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Extension of multisequences and countably uniradial classes of topologies

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Abstract. It is proved that every non trivial continuous map between the sets of extremal elements of monotone sequential cascades can be continuously extended to some subcascades. This implies a result of Franklin and Rajagopalan that an Arens space cannot be continuously non trivially mapped to an Arens space of higher rank. As an application, it is proved that if for a filter $\mathcal{H}$ on ω, the class of $\mathcal{H}$-radial topologies contains each sequential topology, then it includes the class of subsequential topologies.

Keywords: sequential cascade, multisequence, subsequential topology, countably uniradial, Arens topologies of higher order

Classification: Primary 54D55, 54G12; Secondary 54A20

1. Introduction

Multisequences turned out to be very instrumental in numerous problems related to sequentiality [9], [7], [8], [11], [6]. Multisequences were introduced in [10] inspired by a construction in [14] by Fremlin, but were implicitly used before, for example, by Franklin and Rajagopalan in [13]. On the other hand, Kratochvíl used the term *multisequence* in [19], [20] to denote a different but related concept.[1]

A *sequential cascade* is a tree T without infinite branches (equivalently, the set $\max A$ of maximal elements of each nonempty subset A of T, is nonempty), with a least element $\emptyset_T$, and such that for every $t \in T \setminus \max T$, the cofinite filter on the set of immediate successors of t converges to t. The finest topology compatible with so defined a convergence[2] is called the *natural topology* of the cascade.

The elements of $\operatorname{ext} T = \{\emptyset_T\} \cup \max T$ are called *extremal* and the restriction of the natural topology of T to $\operatorname{ext} T$ will be called an *Arens topology*. Each Arens

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[1]We are grateful to Professor M. Hušek who (in June 2002) drew our attention to Kratochvíl's work.

[2]By a convergence on X we understand a relation $x \in \lim \mathcal{F}$ between filters $\mathcal{F}$ on X and elements x of X such that $\mathcal{F} \leq \mathcal{G}$ implies $\lim \mathcal{F} \subset \lim \mathcal{G}$, and $x \in \lim\{x\}^{\uparrow}$ where $\{x\}^{\uparrow}$ stands for the principal ultrafilter of x. If $\mathcal{B}$ is a filter base, then we often abridge $\lim \mathcal{B}$ for the limit of the filter generated by $\mathcal{B}$.

topology is *prime*, that is, such that there is at most one non isolated element. Such spaces were applied in the study of subsequential topologies by Franklin and Rajagopalan in [13]. The (*sequential*) *contour* $\int T$ of a cascade is the trace on $\max T$ of the neighborhood filter of $\emptyset_T$ with respect to the natural topology. Bases of contours were used by Aniskovič in [1] under the name of *decomposable prefilters*. Countable powers of the cofinite filter introduced in [18, 2.7] are also closely related to sequential contours.

The *rank* $r_T(t)$ of $t \in T$ is defined inductively to be 0 if $t \in \max T$, and otherwise the least ordinal greater than the ranks of the successors of t. The rank $r(T)$ of a cascade is by definition the rank of $\emptyset_T$. Each sequential cascade can be embedded into the space $\bigcup_{n<\omega} \omega^n$ of finite sequences of natural numbers endowed with the finest topology for which the sequence $((t,n))_n$ (of immediate successors of t) converges to t for every $t \in \bigcup_{n<\omega} \omega^n$.

A *multisequence* on a set A is a map from $\max T$ of a sequential cascade T to A. The *rank* of a multisequence is that of its domain. A *subcontour* is the image of a sequential contour by a map. The *rank* of a subcontour $\mathcal{F}$ on X is the least ordinal $r(\mathcal{F})$ such that there exists a cascade T of rank $r(\mathcal{F})$ and a map $\Phi : \max T \to X$ with $\mathcal{F} = \Phi(\int T)$. If A is a subset of a topological space X, then a multisequence $\Phi : \max T \to A$ *converges* to an element x of X whenever there exists a continuous extension $\hat{\Phi} : T \to X$ of Φ such that $\hat{\Phi}(\emptyset) = x$. A multisequence on A, in the sense of Kratochvíl [20], is a continuous map Ψ from the Baire space ω^ω to A endowed with the discrete topology. If A is a subset of a topological space X then a *generator* of Ψ is a continuous map Φ from $\bigcup_{n<\omega} \omega^n$ to X such that $\Phi(f\restriction_n) = \Psi(f)$ for all but finitely many n, where $f\restriction_n$ is the restriction of $f \in \omega^\omega$ to n. Notice that the defining map of convergent multisequence (in our sense) is a generator.

Recall that a topological space is called *sequential* if every sequentially closed set is closed. The *sequential order* of a topological space is the least ordinal α such that the α-th iteration of the sequential adherence is idempotent; a topological space is *subsequential* if it is a subspace of a sequential space. The *subsequential order* of a subsequential space X is the least ordinal α such that there exists a sequential space of order α in which X can be homeomorphically embedded. Franklin and Rajagopalan proved in [13, Theorem 6.4] that the subsequential order of each monotone sequential cascade is equal to the rank of the cascade (which is also its sequential order).

Applicability of multisequences to sequential and subsequential topologies hinges on the following two facts, both implicitly contained in [13, Lemmata 7.1.1 and 7.1.2]. One characterizes sequential topologies.

Theorem 1.1. *A topology is sequential if and only if $x \in \operatorname{cl} A$ implies the existence of a multisequence on A that converges to x.*

The other is a special case of the characterizations of subsequential topologies proved in [11] (one implication of another characterization from [11] was announced in [1, Lemma 1]).

Theorem 1.2. *A topology is subsequential if and only if $x \in \operatorname{cl} A$ implies the existence of a subcontour on A that converges to x.*

If a topology is sequential (respectively, subsequential) of order $\alpha < \omega_1$, then a multisequence (respectively, a subcontour) occurring in the theorems above can be taken of rank not greater than α. A filter $\mathcal{F}$ is *subsequential* if the prime topology determined by $\mathcal{F}$ is subsequential. It follows from Theorem 1.2 that a filter is subsequential if and only if it is an infimum of subcontours.

In this paper we prove that rank cannot be increased under a non trivial continuous map from one (monotone) sequential cascade to another. Our main result stipulates that each non trivial continuous map between extremal elements of monotone cascades can be continuously extended to some subcascade. It follows that there is no continuous injection from an Arens space to an Arens space of higher order, which recovers [13, Lemma 8.4]. Some finer results relating cascades to certain indecomposable ordinals can be found in a forthcoming paper [12].

Boldjiev and Malyhin in [2] constructed a free filter $\mathcal{F}$ such that the class of sequential topologies is *$\mathcal{F}$-radial*, that is, if $x \in \operatorname{cl} A$, then there is a map $f : \omega \to A$ such that $f(\mathcal{F})$ converges to x. That filter is a *supercontour* $\mathcal{F}$, that is, the supremum of an ω_1-sequence of ascending sequential contours of increasing rank. We show that no supercontour is subsequential.

We prove that each sequential topology is radial with respect to every supercontour. On the other hand, if there is a filter such that each sequential topology is radial with respect to it, then so is each subsequential topology. Moreover, under continuum hypothesis there exists a supercontour $\mathcal{F}$ such that the class of $\mathcal{F}$-radial topologies strictly (!) includes the class of subsequential topologies.

As for notation and terminology, two families $\mathcal{A}$ and $\mathcal{B}$ (of subsets of a set X) *mesh* ($\mathcal{A}\#\mathcal{B}$) whenever $A \cap B \neq \emptyset$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$; the *grill* $\mathcal{A}^{\#}$ of a family $\mathcal{A}$ of subsets of X is the set of those subsets of X which mesh with each element of $\mathcal{A}$. A family $\mathcal{B}$ is *finer* than a family $\mathcal{A}$ (in symbols, $\mathcal{A} \leq \mathcal{B}$) if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $B \subset A$. A family $\mathcal{A}$ is *isotone* if $A \in \mathcal{A}$ and $A \subset B$ imply $B \in \mathcal{A}$. The *restriction* (or *trace*) of $\mathcal{A}$ to $H \subset X$ is $\mathcal{A}|_H = \{A \cap H : A \in \mathcal{A}\}$ and the *supremum* $\mathcal{A} \vee H$ of $\mathcal{A}$ and $\{H\}$ is the least isotone family in X that includes $\mathcal{A}|_H$. In particular, if $\mathcal{A}$ is a family on X then $\mathcal{A} \vee X$ is the least isotone family that includes $\mathcal{A}$ (the *isotonization* of $\mathcal{A}$).

The preimage by $f : X \to Y$ of $y \in Y, B \subset Y$ and $\mathcal{B}$ (a family of subsets of Y) is denoted respectively by $f^-(y)$, $f^-(B)$, $f^-(\mathcal{B})$. Two filters $\mathcal{F}_0$, $\mathcal{F}_1$ are *homeomorphic* if there exist a bijective map f such that $f(\mathcal{F}_0) = \mathcal{F}_1$. A family $\mathcal{A}$ of subsets (of a given set) is *almost disjoint* if the intersection of two arbitrary distinct elements of $\mathcal{A}$ is finite. A collection $\mathbb{F}$ of filters on X is called *disjoint* if for two distinct elements $\mathcal{F}_0$, $\mathcal{F}_1$ of $\mathbb{F}$, there exists $F \in \mathcal{F}_0$ and $X \setminus F \in \mathcal{F}_1$;

collectionwise disjoint if there is a disjoint family $\{F_{\mathcal{F}} : \mathcal{F} \in \mathbb{F}\}$ of subsets of X such that $F_{\mathcal{F}} \in \mathcal{F}$ for each $\mathcal{F} \in \mathbb{F}$. Finally, the *cofinite filter* on an infinite set X is $\{F \subset X : |X \setminus F| < \infty\}$; we often denote by (n) the cofinite filter on ω.

2. Sequential cascades and multisequences

Let $(T, \sqsubset)$ be a tree with a least element $\emptyset = \emptyset_T$ and without infinite branches, or in other words, such that every nonempty subset has a maximal element. Let $T^{\uparrow}(t) = \{s \in T : t \sqsubseteq s\}$. It follows that for every non maximal $t \in T$, the set $T^{+}(t) = \min\{s \in T : t \sqsubset s\}$ of immediate successors of t, is nonempty. Such a tree is called a *sequential cascade* ([5], [10]) if $T^{+}(t)$ is infinite countable for each non maximal t, and is endowed with the finest topology for which the cofinite filter of $T^{+}(t)$ converges to t for every $t \in T \setminus \max T$.

A subtree S of a sequential cascade T is a *subcascade* of T if $\emptyset_T \in S$ and if for every $t \in S \setminus \max T$ the set $S^{+}(t)$ is infinite. It follows that $\emptyset_S = \emptyset_T$ and that $\max S \subset \max T$.

The trace on $\max T$ of the neighborhood filter of $\emptyset_T$ is called the *contour* of T and is denoted by $\int T$. A filter $\mathcal{F}$ on X is called a *sequential contour* if there is a sequential cascade T and a bijection $f : \max T \to X$ such that $\mathcal{F} = f(\int T)$.

Recall that the *contour of* $\mathcal{G}(\cdot)$ *along* $\mathcal{F}$ (where for each $y \in Y$, $\mathcal{G}(y)$ is an isotone family of subsets of X and $\mathcal{F}$ is an isotone family of subsets of Y) is defined by

$$\int_{\mathcal{F}} \mathcal{G}(\cdot) = \bigcup_{F \in \mathcal{F}} \bigcap_{y \in F} \mathcal{G}(y)$$

[4]. Such constructions have been used by many authors, [15], [16] being the earliest occurrence we could find. If $\mathcal{F}$ is a cofinite filter on $Y = \omega$ and $\mathcal{G}(n) = \mathcal{G}_n \vee X$, then we write $\int_{(n)} \mathcal{G}_n$, and if A is an infinite subset of ω then $\int_{n \in A} \mathcal{G}_n$ stands for the contour along the cofinite filter of A. It follows that if $r(T) > 0$, then

$$\int T = \int_{(n)} \Big(\int T^{\uparrow}(n) \Big). \tag{2.1}$$

Because $T^{+}(t)$ is countably infinite for every $t \in T \setminus \max T$, we can write $T^{+}(t) = \{(t, n) : n \in \omega\}$. The rank of maximal elements of T is null, and if $t \in T \setminus \max T$, then the rank of t is defined as

$$r_T(t) = \sup\{r_T(t, n) + 1 : n \in \omega\}.$$

The rank $r(T)$ of T is the rank of its least element $\emptyset = \emptyset_T$ of T. It follows that the cardinality of a sequential cascade is countable and the rank is always less than ω_1. The sequential contour of rank 1 is the cofinite filter on (a countably infinite set). It is easy to see that if S is a subcascade of T then $r(S) = r(T)$.

The *level* $l_T(\emptyset) = 0$ and for $t \neq \emptyset$ the *level* of t is defined by

$$l_T(t) = \sup\{l_T(s) + 1 : s \sqsubset t\}.$$

It follows that the level of each element of a cascade is finite. Accordingly, for each $t \in T$, there is $k < \omega$ and $n_1, n_2, \ldots, n_k < \omega$ such that $t = (n_1, n_2, \ldots, n_k)$.

A cascade T is *monotone* if for every $t \in T \setminus \max T$, the sequence $(r(t,n))_n$ is increasing; *asymptotically monotone* if

$$r_T(t) = \sup\nolimits_{n<\omega} \inf\nolimits_{k>n}(r_T(t,k) + 1). \tag{2.2}$$

Easy induction shows that there exist monotone sequential cascades of every countable order. From the topological point of view, any permutation of the order of $T^+(t)$ is irrelevant.

Proposition 2.1. *Each asymptotically monotone sequential cascade is homeomorphic to a monotone sequential cascade of the same rank.*

Proof: Each sequential cascade of rank 1 is monotone. Suppose that the claim holds for cascades of rank less than $\beta > 1$ and let T be an asymptotically monotone sequential cascade of rank β. Let $A_\alpha = \{n \in \omega : r_T(n) + 1 = \alpha\}$. Let $n \lhd m$ if either there is α such that $n, m \in A_\alpha$ and $n \leq m$, or $n \in A_{\alpha_n}$, $m \in A_{\alpha_m}$ and $\alpha_n < \alpha_m$. Because of (2.2), A_α is finite for every $\alpha < \beta$, thus the order type of $\lhd$ is ω. Therefore the map r_T is monotone with respect to $\lhd$. By assumption, for every $n \in \omega$ there exists a homeomorphism $f_n : T^\uparrow(n) \to S_n$ where S_n is a monotone sequential cascade, say of rank $r_T(n)$. The cascade S defined by setting $S^\uparrow(n) = S_n$ and ordering $S^+(\emptyset)$ by $\lhd$ is monotone and $f : T \to S$ defined by $f(\emptyset_T) = \emptyset_S$, $f(n) = n$ and $f(n,t) = f_n(t)$ if $t \in T^\uparrow(n)$ is a sought homeomorphism. $\square$

The following proposition is equivalent to [13, Lemma 8.1].

Proposition 2.2. *If $\mathcal{F}$ is a monotone sequential contour of rank β on X and $H \in \mathcal{F}^\#$, then $\mathcal{F}|_H$ is a monotone sequential contour of rank β on H; if $\beta > 1$, then also $\mathcal{F} \vee H$ is a monotone sequential contour of rank β on X.*

Proof: If $r(\mathcal{F}) = 1$ and $H \in \mathcal{F}^\#$, then $\mathcal{F}|_H$ is the cofinite filter of H. If the proposition holds for all the monotone sequential contours of rank less than $\beta > 1$ and if $r(\mathcal{F}) = \beta$, then there exists a partition $\{F_n : n < \omega\}$ of X and monotone sequential contours $\mathcal{F}_n$ on F_n such that $\mathcal{F} = \int_{(n)} \mathcal{F}_n$ and $(r(\mathcal{F}_n))$ is increasing with $\sup_{n<\omega}(r(\mathcal{F}_n) + 1) = \beta$. As $H \in \mathcal{F}^\#$, the set $A = \{n < \omega : H \in \mathcal{F}_n^\#\}$ is infinite, and by the inductive assumption, $\mathcal{F}_n|_H$ is a monotone sequential contour on H for each $n \in A$ so that $\mathcal{F}|_H = \int_{n \in A} \mathcal{F}_n|_H$ and $r(\mathcal{F}|_H) = \beta$. Now, if $\beta > 1$ then the extension $\mathcal{F} \vee H$ of $\mathcal{F}|_H$ to X is still monotone sequential contour of rank β. Indeed, every sequential contour on X of rank greater than 1 contains a set whose complement is infinite: therefore if H is an infinite subset of X and $\mathcal{F}$ is a sequential contour of rank β on H, then whenever $\beta > 1$ or if $\beta = 1$ and $X \setminus H$ is finite, the extension $\mathcal{F} \vee X$ of $\mathcal{F}$ to the whole of X is a sequential contour homeomorphic to $\mathcal{F}$ and hence of the same rank. $\square$

Moreover for every monotone sequential contour $\mathcal{F}$ on a countably infinite set X of rank greater than 1, there is a countable disjoint collection of filters on X and each of them is homeomorphic with $\mathcal{F}$.

Lemma 2.3. *Let $\mathcal{F}$ be a monotone sequential contour on X of rank greater than 1. Then there is a partition $\{X_m : m < \omega\}$ of X such that $\mathcal{F} \vee X_m$ is homeomorphic with $\mathcal{F}$ for every $m < \omega$.*

Proof: Let $\mathcal{F}$ be the contour of a cascade T of rank greater than 1. For every $t \in T$ of rank 1, split $T^+(t)$ into a disjoint union of infinite sets $\{X_{t,m} : m < \omega\}$ and let $p_{t,m} : T^+(t) \to X_{t,m}$ be a bijection. If $X_m = \bigcup_{r_T(t)=1} X_{t,m}$ then $p_m : X = \max T \to X_m$ defined by $p_m(t,n) = p_{t,m}(n)$ if $n \in T^+(t)$, is a bijection for every $m < \omega$. Because $p_{t,m}$ maps the cofinite filter on $T^+(t)$ onto the cofinite filter on $X_{t,m}$, we conclude that for every $m < \omega$, the filter $p_m(\mathcal{F}) = \mathcal{F}|_{X_m}$ is homeomorphic to $\mathcal{F} \vee X_m$. $\square$

Corollary 2.4. *For every monotone sequential contour $\mathcal{F}$ of rank α, there is a partition $\{X_m : m < \omega\}$ of X and for each $m < \omega$ a monotone sequential contour $\mathcal{F}_m$ of rank $\alpha + 1$ such that $X_m \in \mathcal{F}_m \geq \mathcal{F}$.*

Proof: As $\mathcal{F} \leq \mathcal{F} \vee X_m$, also $\mathcal{F} = \int_{(m)} \mathcal{F} \leq \int_{(m)} (\mathcal{F} \vee X_m)$. Take a partition of X assured by Lemma 2.3 and a bijection $p_m : X \to X$ for which $p_m(\mathcal{F}) = \mathcal{F} \vee X_m$ and let $\mathcal{F}_m = p_m(\int_{(n)} (\mathcal{F} \vee X_n))$. Then $\mathcal{F} \leq \mathcal{F} \vee X_m = p_m(\mathcal{F}) \leq \mathcal{F}_m$. $\square$

Corollary 2.5. *There exists an ascending family $\{\mathcal{F}_\alpha : \alpha < \omega_1\}$ of monotone sequential contours such that $r(\mathcal{F}_\alpha) = \alpha$ for each $\alpha < \omega_1$.*

If T is a sequential cascade, then the image of a monotone sequential contour by a mapping $\Phi : \max T \to A$ is called a *multisequence* on A. The rank of a multisequence is by definition the rank of its cascade; a multisequence is monotone if its cascade is monotone.

3. Maps between sequential cascades

Theorem 3.1. *Let (X_n) be a disjoint family of sets such that $X_n \in \mathcal{F}_n \cap \mathcal{G}_n$ for each $n < \omega$. If*

$$\int_{(n)} \mathcal{F}_n \leq \int_{(n)} \mathcal{G}_n \tag{3.1}$$

then $\mathcal{F}_n \leq \mathcal{G}_n$ for almost all n.

Proof: If (3.1) holds, then $\int_{(n)} \mathcal{F}_n \vee A \leq \int_{(n)} \mathcal{G}_n \vee A$ for every set A. In particular if $A = \bigcup_{k<\omega} X_{n_k}$, where (n_k) is an increasing sequence of natural numbers, then $\int_{(k)} \mathcal{F}_{n_k} \leq \int_{(k)} \mathcal{G}_{n_k}$. If there existed an increasing sequence (n_k) such that $\mathcal{F}_{n_k}$ is not greater than $\mathcal{G}_{n_k}$ for each $k < \omega$, then there would be $F_{n_k} \subset X_{n_k}$ such that $F_{n_k} \in \mathcal{F}_{n_k} \backslash \mathcal{G}_{n_k}$. Hence $X_{n_k} \backslash F_{n_k} \in \mathcal{G}^{\#}_{n_k}$ and thus $\bigcup_{k<\omega}(X_{n_k} \backslash F_{n_k}) \in (\int_{(n)} \mathcal{G}_n)^{\#} \subset (\int_{(n)} \mathcal{F}_n)^{\#}$. On the other hand, $\bigcup_{k<\omega} F_{n_k} \in \int_{(n)} \mathcal{F}_n \vee (\bigcup_{k<\omega} X_{n_k}) = \int_{(k)} \mathcal{F}_{n_k}$, which yields a contradiction. □

A subset B of an ordered set $(T, \sqsubseteq)$ is called *saturated* if whenever $s_0 \sqsubseteq s_1 \in B$ then $s_0 \in B$. If T is a sequential cascade, then for each $t \in T$ there is a base $\mathcal{B}$ of saturated neighborhoods of t.

If $\mathcal{F}$ is a filter on an infinite set X, then the *prime topology* of $\mathcal{F}$ is the topology on the disjoint union $\{\infty\} \cup X$ for which all the elements of X are isolated and the trace on X of the neighborhood filter of ∞ is $\mathcal{F}$. If $\mathcal{F}$ is a filter on X and $\mathcal{G}$ is a filter on Y, then we say that a map $f : X \to Y$ is *continuous* provided that $f(\mathcal{F}) \geq \mathcal{G}$, in other words, if f can be extended to a continuous map $\hat{f}$ from the prime topology of $\mathcal{F}$ to the prime topology of $\mathcal{G}$ so that $\hat{f}(\infty) = \infty$. If T is a sequential cascade, then the restriction to $\operatorname{ext} T$ of the natural topology of T is the prime topology of the contour.

We say that a map g between trees is *non trivial* if it maps maximal elements to maximal elements, and minimal elements to minimal elements. In particular, $f : \operatorname{ext} T \to \operatorname{ext} W$ is non trivial if $f(\max T) \subset \max W$ and $f(\emptyset_T) = \emptyset_W$, while $f : \max T \to \max W$ is continuous if $f(\int T) \geq \int W$.

Proposition 3.2. *Let T, W be monotone sequential cascades. If $f : T \to W$ is a non trivial continuous map, then $r_T(t) \geq r_W(f(t))$ for each $t \in T$. If moreover f is injective, then $r_T(t) = r_W(f(t))$ for each $t \in T$.*

Proof: If $r_T(t) = 0$, then $r_W(f(t)) = r_T(t) = 0$ by assumption. If $r_T(t) > 0$, then by the continuity of f, for every $m < \omega$ there is $k_m < \omega$ such that $f(t, k) \in \{(f(t), n) : n > m\} \cup \{f(t)\}$ for $k > k_m$. By the monotonicity of W

$$r_W(f(t)) = \sup_{m<\omega} \inf_{n>m} (r_W((f(t), n)) + 1),$$

that is, for each $\alpha < r_W(f(t))$ there is $m < \omega$ such that $\alpha < r_W((f(t), n)) + 1$ for each $n > m$, hence $\alpha < r_W(f(t, k)) + 1$ for $k > k_m$ and, by the inductive assumption, $\alpha < r_T(t, k) + 1$ for $k > k_m$. Therefore $r_W(f(t)) \leq \sup_{m<\omega} \inf_{k>m}(r_T(t, k) + 1) = r_T(t)$.

If f is injective, then moreover $f(t) \neq f(t, n)$ and by the monotonicity of T and W the inequalities above become equalities. □

Theorem 3.3. *Let T and W be monotone sequential cascades and let $f : \operatorname{ext} T \to \operatorname{ext} W$ be a non trivial (injective) continuous map. Then there exists a subcascade*

R of T and a (injective) continuous map $\hat{f} : R \to W$ such that $\hat{f}|_{\operatorname{ext} R} = f|_{\operatorname{ext} R}$. Moreover $r(T) \geq r(W)$.

Proof: For every $t \in T$, let us consider the set

$$F_T(t) = \{w \in W : \exists_{Q \in \mathcal{N}_T(t)} \forall_{s \in Q}\ w \sqsubseteq f(s)\}. \tag{3.2}$$

The set $\max F_T(t)$ is a singleton and we denote its element by $\hat{f}(t)$. Indeed, $\emptyset_W \in F_T(t)$ and if $w_0, w_1 \in \max F_T(t)$, then by definition there exist $Q_0, Q_1 \in \mathcal{N}_T(t)$ such that $w_0 \sqsubseteq f(s)$ and $w_1 \sqsubseteq f(s)$ for every $s \in Q_0 \cap Q_1 \cap \max T$, and of course $Q_0 \cap Q_1 \in \mathcal{N}_T(t)$. As T is a tree and $w_0, w_1 \sqsubseteq f(s)$ for some s, the elements w_0, w_1 are comparable, hence $w_0 = w_1$ as maximal elements of (3.2).

The map $\hat{f}$ coincides with f on $\operatorname{ext} T$. To prove this, it remains to show that $\hat{f}(\emptyset_T) = \emptyset_W$. Otherwise there would be $Q \in \mathcal{N}_T(\emptyset)$ and $k < \omega$ such that $k \sqsubseteq f(t)$ for every $t \in Q \cap \max T$; hence the neighborhood $\{w \in W : \exists_{m > k}\ m \sqsubseteq w\}$ of $\emptyset_W$ is disjoint from $f(Q \cap \max T)$, contrary to the continuity of f.

The sequence $(\hat{f}(n))_n$ converges to $\hat{f}(\emptyset_T) = \emptyset_W$. Otherwise, there would be a saturated neighborhood S of $\emptyset_W$ and a subsequence $(n_p)_p$ such that $\hat{f}(n_p) \notin S$ for every $p < \omega$. Therefore, for each $p < \omega$ there would be a neighborhood Q_p of n_p in T such that $f(\bigcup_{p<\omega} Q_p \cap \max T) \cap S = \emptyset$. But this contradicts the continuity of f, because $\bigcup_{p<\omega} Q_p \in \mathcal{N}_T(\emptyset)^{\#}$.

We prove now by induction on the rank of T that there is a subcascade R of T such that the restriction of $\hat{f}$ to R is continuous. If $r(T) = 1$, then $R = T$. Assume that $\alpha > 1$, the claim holds for the cascades T of rank less than α and consider T of rank α. As $(\hat{f}(n))_n$ converges to $\hat{f}(\emptyset_T) = \emptyset_W$, there is an increasing sequence (n_p) such that either $\hat{f}(n_p) = \hat{f}(\emptyset_T) = \emptyset_W$ for each $p < \omega$, or $\hat{f}(n_p) = k_p$ for each $p < \omega$, where (k_p) is an increasing sequence. Of course $T^{\uparrow}(n)$ is a cascade of rank smaller than α.

In the former case $\hat{f}$ is a non trivial continuous map and, as $F_T(t) = F_{T^{\uparrow}(n)}(t)$ for each $t \in T^{\uparrow}(n)$, by inductive assumption there is a subcascade R_p of $T^{\uparrow}(n_p)$ such that $\hat{f}|_{R_p}$ is continuous. It is now enough to set $R = \{\emptyset_T\} \cup \bigcup_{p<\omega} R_p$.

In the latter case, for every $p < \omega$ there is $Q_p \in \mathcal{N}_T(n_p)$ such that $f(Q_p \cap \max T) \subset W^{+}(k_p)$. The set

$$H = \{\emptyset_T\} \cup \bigcup_{p<\omega} \{t \in T : \exists_{s \in Q_p}\ t \sqsubseteq s\}$$

meshes with $\mathcal{N}_T(\emptyset)$ and the restriction of f to $\operatorname{ext} H$ is a continuous map valued in $\operatorname{ext} K$ where $K = \bigcup_{p<\omega} W^{\uparrow}(k_p)$. Therefore

$$\int_{(p)} f(\mathcal{N}_H(n_p)) = f(\mathcal{N}_H(\emptyset)) \geq \mathcal{N}_K(\emptyset) = \int_{(p)} \mathcal{N}_K(k_p).$$

Because $W^+(k_p) \in f(\mathcal{N}_H(n_p)) \cap \mathcal{N}_K(k_p)$, by Theorem 3.1 there is q such that $f(\mathcal{N}_H(n_p)) \geq \mathcal{N}_K(k_p)$ for each $p \geq q$. As $H^\uparrow(n_p)$ is an open neighborhood of n_p and since $F_T(t) = F_{H^\uparrow(n_p)}(t)$ for each $t \in H^\uparrow(n_p)$, by inductive assumption, for each $p \geq q$, there is a subcascade R_p of $H^\uparrow(n_p)$ such that $\hat{f}|_{R_p}$ is continuous. It is now enough to set $R = \{\emptyset_T\} \cup \bigcup_{p \geq q} R_p$.

If f is injective, then the former case cannot occur and thus $\hat{f}$ is injective on R by construction of R.

The estimate of the ranks follows from Proposition 3.2. □

Corollary 3.4 ([13, Lemma 8.4])**.** *If $\mathcal{F}$ is a monotone subcontour and $\mathcal{G}$ is a monotone sequential contour such that $r(\mathcal{F}) < r(\mathcal{G})$, then $\mathcal{F}$ must not be finer than $\mathcal{G}$.*

Proof: By definition there exists a sequential contour $\mathcal{E}$ of rank $r(\mathcal{F})$ and a map f such that $\mathcal{F} = f(\mathcal{E})$. If $f(\mathcal{E}) \geq \mathcal{G}$, then by Theorem 3.3 $r(\mathcal{E}) = r(\mathcal{F}) \geq r(\mathcal{G})$ contradicting the assumption. □

In other words, there is no nontrivial continuous map from an Arens space to an Arens space of higher order.

There exists a non trivial continuous map $f : \operatorname{ext} T \to \operatorname{ext} W$ that cannot be extended to a continuous map from T to W (even such that for every subcascade V of T such that $V \in \mathcal{N}_T(\emptyset)$ there is no continuous map $g : V \to W$ which coincides with f on $\operatorname{ext} V$).

Example 3.1. *Let T be a cascade of rank 2 and let A_0, A_1 be complementary infinite subsets of ω and let $h_0 : A_0 \to \omega$ and $h_1 : A_1 \to \omega$ be bijections. Let $f(2n+1, k) = (n, h_0(k))$ if $k \in A_0$ and $f(n, k) = (2n, h_1(k))$ if $k \in A_1$ for every $n < \omega$ (and $f(\emptyset_T) = \emptyset_W$). Then $f : \operatorname{ext} T \to \operatorname{ext} T$ is a homeomorphism but each map $g : V \to W$ that coincides with f on $\operatorname{ext} V$ fails to be continuous at infinitely many elements of rank 1.*

If $S_1, S_2, \dots, S_m$ are sequential cascades, then $\bigoplus_{i=1}^m S_i$ stands for the following cascade: $(\bigoplus_{i=1}^m S_i)^\uparrow(n) = S_p^\uparrow(k)$ if $n = mk + p$ where $k < \omega$ and $1 \leq p \leq k$. In particular, if $S_i = S$ is a monotone cascade for each i, then $\bigoplus_{i=1}^m S$ is a monotone cascade of rank $r(S)$.

Proposition 3.5. *If T, S are monotone sequential cascades such that $1 \leq r(T) \leq r(S)$, then there exists a continuous bijection $f : \operatorname{ext} S \to \operatorname{ext} T$.*

Proof: In every case we set $f(\emptyset_S) = \emptyset_T$. Let us induce on the rank of S. If $r(S) = 1$, then $r(T) = 1$ and thus $\int T$ is the cofinite filter on $\max T$; an arbitrary bijection f from $\max S$ to $\max T$ can be taken, because $\int S$ is a free filter, so is $f(\int S)$, and thus is finer than $\int T$. Suppose that the claim holds for S of rank less than $\beta \geq 2$ and let $1 \leq r(T) \leq r(S) = \beta$. Because T and S are monotone, $(r(\, T^\uparrow(n)) + 1)$ increasingly converges to $r(T)$ and $(r(S^\uparrow(k)) + 1)$ to $r(S)$. Hence for every $n < \omega$

there is $k(n) < \omega$ such that $r(T^{\uparrow}(n)) \leq r(S^{\uparrow}(k))$ for each $k(n) < k \leq k(n+1)$. Let $A_n = \{k : k(n) \leq k < k(n+1)\}$. By the inductive assumption, for every $k \in A_n$ there exists a continuous bijection $f_k : \max S^{\uparrow}(k) \to \max \bigoplus_{i \in A_n}(S^{\uparrow}(k(n)))$ and because $\bigoplus_{i \in A_n}(S^{\uparrow}(k(n)))$ is a monotone sequential cascade of rank not less than $r(T^{\uparrow}(n))$, there exists a continuous bijection $h_n : \max \bigoplus_{i \in A_n}(S^{\uparrow}(k(n))) \to \max T^{\uparrow}(n)$. Take an arbitrary bijection g of $\bigcup_{0 \leq k < k(0)} \max S^{\uparrow}(k)$ onto $\max T^{\uparrow}(0)$.

Let us define $f : \max S \to \max T$ as follows: if $s \in \bigcup_{0 \leq k < k(0)} \max S^{\uparrow}(k)$, then $f(s) = g(s)$; if $s \in \max S^{\uparrow}(k)$ and $k \in A_n$, then $f(s) = h_n \circ f_k(s)$. Then f is a continuous bijection. Indeed, if $B \in \int T$ then there is n_A such that $B \in \int T^{\uparrow}(n)$ for each $n \geq n_A$, hence $f^{-}(B) \in \int S^{\uparrow}(k)$ for each $k \geq k(n_A)$, so that $f^{-}(B) \in \int S$. □

It follows that if T, S are monotone sequential cascades such that $1 \leq r(T) \leq r(S)$, then there exists a continuous bijection from $\operatorname{ext} S$ to $\operatorname{ext} T$, which slightly refines [13, Lemma 8.3].

4. Equivalent filters

A filter $\mathcal{F}_0$ is greater than a filter $\mathcal{F}_1$ for the *Rudin-Keisler order* if there exists a mapping f such that $f(\mathcal{F}_0) \geq \mathcal{F}_1$ (this is an extension to all the filters of the Rudin-Keisler order defined originally on the class of ultrafilters [22, p. 539]). We shall say that filters are *equivalent* if they are such with respect to the Rudin-Keisler order. If there exists a bijective map f such that $f(\mathcal{F}_0) = \mathcal{F}_1$, then we say that $\mathcal{F}_0, \mathcal{F}_1$ are *homeomorphic*. As a consequence of Proposition 3.5,

Proposition 4.1. *All the monotone sequential contours of a given rank are equivalent.*

Monotone sequential contours are homeomorphic if the common rank is finite. On the other hand,

Proposition 4.2. *For every countable infinite ordinal β, there exist two non homeomorphic monotone sequential cascades of rank β.*

Proof: Consider increasing sequences (α_n) and (β_n) of finite ordinals such that $\alpha_n < \beta_n < \alpha_{n+1}$ for each $n < \omega_0$. Let T and W be two monotone cascades such that $r_T(n) = \alpha_n$, $r_W(n) = \beta_n$. If $f : T \to W$ is a homeomorphism, then the isolated points must be mapped onto isolated points: $f(\max T) = \max W$, hence $r_T(t) = r_W(f(t))$ by Proposition 3.2. This means that $\{f(n) : n < \omega_0\}$ are not the successors of $\emptyset_W = f(\emptyset_T)$ and thus $(f(n))_n$ does not converge to $f(\emptyset_T)$, what contradicts the continuity of f. Suppose that the claim holds for every infinite rank less than β and let $r(T) = r(W) = \beta$ be such that for every $t \in T$ and $w \in W$, of rank ω_0, one has $r_T(t, n) = \alpha_n$, $r_W(w, n) = \beta_n$. By Proposition 3.2 there is no injective continuous map from T onto W. □

Corollary 4.3. *For every countable infinite ordinal β, there exist two non homeomorphic monotone sequential contours of rank β.*

Proof: If for T and W of the proof of Proposition 4.2, there existed a homeomorphism $h : \operatorname{ext} T \to \operatorname{ext} W$, then by Theorem 3.3 there would exist a subcascade T_0 of T and a continuous injective map $h_0 : T_0 \to W$ that would coincide with h on $\operatorname{ext} T_0$. But this is impossible in virtue of the proof of Proposition 4.2. $\square$

Lemma 4.4. *If $\{\mathcal{F}_n : n < \omega\}$ is a collectionwise disjoint family of filters and $\mathcal{A}$ is an almost disjoint family on ω, then the family $\{\int_{n \in A} \mathcal{F}_n : A \in \mathcal{A}\}$ is disjoint.*

Proof: Indeed, let $\{X_n : n < \omega\}$ be a partition of X into disjoint infinite sets such that $X_n \in \mathcal{F}_n$ for each $n < \omega$. Let $\mathcal{F}_A = \int_{n \in A} \mathcal{F}_n$. Then for two arbitrary distinct elements A_0 and A_1 of $\mathcal{A}$, the set $\bigcup_{n \in A_0 \setminus A_1} X_n \in \mathcal{F}_{A_0}$ and $X \setminus \bigcup_{n \in A_0 \setminus A_1} X_n \supset \bigcup_{n \in A_1 \setminus A_0} X_n \in \mathcal{F}_{A_1}$. $\square$

Proposition 4.5. *For every $\beta > 1$, the cardinality of the set of sequential contours of rank β on a countably infinite set, is $2^{\aleph_0}$.*

Proof: On a countably infinite set X there is only one sequential contour of rank 1 (this is the cofinite filter on X), while for every $\beta > 1$, there exist at most $2^{\aleph_0}$ sequential contours of rank β. Indeed, if the claim holds for the rank less than $\beta > 1$, then there exist at most $2^{\aleph_0}|\beta| \leq 2^{\aleph_0}\aleph_0 = 2^{\aleph_0}$ of contours of rank less than β. Since every sequential contour of rank β can be obtained from a sequence of contours of lower rank, there exist at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ contours of rank β.

In fact, there are precisely $2^{\aleph_0}$ contours of rank β (for $\beta > 1$). Let $\{X_n : n < \omega\}$ be a partition of X into disjoint infinite sets and let $\mathcal{F}_n$ be the cofinite filter of X_n (that is, the extension to X of the cofinite filter on X_n). By Lemma 4.4, there exist on X at least $2^{\aleph_0}$ disjoint sequential contours of rank 2. It is now enough for every $\beta > 2$ and each $A \in \mathcal{A}$ to pick a sequential contour $\mathcal{F}_{A,\beta}$ on X of rank β and such that $\mathcal{F}_A \leq \mathcal{F}_{A,\alpha} \leq \mathcal{F}_{A,\beta}$ if $2 < \alpha < \beta < \omega_1$. $\square$

A *supercontour* is a filter (on a countably infinite set X) of the form $\bigvee_{\alpha<\omega_1} \mathcal{E}_\alpha$, where $\mathcal{E}_\alpha$ is a monotone sequential contour of rank α and $\mathcal{E}_\alpha \leq \mathcal{E}_\beta$ if $\alpha < \beta$. Let us notice that if $(\mathcal{F}_n)$ is a collectionwise disjoint sequence of supercontours, then $\int_{(n)} \mathcal{F}_n$ is a supercontour. In fact, if $\mathcal{F}_n = \bigvee_{\alpha<\omega_1} \mathcal{E}_{n,\alpha}$, then there exists $\gamma < \omega_1$ such that for each countable $\alpha \geq \gamma$, $(\mathcal{E}_{n,\alpha})_n$ is a collectionwise disjoint sequence of monotone sequential contours of rank α and thus $\mathcal{E}_\alpha = \int_{(n)} \mathcal{E}_{n,\alpha}$ is a monotone sequential contour of rank $\alpha + 1$ and moreover $\mathcal{E}_\alpha \leq \ \mathcal{E}_\beta$ for every countable non limit $\alpha < \beta$. Therefore the supremum of such $\mathcal{E}_\alpha$ is equal to $\int_{(n)} \mathcal{F}_n$.

Theorem 4.6. *On each countably infinite set there are $2^{\aleph_1}$ supercontours.*

Proof: There are $(2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1}$ maps from ω_1 valued in a set of cardinality $2^{\aleph_0}$; since for every $\alpha < \omega_1$ the set of sequential contours of rank α (on a countably infinite set) is of cardinality $2^{\aleph_0}$, we conclude that there exist at most $2^{\aleph_1}$ supercontours. We will prove that this is precisely the cardinality of supercontours. Consider the set $\omega_0^{\omega_1}$ (of maps from ω_1 to ω_0) and let $h|_\beta$ be the restriction of h to $\beta < \omega_1$. For every $1 < \beta < \omega_1$, we will construct a disjoint set

$$\mathbb{F}_\beta = \{\mathcal{F}_{h|\beta} : h \in \omega_0^\beta\}$$

of sequential contours of rank β on an infinite countable set X so that $\mathcal{F}_{h|\alpha} \leq \mathcal{F}_{h|\beta}$ for each $\alpha \leq \beta < \omega_1$. In the special case of $\beta = 1$ the set $\mathbb{F}_1$ consists of a single filter: the cofinite filter $\mathcal{F}_0$ of X. If β is a countable ordinal for which $\alpha + 1 = \beta$, then by Proposition 2.2 and Lemma 2.3, for every $\mathcal{F}_{h|\alpha}$ there exists a disjoint family $\{X_{h|\alpha,n} : n < \omega\}$ of subsets of X such that $X_{h|\alpha,n} \in \mathcal{F}_{h|\alpha}^{\#}$ and for every $n < \omega$, $\mathcal{F}_{h|\alpha} \vee X_{h|\alpha,n}$ is a monotone sequential contour of rank α. By Corollary 2.4 there exists a monotone sequential contour $\mathcal{F}_{h|\beta}$ of rank $\beta = \alpha + 1$ finer than $\mathcal{F}_{h|\alpha} \vee X_{h|\alpha,n}$ hence than $\mathcal{F}_{h|\alpha}$.

If β is a countable limit ordinal, then choose any increasing sequence (β_n) of non limit ordinals such that $\sup_{n<\omega}(\beta_n + 1) = \beta$. Fix $h \in \omega_0^{\omega_1}$. Then by induction we can choose a sequence (X_n) of disjoint subsets of X such that

$$X_n = \bigcup_{k \neq h(\beta_n+1)} X_{h|\beta_n,k} \in \mathcal{F}_{h|\beta_n}^{\#} \quad \text{and} \quad X_{h|\beta_n,h(\beta_n+1)} \in \mathcal{F}_{h|\beta_{n+1}}.$$

Therefore $\mathcal{H}_n = \mathcal{F}_{h|\beta_n} \vee X_n$ defines a collectionwise disjoint sequence of monotone sequential contours, and thus $\mathcal{F}_{h|\beta} = \int_{(n)} \mathcal{H}_n$ is a sequential contour of rank β and moreover $\mathcal{F}_{h|\alpha} \leq \mathcal{F}_{h|\beta}$ for every $\alpha < \beta$.

If $\mathcal{F}_{h|\beta}$ and $\mathcal{F}_{g|\beta}$ are two distinct elements of $\mathbb{F}_\beta$, that is, if $h|\beta \neq g|\beta$, then there is $\alpha < \beta$ such that $h(\alpha) \neq g(\alpha)$ so that by the inductive assumption, there is $F \in \mathcal{F}_{h|\alpha} \subset \mathcal{F}_{h|\beta}$ and such that $X \setminus F \in \mathcal{F}_{g|\alpha} \subset \mathcal{F}_{g|\beta}$, proving that $\mathbb{F}_\beta$ is disjoint.

Now, for every $h \in \omega_0^{\omega_1}$ define $\mathcal{F}_h = \bigvee_{\beta<\omega_1} \mathcal{F}_{h|\beta}$ and let $\mathbb{F}_{\omega_1}$ be the set of all the filters of the form $\mathcal{F}_h$. If $h \neq g$ then, by the same argument as above, $\mathcal{F}_h$ and $\mathcal{F}_g$ do not mesh; in particular, are distinct. As the cardinality of $\omega_0^{\omega_1}$ (the set of maps from ω_1 to ω_0) is $2^{\aleph_1}$ there are at least $2^{\aleph_1}$ supercontours (on each countably infinite set). $\square$

Theorem 4.7. *If $2^{\aleph_0} < 2^{\aleph_1}$, then there are at least $2^{\aleph_1}$ equivalence classes of supercontours.*

Proof: Let X be a countably infinite set and let $\mathbb{F}_{\omega_1}$ be the set supercontours on X. Because the cardinality of $\omega_0^{\omega_0}$ (the set of maps from ω_0 to ω_0) is $2^{\aleph_0}$, every element $\mathcal{F}$ of $\mathbb{F}_{\omega_1}$ can be mapped onto at most $2^{\aleph_0}$ filters on X. As $\mathbb{F}_{\omega_1}$ is disjoint and $\mathcal{G} \neq \mathcal{H} \in \mathbb{F}_{\omega_1}$, then $\mathcal{G}$ and $\mathcal{H}$ do not mesh. Hence if $\mathcal{F} \in \mathbb{F}_{\omega_1}$ then there can be only one element $\mathcal{G}$ of $\mathbb{F}_{\omega_1}$ such that $f(\mathcal{F}) \geq \mathcal{G}$. Therefore by $2^{\aleph_0} < 2^{\aleph_1}$, there are at least $2^{\aleph_1}$ many equivalence classes of elements of $\mathbb{F}_{\omega_1}$. $\square$

If we assume the continuum hypothesis: $2^{\aleph_0} = \aleph_1$ (which implies $2^{\aleph_0} < 2^{\aleph_1}$), then there exists a supercontour which is an ultrafilter.

Theorem 4.8 (CH). *There exists a supercontour which is an ultrafilter.*

PROOF: Let $\mathcal{A}$ be the set of all infinite subsets of ω. By the continuum hypothesis $\mathcal{A}$ can be arranged in $\{A_1^0, A_1^1, \dots, A_\alpha^0, A_\alpha^1, \dots\}$ where $\alpha < \omega_1$ and $A_\alpha^1 = \omega \setminus A_\alpha^0$ for every $\alpha < \omega_1$. Let $\mathcal{F}_1$ be the cofinite filter on ω and let $\mathcal{F}_2$ be a monotone sequential contour of rank 2 such that $A_1^0 \in \mathcal{F}_2$. Suppose $\beta < \omega_1$ and $f : \beta \to \{0,1\}$ that $(\mathcal{F}_\alpha)_{\alpha<\beta}$ is an ascending family of monotone sequential contours satisfying $A_\alpha^{f(\alpha)} \in \mathcal{F}_\alpha$ and $r(\mathcal{F}_\alpha) = \alpha$. Then either A_β^0 or A_β^1 meshes with $\bigvee_{\alpha<\beta} \mathcal{F}_\alpha$; choose $f(\beta)$ so that $A_\beta^{f(\beta)}$ does. By Proposition 2.2 and Corollary 2.5, there exists a monotone sequential contour $\mathcal{F}_\beta$ of rank β such that $A_\beta^{f(\beta)} \in \mathcal{F}_\beta$ and $\bigvee_{\alpha<\beta} \mathcal{F}_\alpha \leq \mathcal{F}_\beta$. The supercontour $\mathcal{F}_{\omega_1} = \bigvee_{\alpha<\omega_1} \mathcal{F}_\alpha$ has the property that for every infinite subset A of ω either $A \in \mathcal{F}_{\omega_1}$ or $\omega \setminus A \in \mathcal{F}_{\omega_1}$, that is, $\mathcal{F}_{\omega_1}$ is an ultrafilter. $\square$

Katětov defined in [18, 2.7] countable powers $\mathcal{N}^\beta$ on $D^{(\beta)}$ of the cofinite filter $\mathcal{N}$ on ω, and showed that for every topological space T, $\operatorname{adh}^\beta_{\mathrm{Seq}} C(T) = \operatorname{adh}_{\mathrm{B}_{\{\mathcal{N}^\beta\}^\flat}} C(T)$ in $\mathbb{R}^T$ endowed with the pointwise convergence (see next section for the meaning of $^\flat$). $\mathcal{N}^0$ is the principal ultrafilter of 0 in ω and $D^{(0)} = \omega$; $\mathcal{N}^1 = \mathcal{N}$ is the cofinite filter on ω and $D^{(1)} = \omega$. If β is a countable isolated ordinal, then $\mathcal{N}^\beta = \mathcal{N} \cdot \mathcal{N}^{\beta-1}$ and $D^{(\beta)} = D \times D^{(\beta-1)}$. If β is a countable limit ordinal, then $\mathcal{N}^\beta = \int_{\mathcal{V}(\beta)} \mathcal{N}^\alpha$, where $\mathcal{V}(\beta)$ is the neighborhood filter of β on $D^{(\beta)}$, the set of all the maps f defined on β and such $f(\alpha) \in D^{(\alpha)}$ for each $\alpha < \beta$.

Let us show that $\mathcal{N}^\beta$ is a subsequential filter of rank β. To this end we shall construct a sequential topology in such a way that $\mathcal{N}^\beta$ will appear as the restriction of a neighborhood filter to the maximal elements of X_β.

Let X_1 be the canonical cascade of rank 1. Let $\beta > 1$ and suppose that X_α have been defined for $\alpha < \beta$; then we define X_β by adjoining to the disjoint union $\bigcup_{\alpha<\beta} X_\alpha$ the space $Y = \{x_\alpha : \alpha \leq \beta\}$, where x_α is the least element of X_α. An order $\sqsubset$ on X_β is defined by adjoining to the disjoint union $\bigcup_{\alpha<\beta} X_\alpha$ ordered pointwise by the orders on X_α for each $\alpha < \beta$, the order on Y defined by $x_\alpha \sqsubset x_\gamma$ whenever $\alpha > \gamma$. Then clearly x_β is the least element of X_β. The topology of X_β is the quotient of the topology of topological sum of $\bigcup_{\alpha<\beta} X_\alpha$ and of the topology of the ordinal $\beta+1$ on Y. Then the topology of X_β is sequential by the inductive assumption as a quotient of sequential topologies. It is also clear that the rank of X_β is β. Hence, the restriction to $\operatorname{ext} X_\beta$ of so defined topology is prime subsequential of order at most β and thus the trace of the neighborhood filter of the least element of X_β is subsequential. It follows from the construction that this trace is precisely $\mathcal{N}^\beta$.

Therefore for every $A \in (\mathcal{N}^\beta)^\#$ and each sequential cascade of rank β, there exists a map $f : \omega \to A$ such that $\mathcal{N}^\beta \vee A \leq f(\int T)$. On the other hand,

Proposition 4.9. *For every $\beta < \omega_1$ the filter $\mathcal{N}^\beta$ is greater than or equal to each monotone contour of rank β with respect to the Rudin-Kreisler order.*

Proof: Each monotone contour of finite rank n is homeomorphic with $\mathcal{N}^n$. Given $\omega_0 \leq \beta < \omega_1$ suppose that the claim holds for the ordinals less than β and let $\mathcal{F}$ be a sequential contour of rank β. If β is isolated, then the equivalence follows immediately from the definition and the inductive assumption. If β is limit, then there exists an increasing sequence (α_n) such that $\sup_{n<\omega} \alpha_n = \beta$ and monotone sequential contours $\mathcal{F}_n$ of rank α_n on mutually disjoint countably infinite X_n such that $\mathcal{F} = \int_{(n)} \mathcal{F}_n$. For every α such that $\alpha_n \leq \alpha < \alpha_{n+1}$, let $X_{n,\alpha}$ be an element of the partition of X_n, assured by Lemma 2.3, such that $\mathcal{F}_n \vee X_{n,\alpha}$ is homeomorphic with $\mathcal{F}_n$. By the inductive assumption, there are maps $f_\alpha : D^{(\alpha)} \to X_{n,\alpha}$ such that $f_\alpha(\mathcal{N}^\alpha) \geq \mathcal{F}_n|_{X_{n,\alpha}}$; we define $f(d) = f_\alpha(d)$ if $d \in D^{(\alpha)}$ and conclude that $\mathcal{N}^\beta = \int_{\mathcal{V}(\beta)} \mathcal{N}^\alpha \geq f(\int_{\mathcal{V}(\beta)} \mathcal{F}|_{X_{n,\alpha}}) \geq f(\int_{(n)} \mathcal{F}_n)$. □

Corollary 4.10. *The filter $\mathcal{N}^\beta$ is equivalent to every monotone sequential contour of rank β.*

5. Concept of radiality

If $\mathbb{E}$ is a class of filters, then a convergence ξ is said to be $\mathbb{E}$-*based* if $x \in \lim_\xi \mathcal{F}$ implies the existence of $\mathcal{E} \in \mathbb{E}$ with $\mathcal{E} \leq \mathcal{F}$ and such that $x \in \lim_\xi \mathcal{E}$. For every convergence ζ there exists the coarsest among those $\mathbb{E}$-based convergences that are finer than ζ; it is denoted by $\mathrm{B}_\mathbb{E}\,\zeta$. A convergence ξ is *upper* $P\,\mathrm{B}_\mathbb{E}$ (*upper* $T\,\mathrm{B}_\mathbb{E}$) if $\xi \geq P\,\mathrm{B}_\mathbb{E}\,\xi$ (respectively, $\xi \geq T\,\mathrm{B}_\mathbb{E}\,\xi$), where P and T are respectively the pretopologizer and the topologizer ([3]).

Given a class $\mathbb{E}$ of filters, we denote by $\mathbb{E}^\flat$ the class of all the images of the elements of $\mathbb{E}$ by maps. If we identify a net with a filter it generates, then a topology is $\mathbb{E}$-*radial* (or $\mathbb{E}$-*Fréchet*) in the sense of Nyikos [21, Definition 3.9] if and only if it is an upper $P\,\mathrm{B}_{\mathbb{E}^\flat}$ convergence; $\mathbb{E}$-*pseudoradial* if and only it is an upper $T\,\mathrm{B}_{\mathbb{E}^\flat}$ convergence.

If λ is a cardinal, then we say that a convergence ξ is λ-*uniconvergent* (respectively, λ-*uniradial*, λ-*unipseudoradial*) if there exists a filter $\mathcal{E}$ such that ξ is $\{\mathcal{E}\}^\flat$-based (respectively, $\{\mathcal{E}\}$-radial, $\{\mathcal{E}\}$-pseudoradial). We shall simplify the terminology by dropping the parentheses in the latter cases.

Finally a class $\mathfrak{X}$ of convergences is λ-*uniconvergent* (respectively, λ-*uniradial*, λ-*unipseudoradial*) if there is a filter $\mathcal{E}$ such that every convergence $\xi \in \mathfrak{X}$ is $\{\mathcal{E}\}^{\flat}$-based (respectively, $\{\mathcal{E}\}$-radial, $\{\mathcal{E}\}$-pseudoradial).

Let us observe that the class of convergences such that their underlying set is of cardinality κ, is $2^{2^{\kappa}}$-uniconvergent (hence, $2^{2^{\kappa}}$-radial and $2^{2^{\kappa}}$-pseudoradial). Indeed, if φX stands for the set of filters on X and $X_{\mathcal{F}}$ is a copy of X for each $\mathcal{F} \in \varphi X$, then $\prod_{\mathcal{F}\in\varphi X} \mathcal{F}$ denotes the filter on $\prod_{\mathcal{F}\in\varphi X} X_{\mathcal{F}}$ generated by $\{p_{\mathcal{F}}^{-}(\mathcal{F}) : \mathcal{F} \in \varphi X\}$ where $p_{\mathcal{G}}$ is the projection of $\prod_{\mathcal{F}\in\varphi X} X_{\mathcal{F}}$ on $X_{\mathcal{G}}$. Then each convergence on X is $\{\prod_{\mathcal{F}\in\varphi X} \mathcal{F}\}^{\flat}$-based.

The notion of $\{\mathcal{E}\}^{\flat}$-based convergence is of course intimately related to that of $\mathcal{E}$-limit of Katětov [17].

6. Countably uniradial classes of topologies

By Proposition 4.1, all the monotone sequential contours of given rank are equivalent. Therefore,

Corollary 6.1. *The class of sequential topologies of order less than or equal to β is radial with respect to each sequential contour of rank β.*

Boldjiev and Malyhin proved in [2] that the class of sequential topologies is countably uniradial ($\mathcal{F}$-*Fréchet-Urysohn* in their terminology) by constructing a filter $\mathcal{F}$ on ω (a supercontour in our terminology) such that each sequential topology is $\mathcal{F}$-radial. The propositions below show that the class of subsequential topologies is countably uniradial with respect to every supercontour; moreover, if a countably uniradial class contains every sequential topology, then it includes the class of subsequential topologies.

Theorem 6.2. *If every sequential topology is $\mathcal{H}$-radial with respect to a filter $\mathcal{H}$ on ω, then the class of $\mathcal{H}$-radial topologies includes that of subsequential topologies.*

Proof: Let $\mathcal{H}$ be a filter on ω. If $x \in \operatorname{cl} A$ with respect to a subsequential topology, then by Theorem 1.2, there is a sequential cascade T and a map $g : \max T \to A$ such that $x \in \lim g(\int T)$. As every sequential topology is $\mathcal{H}$-radial, then in particular is the topologization of T. Because $\emptyset_T \in \operatorname{cl}(\max T)$, it follows that there exists a map $f : \omega \to X$ such that $f(\mathcal{H})$ converges to x, in other words, is finer than the contour $\int T$ of T. Consequently, $x \in \lim g \circ f(\mathcal{H})$ showing that every subsequential topology is $\mathcal{H}$-radial. □

The main result of [2] follows from

Theorem 6.3. *The class of subsequential topologies is uniradial with respect to every supercontour.*

Proof: Indeed, if τ is a subsequential topology and $x \in \operatorname{cl}_\tau A$, then by Theorem 1.2, there exists a sequential contour $\mathcal{F}$ on ω and a map $f : \omega \to A$ such that

$f(\mathcal{F})$ converges to x. By Proposition 3.5, for any $\alpha \geq r(\mathcal{F})$ there is a map h such that $\mathcal{F} \leq h(\mathcal{F}_\alpha) \leq h(\mathcal{F}_\infty)$. Accordingly, $f \circ h(\mathcal{F}_\infty)$ converges to x. □

Proposition 6.4. *If every subsequential topology is $\mathcal{H}$-radial, then the prime topology generated by $\mathcal{H}$ is not subsequential.*

PROOF: Consider a topology on $\{\infty\} \cup X$ which is prime with respect to a filter $\mathcal{H}$. Of course, $\infty \in \operatorname{cl} X$, hence if the topology were subsequential, then by Theorem 1.2 there would exist $\alpha < \omega_1$ and a sequential contour $\mathcal{F}$ on ω of rank α and a map $f : \omega \to X$ such that $\mathcal{H} \leq f(\mathcal{F})$. If $\mathcal{G}$ is a sequential contour of rank $\beta > \alpha$, then since the prime topology generated by $\mathcal{G}$ is $\mathcal{H}$-radial, there exists a map $g : X \to \omega$ such that $\mathcal{G} \leq g(\mathcal{H})$, hence $\mathcal{G} \leq g \circ f(\mathcal{F})$ contrary to Corollary 3.4. □

We call a filter $\mathcal{H}$ on ω *fractal* if for every $A \in \mathcal{H}^\#$ the filters $\mathcal{H}$ and $\mathcal{H}|_A$ are equivalent. Notice that the prime topology determined by $\mathcal{H}$ is $\mathcal{H}$-radial if and only if $\mathcal{H}$ is fractal. It follows from Propositions 2.2 and 3.5 that every monotone sequential contour is fractal. On the other hand,

Proposition 6.5. *Each ultrafilter is fractal.*

PROOF: This is evident for principal ultrafilters. If $\mathcal{U}$ is a free ultrafilter on ω and $A \subset \omega$ fulfills $A \in \mathcal{U}^\#$, then $A \in \mathcal{U}$ and $\mathcal{U} \vee A = \mathcal{U}$. If A_0, A_1 are disjoint infinite sets with $A_0 \cup A_1 = A$, then $A_i \in \mathcal{U}$ either for $i = 0$ or for $i = 1$. If this holds for example, for $i = 0$, then we define a bijection $p : \omega \to A$ by setting $p(n) = n$ if $n \in A_0$ and $p|_{\omega \setminus A}$ is an arbitrary bijection of $\omega \setminus A$ onto A_1. Then $p(\mathcal{U}) = \mathcal{U}|_A$ showing that $\mathcal{U}$ and $\mathcal{U}|_A$ are not only equivalent, but homeomorphic. □

By virtue of Theorem 4.8, Proposition 6.4 and Proposition 6.5,

Theorem 6.6 (CH)**.** *There exists a supercontour $\mathcal{H}$ such that the class of $\mathcal{H}$-radial topologies strictly includes the class of subsequential topologies.*

There remains the problem whether for every supercontour $\mathcal{H}$, the class of $\mathcal{H}$-radial topologies contains a non subsequential topology.

In [13, Theorem 7.3] Franklin and Rajagopalan constructed a countable Hausdorff prime subsequential topology of order ω_1 which *generates* the category of subsequential topologies. It follows from Proposition 6.4 that if $\mathcal{H}$ is the filter generating this topology, then the class of subsequential topologies is not $\mathcal{H}$-radial.

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