On the composition of the integral and derivative operators of functional order

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Abstract. The Integral, I_ϕ, and Derivative, D_ϕ, operators of order ϕ, with ϕ a function of positive lower type and upper type less than 1, were defined in [HV2] in the setting of spaces of homogeneous-type. These definitions generalize those of the fractional integral and derivative operators of order α, where $\phi(t) = t^\alpha$, given in [GSV].

In this work we show that the composition $T_\phi = D_\phi \circ I_\phi$ is a singular integral operator. This result in addition with the results obtained in [HV2] of boundedness of I_ϕ and D_ϕ or the $T1$-theorems proved in [HV1] yield the fact that T_ϕ is a Calderón-Zygmund operator bounded on the generalized Besov, $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$, and Triebel-Lizorkin spaces, $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$, of order $\psi = \psi_1/\psi_2$, where ψ_1 and ψ_2 are two quasi-increasing functions of adequate upper types s_1 and s_2, respectively.

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1. Introduction

In the context of normal spaces of homogeneous-type (X, δ, μ) of order $\theta \le 1$, the integral operator, I_ϕ, and the derivative operator, D_ϕ, of order ϕ, where ϕ is a function of positive lower type and upper type less than θ, were defined in [HV2] in such way that their kernels become equivalent to $\phi(\delta(x,y))/\delta(x,y)$ and $1/(\phi(\delta(x,y))\delta(x,y))$, respectively.

It was proved in that work, by means of the Calderón-type reproduction formulas given in [HS], that I_ϕ is continuous from the Besov spaces $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$, and Triebel-Lizorkin spaces, $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$, into $\dot{B}_p^{\phi\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\phi\psi,q}$, $1 \le p, q < \infty$, respectively. Similarly, it was seen that D_ϕ is continuous from $\dot{B}_p^{\psi,q}$ and $\dot{F}_p^{\psi,q}$ into $\dot{B}_p^{\psi/\phi,q}$ and $\dot{F}_p^{\psi/\phi,q}$, respectively, for the expected range of types of the two functions in each case.

This results generalize the classical ones referred to the fractional integral and derivative operators, I_α and D_α, and their action on the Besov $\dot{B}_p^{\beta,q}$ and $\dot{F}_p^{\beta,q}$ spaces.

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In this work we prove that the composition $T_\phi = D_\phi \circ I_\phi$ is a singular integral operator in the classical sense and, hence, we complete the proof of that it is a Calderón-Zygmund operator bounded on the generalized Besov and Triebel-Lizorkin spaces.

It is worth saying that, once the standard conditions on the kernel of T_ϕ are proved, the same result is obtained by the $T1$-theorems for those spaces proved in [HV1].

This work is organized in the following way:
In Section 2 we define the class of functions involved in the 'order' of the integral and derivative operators. The structure of normal spaces of homogeneous type, the test function space and the notion of continuous approximation to the identity is also set in that section. The definitions of the integral and derivative operators and the main theorem are stated in Section 3. In Section 4 known results on the class of quasi-increasing functions are given and, afterwards, size and smoothness conditions of the kernels of I_ϕ and D_ϕ and the theorems of boundedness on Lipschitz spaces proved in [HV2] are stated. Finally, the proof of the fact that T_ϕ is a Calderón-Zygmund operator is in Section 5.

2. Preliminaries

Let us now consider nonnegative functions ϕ defined on the positive real numbers.

A function $\phi(t)$ is said to be *quasi-increasing* if there is a positive constant C such that if $t_1 < t_2$ then $\phi(t_1) \leq C\phi(t_2)$.

Analogously, $\phi(t)$ is *quasi-decreasing* if there is a positive constant C such that if $t_1 < t_2$ then $\phi(t_2) \leq C\phi(t_1)$.

The functions $\psi(t)$ and $\phi(t)$ are *equivalent*, $\psi \simeq \phi$, if there are positive constants C_1 and C_2 such that $C_1 \leq \phi/\psi \leq C_2$.

The function $\phi(t)$ is said to be of *lower type* α, $0 \leq \alpha < \infty$, if there is a constant $C_1 > 0$ such that

$$\phi(uv) \leq C_1 u^\alpha \phi(v) \quad \text{for} \quad u < 1 \quad \text{and} \quad v > 0. \tag{2.1}$$

Similarly, $\phi(t)$ is of *upper type* α, $0 \leq \alpha < \infty$ if there is a constant $C_2 > 0$ such that

$$\phi(uv) \leq C_2 u^\alpha \phi(v) \quad \text{for} \quad u \geq 1 \quad \text{and} \quad v > 0. \tag{2.2}$$

Clearly, the potential t^α, with $\alpha \geq 0$, is of lower and upper type α. The functions $\max(t^\alpha, t^\beta)$ and $\min(t^\alpha, t^\beta)$, with $\alpha < \beta$, are both of lower type α and upper type β. Also, $t^\beta(1 + \log^+ t)$, with $\beta \geq 0$, is of lower type β and of upper type $\beta + \epsilon$, for every $\epsilon > 0$.

Let us notice that if $\phi(t)$ is of both lower type α and upper type β then $\alpha \leq \beta$. Also, if $\phi(t)$ is of lower type α and $0 \leq \beta < \alpha$ then ϕ is of lower type β. Moreover, since the condition $\phi(t)$ quasi-increasing implies, at least, lower-type 0 for ϕ, a function $\phi(t)$ is quasi-increasing if, and only if, it is of lower type α for some $\alpha \geq 0$.

On the other hand, if $\phi(t)$ is of upper type α and $\beta > \alpha$ then ϕ is of upper type β, and thus, if ϕ is of finite upper type there is a right half line of upper types for ϕ. Let us notice that the condition of having finite upper type is equivalent to the Orlicz condition Δ_2, $\phi(2t) \leq A\phi(t)$ for some positive constant A.

Let us now define the structure of spaces of homogeneous type which is the underlying geometry for the test function spaces defined in this work.

Given a set X, a real valued function $\delta(x,y)$ defined on $X \times X$ is a quasi-distance on X if there exists a constant $A > 1$ such that for all $x, y, z \in X$ it verifies:

$$\begin{aligned}
&\delta(x,y) \geq 0 \ \text{ and } \ \delta(x,y) = 0 \ \text{ if and only if } \ x = y \\
&\delta(x,y) = \delta(y,x) \\
&\delta(x,y) \leq A[\delta(x,z) + \delta(z,y)].
\end{aligned}$$

In a set X endowed with a quasi-distance $\delta(x,y)$, the balls $B_\delta(x,r) = \{y : \delta(x,y) < r\}$ form a basis of neighborhoods of x for the topology induced by the uniform structure on X.

Let μ be a positive measure on a σ-algebra of subsets of X which contains the open set and the balls $B_\delta(x,r)$. The triple $X := (X, \delta, \mu)$ is a *space of homogeneous type* if there exists a finite constant $A' > 0$ such that $\mu(B_\delta(x, 2r)) \leq A'\mu(B_\delta(x,r))$ for all $x \in X$ and $r > 0$. Macías and Segovia [MS] showed how to find a quasi-distance $d(x,y)$ equivalent to $\delta(x,y)$ and $0 < \theta \leq 1$, such that

$$|d(x,y) - d(x',y)| \leq C r^{1-\theta} d(x,x')^\theta \tag{2.3}$$

holds whenever $d(x,y) < r$ and $d(x',y) < r$.

If δ satisfies (2.3) then X is said to be *of order* θ.

X is a *normal space* if $A_1 r \leq \mu(B_\delta(x,r)) \leq A_2 r$ for every $x \in X$ and $r > 0$ and some positive constants A_1 and A_2.

In this work $X := (X, \delta, \mu)$ *means a normal space of homogeneous type of order* θ *and* A *denotes the constant of the triangular inequality associated with* δ.

Given a quasi-increasing function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0} \xi(t) = 0$ and $\lim_{t\to\infty} \xi(t) = \infty$, the Lipschitz space Λ^ξ is the class of all functions $f : X \to \mathbb{C}$ such that

$$|f(x) - f(y)| \leq C\xi(\delta(x,y)) \ \text{ for every } \ x, y \in X,$$

and the number $|f|_\xi$ denoting the infimum of the constants C appearing above, defines a semi-norm on Λ^ξ, since $|f|_\xi = 0$ for all constants functions f.

Furthermore, given a ball B in X, $\Lambda^\xi(B)$ denotes the set of functions $f \in \Lambda^\xi$ with support in B. Since, a function belonging to this space is bounded, the number $\|f\|_\xi = \|f\|_\infty + |f|_\xi$, defines a norm that induces a Banach structure to $\Lambda^\xi(B)$.

We say that a function f belongs to Λ_0^ξ if $f \in \Lambda^\xi(B)$ for some ball B. The space Λ_0^ξ is the inductive limit of the Banach spaces $\Lambda^\xi(B)$.

Finally, $(\Lambda_0^\xi)'$ will mean the space of all continuous linear functionals on Λ_0^ξ.

When $\xi(t) = t^\beta$, with $0 < \beta \le \theta$, we have the classical Lipschitz spaces Λ^β and Λ_0^β.

Finally, we shall consider a symmetric approximation to the identity, that is a family of integral operators $\{S_t\}_{t>0}$, as defined in [GSV], whose kernels $s_t(x,y)$ satisfy the following properties:
There are positive constants, b_1, b_2, c_1, c_2 and c_3, such that for all $x, y \in X$ and $t > 0$, $s_t(x,y)$ satisfies

$$s_t(x,y) = s_t(y,x),$$
$$0 \le s_t(x,y) \le c_1/t,$$
$$s_t(x,y) = 0 \ \text{ if } \ \delta(x,y) > b_1 t \ \text{ and, } \ c_2/t < s_t(x,y) \ \text{ if } \ \delta(x,y) < b_2 t,$$
$$|s_t(x,y) - s_t(x',y)| < c_3 \delta^\theta(x,x')/t^{1+\theta}, \ \text{ for all } \ x, x', y \in X,$$
$$\int s_t(x,y)\, d\mu(y) = 1, \ \text{ for all } \ x \in X,$$

$s_t(x,y)$ is continuously differentiable in t.

3. Integral and derivative operators of order ϕ and main theorem

The general setting for the definition of both operators is that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a quasi-increasing function such that $\lim_{t\to 0^+} \phi(t) = 0$.

We define
$$K_\phi(x,y) = \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)\, dt \ \text{ for } \ x \ne y.$$
Clearly, $K_\phi(x,y) > 0$ and $K_\phi(x,y) = K_\phi(y,x)$ for every (x,y).

For ϕ of positive lower type and upper type $s_\phi < 1$ the *integral operator of order* ϕ, I_ϕ, and its extension $\tilde{I}_\phi$ are defined in the following way:
Given any quasi-increasing function ξ of upper type $\beta > 0$,

if $f \in \Lambda^\xi \cap L^1$ then
$$I_\phi f(x) := \int_X K_\phi(x,y) f(y)\, d\mu(y);$$

if $\beta + s_\phi < \theta$ and $f \in \Lambda^\xi$ then
$$\tilde{I}_\phi f(x) := \int_X (K_\phi(x,y) - K_\phi(x_0,y)) f(y)\, d\mu(y),$$

for every $x \in X$ and an arbitrary fixed $x_0 \in X$.

On the other hand, if ϕ is of finite upper-type we define

$$K_{1/\phi}(x,y) = \int_0^\infty \frac{1}{\phi(t)t} s_t(x,y)\,dt, \quad \text{for} \quad x \neq y.$$

Clearly $K_{1/\phi}$ is also positive and symmetric.

For a function ϕ of lower type $i_\phi > 0$ and upper type s_ϕ, the *derivative operator of order* ϕ, D_ϕ, and its extension, $\tilde{D}_\phi$ are defined as follows:
Given any function ξ of lower type α and of upper type β, such that $s_\phi < \alpha$,

if $f \in \Lambda^\xi \cap L^\infty$, then

$$D_\phi f(x) = \int_X K_{1/\phi}(x,y)(f(y) - f(x))\,d\mu(y) \quad \text{and,}$$

if $f \in \Lambda^\xi$, then

$$\tilde{D}_\phi f(x) = \int_X (K_{1/\phi}(x,y)(f(y) - f(x)) - K_{1/\phi}(x_0,y)(f(y) - f(x_0)))\,d\mu(y)$$

for each $x \in X$ and an arbitrary, but fixed, $x_0 \in X$.

The theorem whose proof is the purpose of this work is stated as follows:

Theorem 3.1. *Let ϕ be of lower type $i_\phi > 0$ and of upper type s_ϕ such that $s_\phi < \epsilon \leq \theta$. Then $T_\phi = D_\phi \circ I_\phi$ is a singular integral operator whose associated kernel is*

$$K(x,y) = \int K_{1/\phi}(x,z)(K_\phi(z,y) - K_\phi(x,y))\,d\mu(z).$$

4. Previous results

A straightforward consequence of the definitions is that if $\phi(t)$ is of upper type s_ϕ then there is a constant $C > 0$ such that

$$\phi(uv) \geq \frac{1}{C} u^{s_\phi} \phi(v), \quad \text{for} \quad u < 1, \ v > 0. \tag{4.4}$$

Similarly, if $\phi(t)$ is of lower type i_ϕ then there is a constant $C > 0$ such that

$$\phi(uv) \geq \frac{1}{C} u^{i_\phi} \phi(v), \quad \text{for} \quad u \geq 1, \ v > 0. \tag{4.5}$$

Also, it is easy to check that

Proposition 4.1. *If $\phi(t)$ is of lower type i_ϕ and $\xi(t)$ is of upper type $\lambda \le i_\phi$ then $\phi(t)/\xi(t)$ is quasi-increasing.*

Proposition 4.2. *If $\phi(t)$ is of lower type $\alpha > 0$ and upper type $\beta \in \mathbb{R}$ and $0 < \gamma < \alpha$ then the function*

$$\psi(t) = t^\gamma \int_0^t \frac{\phi(u)}{u^{\gamma+1}}\, du$$

is equivalent to ϕ, continuous, increasing and invertible. Moreover, its inverse ψ^{-1} is of lower type β^{-1} and of upper type α^{-1}.

The next corollaries of the above Proposition will be needed to define the quasi-metrics associated to the kernels of our operators.

Corollary 4.1. *If ϕ is a quasi-increasing function of upper type $s_\phi < 1$ then there is an equivalent function $\tilde{\phi}$ such that $\tilde{\phi}(t)/t$ is decreasing, continuous and invertible on $t > 0$.*

Corollary 4.2. *If $\phi(t)$ is a quasi-increasing function of finite upper type then there exists a function $\hat{\phi}(t)$ equivalent to $\phi(t)$, such that $t\hat{\phi}(t)$ is increasing, continuous and invertible in $\mathbb{R}^+$.*

The following properties will be useful throughout the proof of the theorem: Let $\phi_i(t)$ be a function of lower type α_i and of upper type β_i, $i = 1, 2$. For every $x \in X$ and $r > 0$ it holds that

$$\text{(4.6)} \qquad \text{If } \alpha_1 > \beta_2 \text{ then } \int_{\delta(x,y)\le r} \frac{\phi_1(\delta(x,y))}{\phi_2(\delta(x,y))\delta(x,y)}\, d\mu(y) \le C\frac{\phi_1(r)}{\phi_2(r)}.$$

$$\text{(4.7)} \qquad \text{If } \beta_1 < \alpha_2 \text{ then } \int_{\delta(x,y)\ge r} \frac{\phi_1(\delta(x,y))}{\phi_2(\delta(x,y))\delta(x,y)}\, d\mu(y) \le C\frac{\phi_1(r)}{\phi_2(r)}.$$

Let us now give a representation of the kernel of I_ϕ in terms of a quasi-metric equivalent to δ.

If ϕ is a quasi-increasing function of upper-type $s_\phi < 1$ consider a fixed function $\tilde{\phi}$, as given in Corollary 4.1. Then

$$K_\phi(x,y) = \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \quad \text{for } x \ne y,$$

where $\delta_\phi(x,y)$ is defined as the unique solution of

$$\frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} = \int_0^\infty \frac{\phi(t)}{t} s_t(x,y)\, dt \quad \text{if } x \ne y, \text{ and}$$
$$\delta_\phi(x,y) = 0 \quad \text{if } x = y.$$

If $\phi(t) = t^\alpha$, $0 < \alpha < 1$, we can choose $\tilde{\phi} = \phi$ and then $\delta_\alpha := \delta_\phi$ is the quasi-metric associated to I_α defined in [GSV].

The following lemmas and theorems are proved in [HV2]. The first one shows that $K_\phi(x, y)$ is equivalent to $\phi(\delta(x, y))/\delta(x, y)$.

Lemma 4.1 ([HV2]). *If ϕ is of upper type $s_\phi < 1$ then there are positive constants C_1 and C_2 such that for $\delta(x, y) > 0$,*

$$C_2 \frac{\phi(\delta(x,y))}{\delta(x,y)} \le \frac{\tilde{\phi}(\delta_\phi(x,y))}{\delta_\phi(x,y)} \le C_1 \frac{\phi(\delta(x,y))}{\delta(x,y)}.$$

In particular,

$$0 < K_\phi(x,y) \le C \frac{\phi(\delta(x,y))}{\delta(x,y)}. \tag{4.8}$$

Moreover, δ_ϕ is a quasi-metric equivalent to δ.

Lemma 4.2 ([HV2]). *Let ϕ be of upper type $s_\phi < 1$. Then*

$$|K_\phi(x,y) - K_\phi(x',y)| + |K_\phi(y,x) - K_\phi(y,x')| \le C \left(\frac{\delta(x,x')}{\delta(x,y)} \right)^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)} \tag{4.9}$$

whenever $\delta(x, y) \ge 2A\delta(x, x')$.

Lemma 4.3 ([HV2]). *Let ϕ be of upper type $s_\phi < \theta$. Then*

$$\int_X [K_\phi(x,y) - K_\phi(x',y)]\, d\mu(y) = 0, \tag{4.10}$$

for every x and $x' \in X$.

Theorem 4.4 ([HV2]). *Let ϕ be of lower type $i_\phi > 0$ and upper type $s_\phi < 1$ and ξ a quasi-increasing function of upper type β.*

If $f \in \Lambda^\xi \cap L^1$ and $\beta > 0$ then $I_\phi f(x)$ converges absolutely for all x and if, also, $\beta + s_\phi < \theta$ then there is a constant $C > 0$, independent of f, such that

$$|I_\phi f|_{\Lambda^{\xi\phi}} \le C|f|_{\Lambda^\xi}.$$

Also, if $f \in \Lambda^\xi$ and $\beta + s_\phi < \theta$ then $\tilde{I}_\phi f(x)$ converges absolutely for all x and there is a constant $C > 0$, independent of f, such that

$$|\tilde{I}_\phi f|_{\Lambda^{\xi\phi}} \le C|f|_{\Lambda^\xi}.$$

Moreover, if $f \in \Lambda^\xi \cap L^1$, then $\tilde{I}_\phi f$ coincides with $I_\phi f$ as an element of $\Lambda^{\xi\phi}$ (since $\tilde{I}_\phi f(x) = I_\phi f(x) - I_\phi f(x_0)$).

From the proof of the above theorems the following results are obtained:

Remark 4.5. *If ϕ is of upper type s_ϕ, ξ is of upper type β and $\beta + s_\phi < \theta$ then I_ϕ maps $\Lambda^\xi \cap L^1 \cap L^\infty$ in $\Lambda^{\xi\phi} \cap L^\infty$ and $\|I_\phi f\|_{\Lambda^{\xi\phi}} \le C(\|f\|_\xi + \|f\|_1)$.*

Remark 4.6. *If $f \in \Lambda_0^\beta$ and $\beta + i_\phi < \theta$ then $I_\phi f \in \Lambda^{\beta + i_\phi} \cap L^\infty$ and $\|I_\phi f\|_{\beta + i_\phi} \le C_{\mu(\operatorname{supp} f)}\|f\|_\beta$. It then follows that I_ϕ is a linear continuous operator from Λ_0^β onto $(\Lambda_0^\beta)'$.*

In an analogous way to the integral operator, a representation of the kernel of D_ϕ in terms of an adequate quasi-metric, size and smoothness properties on the kernel and boundedness of the derivative operator on Lipschitz spaces, proved in [HV2], are given below.

Let ϕ be a quasi-increasing function of finite upper type and consider a fixed function $\hat{\phi}$, as given in Corollary 4.2. Hence we have

$$K_{1/\phi}(x,y) = \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} \quad \text{for} \quad x \neq y,$$

where $\delta_{1/\phi}(x,y)$ is defined as the unique solution of the equation

$$\frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} = \int_0^\infty \frac{1}{\phi(t)t} s_t(x,y)\,dt \quad \text{if} \quad x \neq y, \quad \text{and}$$
$$\delta_{1/\phi}(x,y) = 0 \quad \text{if} \quad x = y.$$

If $\phi(t) = t^\alpha$, $0 < \alpha < 1$, choosing $\hat{\phi} = \phi$ it turns out that $\delta_{-\alpha} := \delta_{t^{-\alpha}}$ is the quasi-metric associated to D_α defined in [GSV].

The next lemma shows the equivalence between $K_{1/\phi}(x,y)$ and $1/(\phi(\delta(x,y))\delta(x,y))$.

Lemma 4.7 ([HV2])**.** *If ϕ is a quasi-increasing function of finite upper type then there are positive constants C_1 and C_2 such that*

$$C_1 \frac{1}{\phi(\delta(x,y))\delta(x,y)} \le \frac{1}{\hat{\phi}(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)} \le C_2 \frac{1}{\phi(\delta(x,y))\delta(x,y)}\,.$$

In particular,

$$0 < K_{1/\phi}(x,y) \le C \frac{1}{\phi(\delta(x,y))\delta(x,y)}\,. \tag{4.11}$$

Moreover, $\delta_{1/\phi}$ is a quasi-metric equivalent to δ.

Lemma 4.8. *If ϕ is a quasi-increasing function of finite upper type then*

$$\begin{aligned} &|K_{1/\phi}(x,y)-K_{1/\phi}(x',y)|+|K_{1/\phi}(y,x)-K_{1/\phi}(y,x')| \\ &\qquad \le C\left(\frac{\delta(x,x')}{\delta(x,y)}\right)^{\theta}\frac{1}{\phi(\delta(x,y))\delta(x,y)} \end{aligned} \tag{4.12}$$

for $\delta(x,y)\ge 2A\delta(x,x')$.

Theorem 4.9 ([HV2])**.** *Let ϕ be a function of lower type $i_\phi>0$ and upper type s_ϕ. Let also ξ be a quasi-increasing function of lower type α and upper type β. If $f\in\Lambda^\xi\cap L^\infty$ and $s_\phi<\alpha$ then $D_\phi f(x)$ is absolutely convergent for every $x\in X$ and if, also, $\beta<\theta+i_\phi$ then*

$$\|D_\phi f\|_{\xi/\phi}\le C\|f\|_\xi.$$

If $f\in\Lambda^\xi$, $s_\phi<\alpha$ and $\beta<\theta+i_\phi$ then $\tilde{D}_\phi f(x)$ is absolutely convergent for every $x\in X$ and

$$|\tilde{D}_\phi f|_{\xi/\phi}\le C|f|_\xi.$$

Moreover, if $f\in\Lambda^\xi\cap L^\infty$, then $\tilde{D}_\phi f$ coincides with $D_\phi f$ as an element of Λ^ξ (since $\tilde{D}_\phi f(x)=D_\phi f(x)-D_\phi f(x_0)$).

Remark 4.10. *Let ξ_i be a function of lower type α_i and upper type β_i for $i=1,2$ and let $s_\phi<\alpha_1$. Then*

$$\langle D_\phi f,g\rangle=\iint K_{1/\phi}(x,y)(f(y)-f(x))g(x)\,d\mu(x)\,d\mu(y),$$

for any $f\in\Lambda^{\xi_1}\cap L^\infty$ and $g\in L^1$.

Furthermore, if $f\in\Lambda^{\xi_1}\cap L^\infty\cap L^1$, $g\in\Lambda^{\xi_2}\cap L^\infty\cap L^1$, and $s_\phi<\alpha_2$ then

$$\langle D_\phi f,g\rangle=\langle D_\phi g,f\rangle.$$

5. Proof of Theorem 3.1

Let us first see that T_ϕ is a linear continuous operator, $T_\phi:\Lambda_0^\beta\to(\Lambda_0^\beta)'$, for every β such that $s_\phi-i_\phi<\beta<\theta-i_\phi$. In fact, by Remark 4.6, I_ϕ is continuous from Λ_0^β to $\Lambda^{\beta+i_\phi}\cap L^\infty$ for $\beta<\theta-i_\phi$ and, by Remark 4.10, D_ϕ is continuous from $\Lambda^{\beta+i_\phi}\cap L^\infty$ to $(\Lambda_0^\beta)'$, if $s_\phi-i_\phi<\beta$.

Let us remark that whenever the size of either K_ϕ or $K_{1/\phi}$ are involved in the following proofs, inequalities (4.8) and (4.11) will be used without being explicitly mentioned.

To prove that

$$|K(x,y)| \leq \frac{C}{\delta(x,y)} \quad \text{for} \quad x \neq y, \tag{5.13}$$

we consider the following partition of X

$$D_1 = \{z : \delta(x,z) \geq 2A\delta(x,y)\},$$
$$D_2 = \{z : \frac{1}{2A}\delta(x,y) < \delta(x,z) < 2A\delta(x,y)\},$$
$$D_3 = \{\delta(x,z) \leq \frac{1}{2A}\delta(x,y)\}.$$

First notice that if $z \in D_1$ then $\delta(z,y) > \delta(x,y)$. Therefore, from $\phi(t)/t$ quasi-decreasing and (4.7), since $i_\phi > 0$, it follows that

$$\begin{aligned}
&\int_{D_1} K_{1/\phi}(x,z)|K_\phi(z,y) - K_\phi(x,y)|\, d\mu(z) \\
&\quad \leq C \int_{D_1} \frac{1}{\phi(\delta(x,z))\delta(x,z)} \left(\frac{\phi(\delta(z,y))}{\delta(z,y)} + \frac{\phi(\delta(x,y))}{\delta(x,y)}\right) d\mu(z) \\
&\quad \leq C \frac{\phi(\delta(x,y))}{\delta(x,y)} \int_{\delta(x,z)\geq 2A\delta(x,y)} \frac{1}{\phi(\delta(x,z))\delta(x,z)}\, d\mu(z) \\
&\quad \leq C \frac{1}{\delta(x,y)}\,.
\end{aligned}$$

Secondly, if $z \in D_2$ then $\delta(z,y) \leq A(\delta(z,x) + \delta(x,y)) < 4A^2\delta(x,y)$, and, from (4.6) it follows that

$$\begin{aligned}
&\int_{D_2} K_{1/\phi}(x,z)|K_\phi(z,y) - K_\phi(x,y)|\, d\mu(z) \\
&\quad \leq 2C \frac{1}{\phi(\delta(x,y))\delta(x,y)} \int_{\delta(z,y)<4A^2\delta(x,y)} \frac{\phi(\delta(z,y))}{\delta(z,y)}\, d\mu(z) \\
&\quad \leq C \frac{1}{\delta(x,y)}\,.
\end{aligned}$$

Finally, if $z \in D_3$, use Lemma 4.2 and (4.6), since $s_\phi < \theta$, to get

$$\begin{aligned}
&\int_{D_3} K_{1/\phi}(x,z)|K_\phi(z,y) - K_\phi(x,y)|\, d\mu(z) \\
&\quad \leq C \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}} \int_{\delta(x,z)\leq \frac{1}{2A}\delta(x,y)} \frac{\delta(z,x)^\theta}{\phi(\delta(x,z))\delta(x,z)}\, d\mu(z) \\
&\quad \leq C \frac{1}{\delta(x,y)}\,.
\end{aligned}$$

The proof of (5.13) is thus finished.

It will now be shown that $T_\phi = D_\phi \circ I_\phi$ has K as associated kernel.

Let f and $g \in \Lambda_0^\beta$ have disjoint supports. Then

$$\begin{aligned} D_\phi \circ I_\phi f(x) &= \int K_{1/\phi}(x,z)(I_\phi f(z) - I_\phi f(x))\, d\mu(z) \\ &= \int K_{1/\phi}(x,z) \int (K_\phi(z,y) - K_\phi(x,y)) f(y)\, d\mu(y)\, d\mu(z). \end{aligned}$$

If $x \notin \operatorname{supp} f$ then using (5.13), this last integral is absolutely convergent. Applying Fubini's theorem it follows that

$$\begin{aligned} D_\phi \circ I_\phi f(x) &= \int \left(\int K_{1/\phi}(x,z)(K_\phi(z,y) - K_\phi(x,y))\, d\mu(z) \right) f(y)\, d\mu(y) \\ &= \int K(x,y) f(y)\, d\mu(y). \end{aligned}$$

Moreover, if $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ then $\int |K(x,y)||f(y)|\, d\mu(y)$ is bounded for $x \in \operatorname{supp} g$, and therefore

$$\begin{aligned} \langle T_\phi f, g \rangle &= \int_X T_\phi f(x) g(x)\, d\mu(x) \\ &= \iint K(x,y) f(y) g(x)\, d\mu(y)\, d\mu(x). \end{aligned}$$

We will now prove that there are constants $C > 0$, $\nu > 1$ and $0 < \gamma < 1$, such that

$$|K(x,y) - K(x',y)| \le C \frac{\delta(x,x')^\gamma}{\delta(x,y)^{1+\gamma}}, \quad \text{if} \quad \delta(x,y) > \nu\delta(x,x'). \tag{5.14}$$

Notice that

$$\begin{aligned} &|K(x,y) - K(x',y)| \\ &\le \int \left| K_{1/\phi}(x,z)(K_\phi(z,y) - K_\phi(x,y)) - K_{1/\phi}(x',z)(K_\phi(z,y) - K_\phi(x',y)) \right| d\mu(z). \end{aligned} \tag{5.15}$$

Denoting by $h(z)$ the function inside the above integral, choosing k and ν such that $2 \le 3A^2 < k < \frac{\nu}{2A}$, and setting

$$\delta(x,y) > \nu\delta(x,x'), \tag{5.16}$$

we consider the partition of X defined by $A = \{z : \delta(x,z) > \frac{1}{k}\delta(x,y)\}$, and its complement A^c. To obtain a bound for the integral on the set A we display $h(z)$ in the form

$$\begin{aligned} h(z) &= (K_{1/\phi}(x,z) - K_{1/\phi}(x',z))K_\phi(z,y) \\ &\quad + K_{1/\phi}(x,z)(K_\phi(x',y) - K_\phi(x,y)) \\ &\quad + K_\phi(x',y)(K_{1/\phi}(x',z) - K_{1/\phi}(x,z)) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Notice that if $z \in A$ then, by (5.16), it holds that $\delta(x,z) > \frac{1}{k}\delta(x,y) > \frac{\nu}{k}\delta(x,x')$. Now, from (4.7) — since ϕ is quasi-increasing — it turns out that

$$\begin{aligned} \int_A |I_3|\, d\mu(z) &\le C\frac{\phi(\delta(x',y))}{\delta(x',y)}\delta(x,x')^\theta \int_{\delta(x,z)>\frac{1}{k}\delta(x,y)} \frac{1}{\phi(\delta(x,z))\delta(x,z)^{1+\theta}}\, d\mu(z) \\ &\le C\frac{\phi(\delta(x',y))}{\delta(x',y)} \frac{\delta(x,x')^\theta}{\phi(\delta(x,y))\delta(x,y)^\theta}\,. \end{aligned}$$

Nevertheless, from (5.16) it holds that $\delta(x,y) \le A(\delta(x,x')+\delta(x',y)) \le \frac{A}{\nu}\delta(x,y)+A\delta(x',y)$ and, as $\nu > A$, $\delta(x',y) > (\frac{1}{A} - \frac{1}{\nu})\delta(x,y) > C\delta(x,y)$, with $C > 0$. Moreover, since $\phi(t)/t$ is quasi-decreasing then by (4.7)

$$\int_A |I_3|\, d\mu(z) \le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\,. \tag{5.17}$$

On the other hand, using (5.16) and (4.7) — since ϕ is of positive lower type — it follows that

$$\begin{aligned} \int_A |I_2|\, d\mu(z) &\le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\phi(\delta(x,y)) \int_{\delta(x,z)>\frac{1}{k}\delta(x,y)} \frac{d\mu(z)}{\phi(\delta(x,z))\delta(x,z)} \\ &\le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\,. \end{aligned} \tag{5.18}$$

Finally to obtain a bound for $\int_A |I_1|$, the following partition of A is considered

$$\begin{aligned} D_1 &= \{z : \delta(x,z) > k\delta(x,y)\}, \\ D_2 &= \Big\{z : \frac{1}{k}\delta(x,y) < \delta(x,z) \le k\delta(x,y)\Big\}. \end{aligned}$$

First notice that if $z \in D_1$ and (5.16) holds then $\delta(x,z) > k\delta(x,y) > \nu k\delta(x,x')$ and $\nu k > 2A$.

Therefore, use (4.12) to get

$$\int_{D_1} |I_1|\, d\mu(z) \le C\delta(x,x')^\theta \int_{D_1} \frac{1}{\phi(\delta(x,z))\delta(x,z)^{1+\theta}} \frac{\phi(\delta(z,y))}{\delta(z,y)}\, d\mu(z),$$

but for $z \in D_1$ it also holds that $\delta(x,z) \le A(\delta(x,y)+\delta(y,z)) \le A(\frac{1}{k}\delta(x,z) + \delta(y,z))$, and then $\delta(y,z) > (\frac{1}{A} - \frac{1}{k})\delta(x,z)$, with $1/A - 1/k > 0$. Since $\phi(t)/t$ is quasi-decreasing, we have

$$\begin{aligned}\int_{D_1} |I_1|\, d\mu(z) &\le C\delta(x,x')^\theta \int_{\delta(x,z)>k\delta(x,y)} \frac{1}{\delta(x,z)^{2+\theta}}\, d\mu(z)\\ &\le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\,.\end{aligned} \tag{5.19}$$

On the other hand, if $z \in D_2$ and (5.16) holds then $\nu\delta(x,x') < \delta(x,y) < k\delta(x,z)$. Therefore,

$$\int_{D_2} |I_1|\, d\mu(z) \le \delta(x,x')^\theta \int_{D_2} \frac{1}{\delta(x,z)^{1+\theta}\phi(\delta(x,z))} \frac{\phi(\delta(z,y))}{\delta(z,y)}\, d\mu(z).$$

Nevertheless, for $z \in D_2$ it also holds that $\delta(z,y) \le A(\delta(x,z)+\delta(x,y)) \le A(k+1)\delta(x,y)$, and $\delta(x,z) > \frac{1}{k}\delta(x,y)$. Therefore,

$$\begin{aligned}\int_{D_2} |I_1|\, d\mu(z) &\le C\frac{\delta(x,x')^\theta}{\phi(\delta(x,y))\delta(x,y)^{1+\theta}} \int_{\delta(z,y)\le C\delta(x,y)} \frac{\phi(\delta(z,y))}{\delta(z,y)}\, d\mu(z)\\ &\le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\,.\end{aligned} \tag{5.20}$$

We conclude from (5.19) and (5.20) that

$$\int_A |I_1|\, d\mu(z) \le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\,, \tag{5.21}$$

and, (5.17), (5.18) and (5.21) imply

$$\int_A |h(z)|\, d\mu(z) \le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\,. \tag{5.22}$$

To bound $\int_{A^c}$, we consider the following partition of $A^c = \{z : \frac{1}{k}\delta(x,y) \ge \delta(x,z)\}$,

$$\begin{aligned}B_1 &= \{z : \delta(x,z) \le \nu/k\delta(x,x')\},\\ B_2 &= \Big\{z : \nu/k\delta(x,x') \le \delta(x,z) \le \frac{1}{k}\delta(x,y)\Big\}.\end{aligned}$$

Firstly notice that

$$\begin{aligned}
&\int_{B_1} |h(z)|\, d\mu(z) \\
&\quad \le \int_{B_1} \frac{1}{\phi(\delta(x,z))\delta(x,z)} |K_\phi(z,y) - K_\phi(x,y)|\, d\mu(z) \\
&\quad\quad + \int_{B_1} \frac{1}{\phi(\delta(x',z))\delta(x',z)} |K_\phi(z,y) - K_\phi(x',y)|\, d\mu(z) \\
&\quad = F_1 + F_2.
\end{aligned}$$

Nevertheless, if $z \in A^c$ and $\delta(x,y) > \nu\delta(x,x')$ then $\delta(x,y) \ge k\delta(x,z)$ and it also holds that

$$\delta(x',y) \ge C\delta(x',z), \tag{5.23}$$

with $C > 1$. Indeed, by (5.16), it holds that

$$\delta(x,y) \le A(\delta(x,x') + \delta(x',y)) \le A(\nu^{-1}\delta(x,y) + \delta(x',y)),$$

and, since $A < \nu$, then

$$\delta(x,y) \le \frac{\nu A}{\nu - A}\delta(x',y). \tag{5.24}$$

Therefore, for $z \in A^c$ and $\delta(x,y) > \nu\delta(x,x')$ it holds that

$$\begin{aligned}
\delta(x',z) &\le A(\delta(x,x') + \delta(x,z)) \\
&\le A(\nu^{-1} + k^{-1})\delta(x,y) \le \frac{A(1/\nu + 1/k)}{1/A - 1/\nu}\delta(x',y);
\end{aligned} \tag{5.25}$$

and since $A(1/\nu+1/k)/(1/A-1/\nu) < 1$, (5.23) is now clear. On A^c, the smoothness condition on K_ϕ can be used to get

$$\begin{aligned}
F_1 &\le C\frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}} \int_{\delta(x,z)<\frac{\nu}{k}\delta(x,x')} \frac{1}{\phi(\delta(x,z))\delta(x,z)} \delta(x,z)^\theta\, d\mu(z) \\
&\le C\frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}} \frac{\delta(x,x')^\theta}{\phi(\delta(x,x'))}\,.
\end{aligned}$$

Moreover, by (5.16) and (4.4), it holds that

$$F_1 \le C\frac{\delta(x,x')^{\theta - s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,. \tag{5.26}$$

On the other hand, from (5.25) it follows that

$$F_2 \leq C\frac{\phi(\delta(x',y))}{\delta(x',y)^{1+\theta}}\int_{\delta(x,z)<\nu/k\delta(x,x')}\frac{1}{\phi(\delta(x',z))\delta(x',z)^{1+\theta}}\,d\mu(z);$$

but, for $z \in B_1$, $\delta(x',z) \leq A(\delta(x',x)+\delta(x,z)) < A(1+\nu/k)\delta(x,x')$ holds and then,

$$F_2 \leq C\frac{\phi(\delta(x',y))}{\delta(x',y)^{1+\theta}}\frac{\delta(x,x')^{\theta}}{\phi(\delta(x,x'))}\,.$$

Nevertheless, from (5.24) and (5.16), we get that $\delta(x',y) > \frac{\nu-A}{A}\delta(x,x')$, and from (4.4) and, again (5.24), it follows that

$$F_2 \leq C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x',y)^{1+\theta-s_\phi}} \leq C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,. \tag{5.27}$$

We then conclude from (5.26) and (5.27) that

$$\int_{B_1}|h(z)|\,d\mu(z) \leq C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,. \tag{5.28}$$

On the other hand reordering $h(z)$ in (5.15), we get

$$\begin{aligned}
&\int_{B_2}|h(z)|\,d\mu(z)\\
&\quad\leq \int_{B_2}|K_{1/\phi}(x,z)-K_{1/\phi}(x',z)||K_\phi(z,y)-K_\phi(x,y)|\,d\mu(z)\\
&\qquad+\int_{B_2}\frac{1}{\phi(\delta(x',z))\delta(x',z)}|K_\phi(x',y)-K_\phi(x,y)|\,d\mu(z)\\
&\quad= J_1+J_2.
\end{aligned}$$

Using the smoothness conditions on both kernels, K_ϕ and $K_{1/\phi}$, and (5.16) we obtain that

$$\begin{aligned}
J_1 &\leq C\frac{\delta(x,x')^{\theta}}{\delta(x,y)^{1+\theta}}\phi(\delta(x,y))\int_{\frac{\nu}{k}\delta_{1/\phi}(x,x')\leq\delta_{1/\phi}(x,z)}\frac{1}{\phi(\delta(x,z))\delta(x,z)}\,d\mu(z)\\
&\leq C\frac{\delta(x,x')^{\theta}}{\delta(x,y)^{1+\theta}}\frac{\phi(\delta(x,y))}{\phi(\delta(x,x'))}\\
&\leq C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,.
\end{aligned} \tag{5.29}$$

On the other hand, since $\delta(x,x') \leq \frac{1}{\nu}\delta(x,y)$, we have

$$J_2 \leq \frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\phi(\delta(x,y)) \int_{B_2} \frac{1}{\phi(\delta(x',z))\delta(x',z)}\, d\mu(z),$$

but $\frac{\nu}{k}\delta(x,x') \leq \delta(x,z) \leq A(\delta(x',z)+\delta(x,x'))$ and, therefore, $\delta(x',z) \geq \frac{1}{A(\frac{\nu}{k}-A)}\delta(x,x')$. We then conclude that

$$\begin{aligned} J_2 &\leq C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\phi(\delta(x,y)) \int_{\delta(x',z)\geq C\delta(x,x')} \frac{1}{\phi(\delta(x',z))\delta(x',z)}\, d\mu(z) \\ &\leq C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}} \frac{\phi(\delta(x,y))}{\phi(\delta(x,x'))} \\ &\leq C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,. \end{aligned} \tag{5.30}$$

By (5.29) and (5.30), we have proved that

$$\int_{B_2} |h(z)|\, d\mu(z) \leq C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,; \tag{5.31}$$

and, by (5.28) and (5.31), we have got that

$$\int_{A^c} |h(z)|\, d\mu(z) \leq C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,. \tag{5.32}$$

From (5.22) and (5.32), choosing $\gamma = \theta - s_\phi$, inequality (5.14) is obtained.

It remains to prove that there are constants $C' > 0$, $\nu' > 1$ and $0 < \gamma' < 1$, such that

$$|K(y,x) - K(y,x')| \leq C\frac{\delta(x,x')^{\gamma'}}{\delta(x,y)^{1+\gamma'}} \quad \text{if} \quad \delta(x,y) > \nu'\delta(x,x'). \tag{5.33}$$

Notice that if

$$\delta(x,y) > 2A\delta(x,x') \tag{5.34}$$

holds then $\delta(x',y) \leq (A+1/2)\delta(x,y)$. We may thus consider the partition of X in the family of sets

$$\begin{aligned} A &= \{z : \delta(y,z) < \frac{1}{2A}\min(\delta(x',y),\delta(x,y))\} \\ B &= \Big\{z : \frac{1}{2A}\min(\delta(x',y),\delta(x,y)) \leq \delta(z,y) < 2A\delta(x,y)\Big\}, \\ C &= \{z : 2A\delta(x,y) \leq \delta(z,y)\}. \end{aligned}$$

Moreover, from (5.34) it follows that

$$\delta(x,x') < \frac{1}{2A}\delta(x,y) < \delta(x',y), \tag{5.35}$$

and thus $\delta(x,x') < \min(\delta(x,y),\delta(x',y))$. Therefore, the set A may be partitioned into the nonempty sets

$$\begin{aligned} A_1 &= \Big\{z : \delta(y,z) \le \frac{1}{2A}\delta(x,x')\Big\},\\ A_2 &= \Big\{z : \frac{1}{2A}\delta(x,x') \le \delta(y,z) < \frac{1}{2A}\min(\delta(x',y),\delta(x,y))\Big\}. \end{aligned}$$

On the other hand, notice that the left side of (5.3) is

$$\begin{aligned} &|K(y,x) - K(y,x')|\\ &\le \int K_{1/\phi}(y,z)\,\big|(K_\phi(z,x) - K_\phi(y,x)) - (K_\phi(z,x') - K_\phi(y,x'))\big|\,d\mu(z). \end{aligned}$$

Denoting $g(z)$ the function inside the above integral, the smoothness estimate on K_ϕ, inequalities (4.6), since $s_\phi < \theta$, and (5.35), the fact that $\phi(t)/t^{1+\theta}$ is quasi-decreasing and, finally, (4.4) lead to the bound

$$\begin{aligned} &\int_{A_1} g(z)\,d\mu(z)\\ &\quad\le \int_{A_1} \frac{1}{\phi(\delta(y,z))\delta(y,z)}|K_\phi(z,x) - K_\phi(y,x)|\,d\mu(z)\\ &\quad\quad + \int_{A_1} \frac{1}{\phi(\delta(y,z))\delta(y,z)}|K_\phi(z,x') - K_\phi(y,x')|\,d\mu(z)\\ &\quad\le C\left(\frac{\phi(\delta(y,x))}{\delta(y,x)^{1+\theta}} + \frac{\phi(\delta(y,x'))}{\delta(y,x')^{1+\theta}}\right)\times\\ &\quad\quad \times \int_{\delta(y,z)\le\frac{1}{2A}\delta(x,x')} \frac{\delta(y,z)^\theta}{\phi(\delta(y,z))\delta(y,z)}\,d\mu(z)\\ &\quad\le C\frac{\phi(\delta(y,x))}{\delta(y,x)^{1+\theta}}\frac{\delta(x,x')^\theta}{\phi(\delta(x,x'))}\\ &\quad\le C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}\,. \end{aligned} \tag{5.36}$$

We now reorder $g(z)$ to write

$$\begin{aligned}\int_{A_2} & g(z)\,d\mu(z) \\ &\le \int_{A_2} \frac{1}{\phi(\delta(y,z))\delta(y,z)} |K_\phi(z,x) - K_\phi(z,x')|\,d\mu(z) \\ &\quad + \int_{A_2} \frac{1}{\phi(\delta(y,z))\delta(y,z)} |K_\phi(y,x') - K_\phi(y,x)|\,d\mu(z) \\ &= H_1 + H_2.\end{aligned} \tag{5.37}$$

Nevertheless, for $z \in A_2$, $\delta(x,y) \le A(\delta(x,z) + \delta(y,z)) \le A(\delta(x,z) + \frac{1}{2A}\delta(x,y))$ holds, and then $\delta(x,y) \le \frac{1}{2A}\delta(x,z)$. Therefore, from the fact that $\phi(t)/(t^{1+\theta})$ is quasi-decreasing, (5.34) and (4.4) it follows that

$$\begin{aligned}H_1 &\le C\delta(x,x')^\theta \int_{\delta(y,z)\ge\frac{1}{2A}\delta(x,x')} \frac{1}{\phi(\delta(y,z))\delta(y,z)} \frac{\phi(\delta(x,z))}{\delta(x,z)^{1+\theta}}\,d\mu(z) \\ &\le C\delta(x,x')^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}} \int_{\delta(y,z)\ge\frac{1}{2A}\delta(x,x')} \frac{1}{\phi(\delta(y,z))\delta(y,z)}\,d\mu(z) \\ &\le C\delta(x,x')^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}} \frac{1}{\phi(\delta(x,x'))} \\ &\le C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}.\end{aligned} \tag{5.38}$$

Similarly

$$\begin{aligned}H_2 &\le C\delta(x,x')^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}} \int_{\delta(y,z)\ge\frac{1}{2A}\delta(x,x')} \frac{1}{\phi(\delta(y,z))\delta(y,z)}\,d\mu(z) \\ &\le C\delta(x,x')^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}} \frac{1}{\phi(\delta(x,x'))} \\ &\le C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}.\end{aligned} \tag{5.39}$$

Thus, (5.36), (5.37), (5.38) and (5.39) give

$$\int_A g(z)\,d\mu(z) \le C\frac{\delta(x,x')^{\theta-s_\phi}}{\delta(x,y)^{1+\theta-s_\phi}}. \tag{5.40}$$

On the other hand,

$$\begin{aligned}
&\int_B g(z)\,d\mu(z)\\
&\quad\le \int_B \frac{1}{\phi(\delta(y,z))\delta(y,z)} |K_\phi(z,x)-K_\phi(z,x')|\,d\mu(z)\\
&\qquad+\int_B \frac{1}{\phi(\delta(y,z))\delta(y,z)} |K_\phi(y,x')-K_\phi(y,x)|\,d\mu(z)\\
&\quad= G_1+G_2.
\end{aligned}$$

From (5.34), it follows that

$$G_2 \le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\phi(\delta(x,y))\int_{\delta(y,z)\ge\frac{1}{2A}\min(\delta(x',y),\delta(x,y))} \frac{1}{\phi(\delta(y,z))\delta(y,z)}\,d\mu(z).$$

But from (5.35), for $z\in B$ we have

$$\delta(y,z)\ge \frac{1}{4A^2}\delta(x,y)=C\delta(x,y), \tag{5.41}$$

and thus,

$$\begin{aligned}
G_2 &\le C\delta(x,x')^\theta \frac{\phi(\delta(x,y))}{\delta(x,y)^{1+\theta}}\int_{\delta(z,y)\ge C\delta(x,y)} \frac{1}{\phi(\delta(y,z))\delta(y,z)}\,d\mu(z)\\
&\le C\frac{\delta(x,x')^\theta}{\delta(x,y)^{1+\theta}}\,.
\end{aligned}\tag{5.42}$$

To get a bound for G_1, we first notice that from (5.34) it follows that $\delta(y,x)$ and $\delta(y,x')$ are equivalent, since (5.35) holds and, also, $\delta(y,x')\le A(\delta(y,x)+\delta(x,x'))\le (A+\frac{1}{2})\delta(y,x)$.

We now cut the set B in

$$\begin{aligned}
&D_1 = B\cap\{z:\delta(z,x)<4A^2\delta(x,x')\},\\
\text{and}\quad &D_2 = B\cap\{z:\delta(z,x)\ge 4A^2\delta(x,x')\},
\end{aligned}$$

and thus we write

$$G_1\le\left(\int_{D_1}+\int_{D_2}\right)\frac{1}{\phi(\delta(y,z))\delta(y,z)}|K_\phi(z,x)-K_\phi(z,x')|\,d\mu(z)=G_{11}+G_{12}.$$

From (5.41), $1/(\phi(t)t)$ quasi-decreasing, (4.6), as $i_\phi > 0$, and since for $z \in D_1$ it holds that $\delta(z,x') \le A(\delta(z,x)+\delta(x,x')) \le A(4A^2+1)\delta(x,x')$, it follows that

$$\begin{aligned}
G_{11} &\le C \int_{\delta(z,x)<4A^2\delta(x,x')} \frac{1}{\phi(\delta(y,z))\delta(y,z)} \left(\frac{\phi(\delta(z,x))}{\delta(z,x)} + \frac{\phi(\delta(z,x'))}{\delta(z,x')} \right) d\mu(z) \\
&\le C \frac{1}{\phi(\delta(y,x))\delta(y,x)} \Big(\int_{\delta(z,x)<4A^2\delta(x,x')} \frac{\phi(\delta(z,x))}{\delta(z,x)}\, d\mu(z) \\
&\quad + \int_{\delta(z,x')<A(4A^2+1)\delta(x,x')} \frac{\phi(\delta(z,x'))}{\delta(z,x')}\, d\mu(z) \Big) \\
&\le C \frac{1}{\phi(\delta(y,x))\delta(y,x)} \phi(\delta(x,x')).
\end{aligned}$$

Furthermore, from (5.34) and (2.1) it follows that

$$G_{11} \le C \frac{\delta(x,x')^{i_\phi}}{\delta(y,x)^{i_\phi+1}}\,. \tag{5.43}$$

On the other hand, (5.41) and (2.1) lead to

$$\begin{aligned}
G_{12} &\le \delta(x,x')^\theta \int_{\delta(z,x)\ge 4A^2\delta(x,x')} \frac{1}{\phi(\delta(y,z))\delta(y,z)} \frac{\phi(\delta(z,x))}{\delta(z,x)^{1+\theta}}\, d\mu(z) \\
&\le C \frac{\delta(x,x')^\theta}{\phi(\delta(y,x))\delta(y,x)} \int_{\delta(z,x)\ge 4A^2\delta(x,x')} \frac{\phi(\delta(z,x))}{\delta(z,x)^{1+\theta}}\, d\mu(z) \\
&\le C \frac{1}{\phi(\delta(y,x))\delta(y,x)} \phi(\delta(x,x')) \\
&\le C \frac{\delta(x,x')^{i_\phi}}{\delta(x,y)^{i_\phi+1}}\,.
\end{aligned} \tag{5.44}$$

Thus, looking at (5.42), (5.43) and (5.44), and since $i_\phi < \theta$, we conclude that

$$\int_B g(z)\, d\mu(z) \le C \frac{\delta(x,x')^{i_\phi}}{\delta(x,y)^{i_\phi+1}}\,. \tag{5.45}$$

At last, to get a bound on the set C we write

$$\begin{aligned}
&\int_C g(z)\, d\mu(z) \\
&\quad \le \int_{\delta(y,z)\ge 2A\delta(y,x)} \frac{1}{\phi(\delta(y,z))\delta(y,z)} |K_\phi(z,x) - K_\phi(z,x')|\, d\mu(z) \\
&\quad\quad + \int_{\delta(y,z)\ge 2A\delta(y,x)} \frac{1}{\phi(\delta(y,z))\delta(y,z)} |K_\phi(y,x') - K_\phi(y,x)|\, d\mu(z) \\
&\quad = J_1 + J_2.
\end{aligned} \tag{5.46}$$

Notice that for $z \in C$ it holds that $\delta(y,x) \leq \frac{1}{2A}\delta(y,z) \leq \frac{1}{2}(\delta(y,x)+\delta(x,z))$, hence $\delta(y,x) \leq \delta(x,z)$, and, from (5.34), it follows that $\delta(x,z) \geq 2A\delta(x,x')$. Furthermore, since $1/\phi(t)t$ is quasi-decreasing, we have

$$
\begin{aligned}
J_1 &\leq \delta(x,x')^\theta \int_{\delta(y,z)\geq 2A\delta(y,x)} \frac{1}{\phi(\delta(y,z))\delta(y,z)} \frac{\phi(\delta(z,x))}{\delta(z,x)^{1+\theta}}\, d\mu(z) \\
&\leq C\frac{\delta(x,x')^\theta}{\phi(\delta(x,y))\delta(x,y)} \int_{\delta(x,z)\geq\delta(y,x)} \frac{\phi(\delta(z,x))}{\delta(z,x)^{1+\theta}}\, d\mu(z) \qquad (5.47)\\
&\leq C\frac{\delta(x,x')^\theta}{\delta(y,x)^{1+\theta}}\,.
\end{aligned}
$$

Finally, from (5.34) we deduce that

$$
\begin{aligned}
J_2 &\leq \frac{\delta(x,x')^\theta}{\delta(y,x)^{1+\theta}}\phi(\delta(y,x)) \int_{\delta(y,z)\geq 2A\delta(y,x)} \frac{1}{\phi(\delta(y,z))\delta(y,z)}\, d\mu(z) \qquad (5.48)\\
&\leq C\frac{\delta(x,x')^\theta}{\delta(y,x)^{1+\theta}}\,.
\end{aligned}
$$

From (5.46), (5.47) and (5.48) we have got that

$$
\int_C g(z)\, d\mu(z) \leq C\frac{\delta(x,x')^\theta}{\delta(y,x)^{1+\theta}}\,. \tag{5.49}
$$

Nevertheless, since $0 < i_\phi < \theta$ and $\theta - s_\phi < \theta$, from (5.40), (5.45) and (5.49) it turns out that

$$
|K(y,x) - K(y,x')| \leq C\frac{\delta(x,x')^{\min(i_\phi,\theta-s_\phi)}}{\delta(y,x)^{1+\min(i_\phi,\theta-s_\phi)}}\,,
$$

for $\delta(x,y) > 2A\delta(x,x')$. The proof of this theorem is thus finished. $\square$

We remark that once the standard conditions of size and smoothness on the kernel of T_ϕ have been proved, the $T1$-theorems stated in [HV1] give an alternative proof of the fact that T_ϕ is a Calderón-Zygmund operator bounded on the generalized Besov and Triebel-Lizorkin spaces. In fact, it was proved in [H] that $T_\phi 1 = T_\phi^* 1 = 0$ and T_ϕ is a weakly bounded operator, that is, $|\langle T_\phi f, g\rangle| \leq C\|f\|_\beta \|g\|_\beta (\mu(B))^{1+2\beta}$, for f and $g \in \Lambda_0^\beta(B)$ and B a ball.

References

[GSV] Gatto A.E., Segovia C., Vági S., *On fractional differentiation and integration on spaces of homogeneous type*, Rev. Mat. Iberoamericana **12** (1996), no. 2, 111–145.

[H] Hartzstein S.I., *Acotación de operadores de Calderón-Zygmund en espacios de Triebel-Lizorkin y de Besov generalizados sobre espacios de tipo homogéneo*, Thesis, 2000, UNL, Santa Fe, Argentina.

[HV1] Hartzstein S.I., Viviani B.E., *$T1$ theorems on generalized Besov and Triebel-Lizorkin spaces over spaces of homogeneous type*, Revista de la Unión Matemática Argentina, **42** (2000), no. 1, 51–73.

[HV2] Hartzstein S.I., Viviani B.E., *Integral and derivative operators of functional order on generalized Besov and Triebel-Lizorkin spaces in the setting of spaces of homogeneous type*, Comment. Math. Univ. Carolinae **43** (2002), 723–754.

[MS] Macías R.A., Segovia C., *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), 257–270.

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