Equicontinuity of power maps in locally pseudo-convex algebras

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Abstract. We show that, in any unitary (commutative or not) Baire locally pseudo-convex algebra with a continuous product, the power maps are equicontinuous at zero if all entire functions operate. We obtain the same conclusion if every element is bounded. An immediate consequence is a result of A. Arosio on commutative and complete metrizable locally convex algebras.

Keywords: locally pseudo-convex algebra, continuous product, m-p-convexity, Baire space, power maps

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1. Introduction and preliminaries

P. Turpin showed ([12]) that a commutative locally convex Q-algebra with continuous inverse is actually m-convex. He also exhibits ([12]) an example of a complete metrizable non m-p-convex commutative locally p-convex Q-algebra with continuous inverse. In his example, the power maps are equicontinuous at zero. Using a Baire type argument and the Mazur-Orlicz formula, we obtain the equicontinuity of the power maps in a general context of non necessarily commutative locally pseudo-convex algebra. More precisely, we show that the power maps $(x \longmapsto x^n)_n$ are equicontinuous at zero in any Baire locally pseudo-convex algebra with a continuous product and in which all entire functions operate. As a consequence, we obtain the result of [5] for commutative Baire locally convex algebras and hence the result of Mityagin, Rolewicz and Zelazko for commutative and complete metrizable locally convex algebras ([9]). We also obtain our result of [4] in the non commutative case. We generalize a result of E.A. Michael [8] on m-convex algebras. We prove that entire functions operate in any M-complete locally p-convex algebra in which the power maps are equicontinuous at zero; hence the same result holds for any M-complete locally A-p-convex algebra. Finally we obtain that the power maps are equicontinuous at zero in any Baire and locally pseudo-convex algebra $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ with a continuous product and in which every element is bounded. Therefore $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ is already m-convex in the commutative locally convex case; this includes the result of A. Arosio for commutative and complete metrizable locally convex algebras ([1]).

Let E be a vector space and $0 < p \leq 1$. As usual, a p-seminorm on E is a subadditive function $\|\cdot\|_p : E \longrightarrow \mathbb{R}_+$ such $\|\lambda x\|_p = |\lambda|^p \|x\|_p$, for any $\lambda \in \mathbb{C}$ and $x \in E$. If, in addition, $\|x\|_p = 0$ implies $x = 0$, then $\|\cdot\|_p$ is a p-norm. By a p-normed space, we mean a space endowed with a p-norm. A complete p-normed space is called a p-Banach space. Moreover, if E is an algebra and $\|\cdot\|_p$ is submultiplicative (i.e., $\|xy\|_p \leq \|x\|_p \|y\|_p$, for all $x, y \in E$), then $\|\cdot\|_p$ is called an algebra p-norm. A p-normed algebra is an algebra endowed with an algebra p-norm. A complete p-normed algebra is called a p-Banach algebra. Let (E, τ) be a locally pseudo-convex space ([11], [13]) the topology of which is given by a family $\{|\cdot|_\lambda : \lambda \in \Lambda\}$ of p_λ-seminorms, $0 < p_\lambda \leq 1$. If E is endowed with an algebra structure such that the product is separately continuous, we say that $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ is a locally pseudo-convex algebra (l-pseudo-$c.a.$ in short). It is said to be with continuous product if the product is continuous in two variables. Recall that a l-pseudo-$c.a.$ E is called a Q-algebra if the group $G(E)$ of its invertible elements is open. Notice that l-pseudo-$c.a.$'s are usual locally convex algebras ($l.c.a.$ in short) when $p_\lambda = 1$, for every $\lambda \in \Lambda$.

For the notions used here in the context of a general locally pseudo-convex algebras (l-pseudo-$c.a.$), the reader is referred to [12], [13]. Concerning general locally convex algebras ($l.c.a.$) see [7] and for locally m-convex algebras ($l.m.c.a.$) see [7], [8], [14]. All algebras considered here are over a field K ($K = \mathbb{R}$ or $K = \mathbb{C}$).

2. Equicontinuity of power maps in locally pseudo-convex algebras

Recall that an entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $a_n \in K$, operates in a unitary l-pseudo-$c.a.$ $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$ if, for every x in E, $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ converges in $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$. It is known that a commutative and complete metrizable locally convex algebra in which all entire functions operate is necessarily m-convex ([9]). The condition of local convexity cannot be replaced by local p-convexity. Indeed in [12], P. Turpin gives an example of a commutative non m-p-convex locally p-convex Q-algebra with continuous inverse. This is also a metrizable and complete algebra and hence all entire functions operate. In this example, the power maps are equicontinuous at zero. This fact remains valid in the more general context of locally pseudo-convex algebras as the following result shows.

Theorem 2.1. *Let $(E, (|\cdot|_\lambda)_{\lambda \in \Lambda})$, $0 < p_\lambda \leq 1$, be a unitary (commutative or not) l-pseudo-c.a. with a continuous product which is a Baire space. If entire functions operate in E, then the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero.*

Proof: Observe first that, for every $x \in E$, $\sup_n |x^n|_\lambda^{\frac{1}{np_\lambda}} < +\infty$, for every $\lambda \in \Lambda$. Indeed if it does not hold, then there exist $a_0 \in E$ and $\lambda_0 \in \Lambda$ such that $\left|a_0^{k_n}\right|_{\lambda_0} > n^{p_{\lambda_0} k_n}$, for a certain increasing sequence $(k_n)_n$ of integers. In this case the entire function $\sum_{n=0}^{\infty} n^{-k_n} z^{k_n}$ diverges at a_0. Let $\lambda \in \Lambda$ and $f_\lambda : E \longrightarrow \mathbb{R}_+$, be the map given by $f_\lambda(x) = \sup_n |x^n|_\lambda^{\frac{1}{np_\lambda}}$. The function f_λ is lower semicontinuous because the product is continuous. For every integer m, set $E_m = \{a \in E : f_\lambda(a) \leq m\}$. Since f_λ is lower semicontinuous, the set E_m is a closed subset of E, for every integer m. By a Baire type argument, there is an integer k such that E_k is of non void interior. It follows that there is $x_0 \in E_k$ and a neighborhood V of zero such that $x_0 + V \subset E_k$. So for every x in V, we have

$$|(x_0 + x)^n|_\lambda \leq k^{np_\lambda}, \quad n = 1, 2, \ldots, \tag{1}$$

whence

$$\left(\frac{x_0}{k} + \frac{x}{k}\right)^n \in U_\lambda, \quad n = 1, 2, \ldots,$$

where $U_\lambda = \{x \in E : |x|_\lambda \leq 1\}$. On the other hand, by the Mazur-Orlicz formula ([3]), we have

$$\left(\frac{x}{kn}\right)^n = \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j}\, C_n^j \left(\frac{x_0}{k} + \frac{j}{kn} x\right)^n, \quad x \in V, \quad n = 1, 2, \ldots .$$

Then

$$\left(\frac{x}{k}\right)^n \in \frac{n^n}{n!} \sum_{j=0}^{n} C_n^j U_\lambda, \quad \text{since } U_\lambda \text{ is balanced.}$$

There exists a constant $M > 0$ such that $\frac{(2n)^n}{n!} \leq M^n$, for every integer n. Thus

$$\left(\frac{x}{k}\right)^n \in M^n U_\lambda, \quad x \in V,\ n = 1, 2, \ldots, \quad \text{for } U_\lambda \text{ is balanced,}$$

whence

$$x^n \in U_\lambda; \quad \text{for every } x \in \frac{1}{Mk} V, \quad n = 1, 2, \ldots .$$

□

Remarks 2.2.

1) If E is commutative, we can obtain the conclusion without using the Mazur-Orlicz formula. Indeed, since the product is continuous, for $\lambda \in \Lambda$ there is $\lambda' \in \Lambda$ such that

$$|xy|_\lambda^{p_{\lambda'}} \leq |x|_{\lambda'}^{p_\lambda} |y|_{\lambda'}^{p_\lambda}, \quad \text{for all } x, y \in E. \tag{2}$$

For every integer m, set $E'_m = \{a \in E : f_{\lambda'}(a) \leq m\}$. Arguing as above, we see that there is an integer k such that E'_k is of non void interior. Hence, there is an x_0 in E'_k and a neighborhood V' of zero such that

$$x_0 + V' \subset E'_k.$$

So for every $x \in V'$, we have

$$|(x_0 + x)^n|_{\lambda'} \le k^{np_{\lambda'}}, \quad n = 1, 2, \dots . \tag{3}$$

Now using (2) and (3), one has

$$|x^n|_\lambda = |(x_0 + x - x_0)^n|_\lambda \le (2k^{p_\lambda})^n, \quad \text{for every} \;\; x \in V',$$

whence

$$x^n \in U_\lambda \;\; \text{for every} \;\; x \in \frac{1}{2^{\frac{1}{p_\lambda}} k} V', \quad n = 1, 2, \dots .$$

2) If moreover $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ is a commutative *l.c.a.*, then $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ is an m-convex algebra. Indeed, consider the polarization formula

$$x_1 x_2 \dots x_n = \frac{1}{n!} \sum_I (-1)^{n-c(I)} \left(\sum_{i\in I} x_i \right)^n,$$

where I runs over the collection of all finite subsets of $\{1, 2, \dots, n\}$, $c(I)$ is the cardinality of I, and x_1, x_2, …, x_n are elements of E. For $t > 0$, if $x_i \in \frac{1}{Mk} tV$, $1 \le i \le n$, we have

$$x_1 x_2 \dots x_n \in \frac{(2nt)^n}{n!} U_\lambda.$$

Then, for t small enough, U_λ contains an idempotent neighborhood of zero.

3) There exists a unitary commutative Baire l-pseudo-*c.a.* with a continuous product in which all entire functions operate and which is not topologizable as a locally convex algebra on which the power maps are equicontinuous at zero. Indeed, let E be the commutative topological algebra exhibited by V. Müller [10, Theorem 2]. This algebra is a p-normed algebra which is not topologizable as a locally convex algebra with continuous product. The closure $\overline{E}$ of E satisfies the hypothesis of Theorem 2.1. But, by another result of P. Turpin [12, Proposition 3], the algebra $\overline{E}$ is not topologizable as a locally convex algebra on which the power maps are equicontinuous at zero.

As a consequence of Theorem 2.1, we obtain the following results.

Corollary 2.3. *Let $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ be a unitary and complete metrizable l.p.c.a., $0 < p \le 1$. If E is Q-algebra, then the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero.*

Proof: Since $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ is a complete metrizable Q-algebra, it is with continuous product and inverse. By a result of Waelbroeck [13, p. 90], $\rho = \beta$, where ρ and β are respectively the spectral radius and radius of boundedness. The Q-algebra property implies the boundedness of every element. Hence entire functions operate on E. The conclusion follows from Theorem 2.1.

□

Corollary 2.4. *Let $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ be a unitary and complete metrizable l.p.c.a., $0 < p \leq 1$. If* $\operatorname{Rad} E$ *is closed, then, in this radical, the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero.*

Proof: The unitary subalgebra $(\operatorname{Rad} E)^1 = \operatorname{Rad} E \oplus \mathbb{C}e$, of E, is closed and such that $\operatorname{Rad}\left[(\operatorname{Rad} E)^1\right] = \operatorname{Rad} E$. It follows that $(\operatorname{Rad} E)^1$ is a Q-algebra. So, by the previous corollary, the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero on $(\operatorname{Rad} E)^1$ and so on $\operatorname{Rad} E$. □

To make the paper self-contained, we give the following definitions: Let E be a locally p-convex space, $0 < p \leq 1$. A sequence $(x_n)_n$ in E is said to be Mackey-Cauchy if there exists a bounded p-disk $B \subset E$ such that $(x_n)_n$ is a Cauchy sequence in the p-normed space $(E_B, \|\cdot\|_B)$, where $E_B = \bigcup_{\lambda>0} \lambda B$ is the span of B and $\|\cdot\|_B$ is the p-gauge of B. The space E is said to be Mackey-complete (M-complete) if every Mackey-Cauchy sequence is convergent. As in the locally convex case, one can prove that E is M-complete if and only if every bounded and closed p-disk is a completant p-disk i.e., the space $(E_B, \|\cdot\|_B)$ is a p-Banach space. Moreover, if $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ is a *l-p-c.a.*, it is said to be *m-p*-complete if every bounded and closed idempotent p-disk is a completant p-disk.

It is clear that entire functions operate in sequentially complete *l.m-p-c.a.*'s. This result is not in general true in the *m-p*-complete case. We show that M-completeness is sufficient in any *l.p.c.a.* for which the power maps are equicontinuous at zero.

Theorem 2.5. *Entire functions operate in any unitary and M-complete l.p.c.a. (E, τ) in which the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero. In particular, entire functions operate in any unitary and M-complete l.m.p.c.a.*

Proof: Let $(|\cdot|_\lambda)_{\lambda\in\Lambda}$ be a family of p-seminorms defining τ and let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function. We first prove that the sequence $(a_n x^n)_n$ is bounded for every $x \in E$. Since $(x \longmapsto x^n)_n$ is equicontinuous at zero, there is, for each $\lambda \in \Lambda$, an open neighborhood U_λ of zero such that $|y^n|_\lambda \leq 1$, for every $y \in U_\lambda$ and $n = 1, 2, \ldots$. Let $x \in E$ and $r > 0$ be such that $rx \in U_\lambda$. Then since $\lim_n |a_n|^{\frac{1}{n}} = 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n| \leq \left(\frac{r}{2}\right)^n$, for every $n \geq n_0$. It follows that $|a_n x^n|_\lambda \leq \frac{1}{2^{np}}$, for every $n \geq n_0$. Hence the sequence $(a_n x^n)_n$ is bounded. Now there exists a bounded and completant p-disk B such that $a_n(2x)^n \in B$, for every $n = 1, 2, \ldots$. In the p-Banach space $(E_B, \|\cdot\|_B)$, one has $\|a_n x^n\|_B \leq \frac{1}{2^{np}}$, where E_B is the span of B and $\|\cdot\|_B$ is the p-gauge of B. Therefore the series $\sum_{n=0}^{+\infty} a_n x^n$ converges in $(E_B, \|\cdot\|_B)$, and hence in $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$. □

A *l-p-c.a.* $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$, $0 < p \leq 1$, is said to be a locally A-p-convex algebra (*l.A-p-c.a.* in short) if for each λ and x, there exists $N(x, \lambda) > 0$ such that

$$\max\left(|xy|_\lambda, |yx|_\lambda\right) \leq N(x, \lambda)|y|_\lambda, \quad \text{for every} \quad y \in E.$$

We then have:

Corollary 2.6. *Entire functions operate in any unitary and M-complete $l.A$-p-$c.a.$ (E,τ).*

Proof: Let $(|\cdot|_\lambda)_{\lambda\in\Lambda}$ be a family of p-seminorms defining τ. There exists on E an m-p-convex topology $M(\tau)$, finer than τ, given by the p-seminorms $(\|\cdot\|_\lambda)_{\lambda\in\Lambda}$ defined by $\|x\|_\lambda = \sup\{|xy|_\lambda : |y|_\lambda \leq 1\}$. Clearly, bounded sets for $M(\tau)$ are bounded for τ. On the other hand, since every closed and bounded p-disk in a M-complete locally p-convex space is necessarily completant, the two topologies have the same bounded sets. Hence $(E,(\|\cdot\|_\lambda)_{\lambda\in\Lambda})$ is also M-complete. Moreover the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero in the algebra $(E,(\|\cdot\|_\lambda)_{\lambda\in\Lambda})$. The conclusion follows from Theorem 2.5. □

A l-pseudo-$c.a.$ is said to be strongly sequential if there is a 0-neighborhood U such that for all $x \in U$, the sequence $(x^n)_n$ converges to zero. The example of W. Zelazko [15] is a strongly sequential non m-convex algebra on which the power maps are equicontinuous at zero. We obtain the same in locally pseudo-convex algebras.

Proposition 2.7. *Let $(E,(|\cdot|_\lambda)_{\lambda\in\Lambda})$, $0 < p_\lambda \leq 1$, be a unitary l-pseudo-$c.a.$ with a continuous product which is a Baire space. If E is strongly sequential, then the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero.*

Proof: Since E is strongly sequential, there is an open neighborhood Ω of zero in E such that $\lim_k x^k = 0$, for every $x \in \Omega$. Let U be a balanced and closed neighborhood of zero in E and put

$$F = \{x \in E : x^n \in U, \quad n = 1,2,\dots\}.$$

The product being continuous, the set F is closed. Since $\lim_k x^k = 0$, for every $x \in \Omega$, we have

$$\Omega \subset \bigcup \{mF : m = 1,2,\dots\}.$$

But Ω being open is also a Baire space. By Baire's theorem, F has non-void interior. Let $x_0 \in F$ and let V be a balanced neighborhood of zero such that $x_0 + V \subset F$. So

$$(x_0 + x)^n \in U;\ x_0 \in V,\ n = 1,2,\dots\ .$$

Now using the Mazur-Orlicz formula ([3])

$$\left(\frac{x}{n}\right)^n = \frac{1}{n!}\sum_{k=0}^{n} (-1)^{n-k}\, C_n^k \left(x_0 + \frac{k}{n}x\right)^n, \quad x \in V, \quad n = 1,2,\dots,$$

and following the same lines of argument as in the proof of Theorem 2.3, we have

$$x^n \in U; \quad x \in \frac{1}{2e}V, \quad n = 1,2,\dots\ .$$

□

In the commutative locally convex case, we obtain the following

Corollary 2.8. *Let $(E, (|\cdot|_i)_{i\in I})$ be a unitary commutative l.c.a. with a continuous product which is a Baire space. If E is strongly sequential, then it is m-convex.*

A. Arosio [1] showed that a commutative and complete metrizable locally convex algebra every element of which is bounded is locally m-convex. In the non commutative case, the result is not valid as the example of Zelazko [15] shows. In Baire *l.c.a.*'s with continuous product, we have obtained the equicontinuity of the power maps under an additional condition ([5]). In the context of l-pseudo-*c.a.*'s we have the following

Theorem 2.9. *Let $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$, $0 < p_\lambda \le 1$, be a unitary Baire and l-pseudo-c.a. with a continuous product. If every element of E is bounded, then the sequence $(x \longmapsto x^n)_n$ of power maps is equicontinuous at zero. In particular, if $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ is a commutative l.c.a., then $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$ is an m-convex algebra.*

Proof: Let $x \in E$ and $\alpha > 0$ be such that $\{(\alpha^{-1}x)^n : n = 1, 2, \dots\}$ is a bounded subset of $(E, (|\cdot|_\lambda)_{\lambda\in\Lambda})$. Let U_λ be an absolutely p_λ-convex and closed neighborhood of zero in E. There exists $\delta > 0$ such that $(\alpha^{-1}x)^n \in \delta U_\lambda$, for every $n = 1, 2, \dots$. Since the product is continuous, the set

$$E_{h,l} = \{x \in E : x^n \in hl^n U_\lambda : n = 1, 2, \dots\}$$

is closed, for every pair (h, l) of integers. By a Baire type argument, there are $s, t \in \mathbb{N}$ such that $E_{s,t}$ is of non void interior. It follows that there is an $x_0 \in E_{s,t}$ and a neighborhood of zero V such that $x_0 + V \subset E_{s,t}$. So for every $x \in V$, we have

$$(x_0 + x)^n \in s\ t^n U_\lambda, \quad \text{for every} \quad n \in \mathbb{N}$$

and by the Mazur-Orlicz formula ([3]),

$$\left(\frac{x}{n}\right)^n = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k}\, C_n^k \left(x_0 + \frac{k}{n}x\right)^n, \quad x \in V_{\lambda'}, \quad n = 1, 2, \dots,$$

so that

$$x^n \in \frac{n^n}{n!} \sum_{k=0}^{k} s\ t^n C_n^k U_\lambda \subset \frac{(2stn)^n}{n!} U_\lambda, \quad \text{for every} \quad x \in V_{\lambda'},\ n = 1, 2, \dots .$$

There exists $c > 0$ such that $\frac{(2stn)^n}{n!} \le c^n$, for every integer n. Thus

$$x^n \in U_\lambda; \quad x \in \frac{1}{c}V, \quad n = 1, 2, \dots,$$

holds. □

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