \in (W_n \cup X) \setminus W_n$, then, $z \in cX \setminus \overline{cX \setminus X} \subset W_n \cup X$. Therefore, $\{W_n \cup X\}_{n\geq 1}$ is an open base of X in cX.

The implication (2) $\Rightarrow$ (5) is evident and the implication (5) $\Rightarrow$ (2) is proved by applying Lemma 3.7 and Theorem 3.2 to $X = B \cup \overline{A}$. □

Remark. (4) $\Rightarrow$ (2) of the above theorem holds for any space. But, in Example 3.10, we will give a metrizable, non ultracomplete space X satisfying $\chi(X_C, X) \leq \omega_0$.

Theorem 3.9. *For a collectionwise normal γ-space X, the following conditions are equivalent:*

(1) *X is c-ultracomplete;*
(2) *X is the union of two ultracomplete subsets P and Q, and $\chi(L, X) \leq \omega_0$ for some subset L satisfying $\partial Q \subset L \subset Q$;*

(3) *X contains a locally compact dense subspace Y such that $X \setminus Y$ is of countable character.*

Proof: Since the implication (1) $\Rightarrow$ (2) is evident, we show the implication (2) $\Rightarrow$ (1). If $\partial Q = \emptyset$, then Q is clopen in X. Also $X \setminus Q$ is a closed subset of P and hence, $X \setminus Q$ is a closed ultracomplete subset. Therefore, by Proposition 2.4 and Theorem 3.8, $X = Q \cup (X \setminus Q)$ is c-ultracomplete. Now, suppose that $\partial Q \neq \emptyset$. Then, ∂Q is countably compact. Indeed, if ∂Q contains a discrete sequence $\{x_n\}_{n\geq 1}$, then it is discrete in X. By collectionwise normality, there exists a discrete open sequence $\{H_n\}_{n\geq 1}$ in X such that $x_n \in H_n$ for every $n \in \mathbb{N}$ ([12, p. 209]). On the other hand, since $\chi(L, X) \leq \omega_0$, there exists a decreasing open base $\{W_n\}_{n\geq 1}$ of L in X. We put $A_n = H_n \cap W_n$ for every $n \in \mathbb{N}$. There exists $y_n \in A_n \cap (X \setminus Q) \subset H_n$ because of the fact that $x_n \in \partial Q \subset \overline{X \setminus Q}$ for every $n \in \mathbb{N}$. Then, the subset $Y = \{y_n\}_{n\geq 1}$ is closed in X and $L \cap Y \subset Q \cap Y = \emptyset$. Therefore, there exists an $m \in \mathbb{N}$ such that $y_m \in W_m \subset X \setminus Y$. This contradiction asserts countable compactness of ∂Q. Since X is a γ-space, by Lemma 3.7, ∂Q is compact and $\chi(\partial Q, X) \leq \omega_0$. Therefore, X is c-ultracomplete by Theorems 3.2 and 3.8.

(1) $\Rightarrow$ (3): By Theorem 1.21, there exists a locally compact open dense subspace $Y = X \setminus X_C$ and $X \setminus Y = X_C$ is compact. Therefore, $X \setminus Y$ is of countable character from Lemma 3.7.

(3) $\Rightarrow$ (1): Since $\partial(X \setminus Y) = X \setminus Y$ and $\chi(X \setminus Y, X) \leq \omega_0$, $X \setminus Y$ is compact by analogy with the proof of (2) $\Rightarrow$ (1). Therefore, X is c-ultracomplete by Theorem 1.7. □

Example 3.10. Let X be the space of the rationals with the subspace topology of $\mathbb{R}$. Then, X is a metrizable, non ultracomplete space such that $\chi(X_C, X) \leq \omega_0$.

Theorem 3.11. *Let X be a collectionwise normal γ-space with $\chi(X_C, X) \leq \omega_0$. Then, the following conditions are equivalent:*

(1) *X is c-ultracomplete;*
(2) *there exists a locally compact space Z such that X is dense in Z and $Z \setminus X$ is locally compact.*

Proof: (1) $\Rightarrow$ (2): $\beta X \setminus X = \overline{\beta X \setminus X} \setminus X_C$ is locally compact by compactness of X_C.

(2) $\Rightarrow$ (1): $Z \setminus X$ is open in $\overline{Z \setminus X}$. Hence, by Lemma 1.5, $X_C = \overline{Z \setminus X} \setminus (Z \setminus X)$ is locally compact. This implies that $X_C = \partial_X X_C$ and $\partial_X X_C$ is compact by the proof of (2) $\Rightarrow$ (1) of Theorem 3.9. Therefore, $X = (X \setminus X_C) \cup X_C$ is c-ultracomplete by Theorem 3.2.

□

Remarks. (1) The above theorem is also true for a paracompact space.

(2) For the ultracomplete space X of Example 1.11, $\beta X \setminus X$ is not locally compact. But, if X is a c-ultracomplete dense subspace of a locally compact space Z, then $Z \setminus X$ is locally compact by Lemma 1.5.

Before we consider ultracompleteness in the realm of ks-spaces, a definition is given.

Definition 3.12 ([9], [13]). Let $(X,\mathcal{U})$ be a uniform space (i.e. diagonal uniform space). A filter base $\mathcal{F}$ of X is called a *weakly Cauchy filter base* if for any $U \in \mathcal{U}$, there exists an $x \in X$ such that $F \cap U(x) \neq \emptyset$ for each $F \in \mathcal{F}$. A uniform space $(X,\mathcal{U})$ is called *cofinally complete* if every weakly Cauchy filter base has a cluster point. A space X is called *cofinally completely metrizable* if the uniform space $(X,\mathcal{U}_d)$, with the metric uniformity $\mathcal{U}_d$ for some compatible metric d on X, is cofinally complete.

S. Romaguera [13] proved the following theorem.

Theorem 3.13. *A metric space X is cofinally completely metrizable if and only if it is ultracomplete.*

Theorem 3.14 ([15]). *Every ultracomplete ks-space is cofinally completely metrizable.*

Theorem 3.15. *For a ks-space X, the following conditions are equivalent:*

(1) *X is c-ultracomplete;*
(2) *X is ultracomplete;*
(3) *X is cofinally completely metrizable;*
(4) *there exists a compatible metric d on X such that if $\mathcal{F}$ is a filter base of X satisfying the condition: for every $n \in \mathbb{N}$ there exist $x_n \in X$ with $d(x_n, F) < 1/n$ for each $F \in \mathcal{F}$, then $\mathcal{F}$ has a cluster point.*

Proof: The implication (1) $\Rightarrow$ (2) is evident and the implication (2) $\Rightarrow$ (3) follows from Theorem 3.14.

(3) $\Rightarrow$ (4): By Definition 3.12, there exists a compatible metric d such that the uniform space $(X,\mathcal{U}_d)$ with the metric uniformity $\mathcal{U}_d$ is cofinally complete. Let $\mathcal{F}$ be a filter base of X satisfying condition (4). Then, for every $U \in \mathcal{U}_d$, there exists $U_m = \{(x,y) \mid d(x,y) < 1/m\}$ with $U_m \subset U$. Also, for every $F \in \mathcal{F}$, $d(x_m, F) < 1/m$. Therefore, there exists $y(F) \in F$ such that $d(x_m, y(F)) < 1/m$ for every $F \in \mathcal{F}$. This implies that $y(F) \in U_m(x_m) \cap F \subset U(x_m) \cap F$. Hence, $\mathcal{F}$ is a weakly Cauchy filter base with respect to $\mathcal{U}_d$ and therefore, $\mathcal{F}$ has a cluster point.

(4) $\Rightarrow$ (3): We need to see that the uniform space $(X,\mathcal{U}_d)$ is cofinally complete. Let $\mathcal{F}$ be a weakly Cauchy filter base of X. Then for every $n \in \mathbb{N}$, there exists $x_n \in X$ such that $U_n(x_n) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Therefore, $d(x_n, F) < 1/n$ for every $n \in \mathbb{N}$. Hence, $\mathcal{F}$ has a cluster point, which implies that $(X,\mathcal{U}_d)$ is cofinally complete for a compatible metric d on X.

Since a metrizable space is a normal γ-space, the implication (3) $\Rightarrow$ (1) follows from Theorems 3.8 and 3.13. □

4. Products of c-ultracomplete spaces

We begin this section by showing that the product of two c-ultracomplete spaces need not to be c-ultracomplete.

Theorem 4.1. *The product space $X \times Y$ is c-ultracomplete if and only if X and Y are both locally compact or, one of X and Y is compact and the other is c-ultracomplete.*

Proof: One can see that $(X \times Y)_C = (X_C \times Y) \cup (X \times Y_C)$. Thus, the proof follows from Proposition 1.17 and the fact that $X_C \times Y$ and $X \times Y_C$ are closed in $(X \times Y)_C$. □

For infinite products, we have

Theorem 4.2. *Let A be some indexing set and let X_α be a space for any $\alpha \in A$. Then, the product space $X = \prod_{\alpha \in A} X_\alpha$ is c-ultracomplete if and only if one of the following conditions holds:* (a) *there exists a finite subset $A_0 \subset A$ such that X_α is locally compact for any $\alpha \in A_0$ and X_α is compact for any $\alpha \in A \setminus A_0$ or,* (b) *there exists $\alpha_0 \in A$ such that X_{α_0} is c-ultracomplete and X_α is compact for any $\alpha \in A \setminus \{\alpha_0\}$.*

Proof: The *if* part is evident and so we are left with the *only if* part. Put $A_0 = \{\alpha \in A \mid X_\alpha \text{ is noncompact}\}$. If (b) does not hold, then $|A_0| \geq 2$. Pick $\alpha_0 \in A_0$. As $X_{\alpha_0} \times \prod_{\alpha \in A_0 \setminus \{\alpha_0\}} X_\alpha$ is c-ultracomplete, Theorem 4.1 implies that X_{α_0} and $\prod_{\alpha \in A_0 \setminus \{\alpha_0\}} X_\alpha$ are locally compact. Hence, A_0 is finite and (a) holds. □

The following corollaries can be easily proved.

Corollary 4.3. *For a space X, the following are true.*

(1) *X^2 is c-ultracomplete if and only if X is locally compact.*
(2) *X^∞ is c-ultracomplete if and only if X is compact.*

Corollary 4.4. *If X is a c-ultracomplete, countably compact and non locally compact space* (*such a space is given in Example* 1.15), *then X^2 is ultracomplete* [3] *but not c-ultracomplete.*

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