Moment sequences and abstract Cauchy problems

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Abstract. We give a new characterization of the solvability of an abstract Cauchy problems in terms of moment sequences, using the resolvent operator at only one point.

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Introduction

One idea to solve abstract Cauchy problems is to apply the Laplace transform to the equation $Au(t) + f(t) = u'(t)$, which leads to an equation where the unknown function only appears once[1]. Now various characterizations of Laplace transforms yield solvability conditions of the problem[2].

In this paper the same aim is reached by using the concept of moment sequences in Banach spaces.

We extend Widder's condition on scalar moment sequences to the Banach space case and give a characterization of finite Laplace transforms. Then we apply moment sequences to the abstract Cauchy problem. We deal with (almost) arbitrary inhomogenities, we only need the existence of the resolvent at one point and do not assume that $D(A)$ is dense in X.

It turns out that an abstract Cauchy problem has a "p-regular" solution if and only if a special sequence is a p-moment sequence. The notation "p-regular" will be precised in Theorems 2.2 and 2.4.

1. Finite Laplace transform and moment sequences

With $g^{[k]}$ we denote the k-th antiderivative of some integrable function g, and if $f : [0, \infty) \to X$ is some Banach space valued locally integrable and exponentially bounded function we let

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt \quad (\Re\, \lambda \text{ large})$$

[1] see [ArBatNeu], [NeubrBaeum94].

[2] see [Arendt87], [ArBatNeu], [ArMennKey94], [Cioranescu95], [Cio+Lumer94], [Cio+Lumer95], [Hieber91], [NeubrBaeum94], [Phillips54].

be the Laplace transform of f.

A sequence $(\mu_n)_{n\geq 0}$ in some Banach space X is called a p-moment sequence if there is some function $\psi \in V_p([0,1],X)$ of bounded p-variation such that

$$\mu_n = \int_0^1 t^n \, d\psi(t) \quad \text{for all} \quad n \geq 0.$$

In this case ψ is called the representing function of $(\mu_n)_n$.

We recall that if $p < \infty$ the p-variation of ψ is given by

$$\|\psi\|_{V_p} = \sup \left\{ \left(\sum_{k=0}^{n-1} \frac{\|\psi(t_{k+1}) - \psi(t_k)\|^p}{(t_{k+1} - t_k)^{p-1}} \right)^{1/p} \Big| \, n \in \mathbb{N},\, 0 \leq t_0 < t_1 \ldots t_n \leq 1 \right\},$$

and the ∞-variation of ψ by

$$\|\psi\|_{V_\infty} = \sup \left\{ \frac{\|\psi(s) - \psi(t)\|}{s-t} \Big| \, 0 \leq s < t \leq 1 \right\}.$$

A function ψ is of bounded p-semi variation if $x^*\psi$ if of bounded p-variation for all $x^* \in X^*$.

The space $(P)V_p([0,1],X)$ consists of all functions ψ of bounded p-(semi) variation with $\psi(0) = 0$.

Widder [13] gave a characterization of scalar moment sequences. His conditions are valid also in the Banach space case (see [9]).

Definition 1.1. Let $(\mu_n)_{n\geq 0}$ be a sequence in X.

(1) For $k \geq m \geq 0$ we define vectors $\lambda_{k,m}$ by

$$\lambda_{k,m} = (-1)^{k-m} \binom{k}{m} \sum_{r=0}^{k-m} (-1)^r \binom{k-m}{r} \mu_{k-r}.$$

(2) For $1 \leq p < \infty$ the B_p-norm of $(\mu_n)_n$ is given by

$$\|(\mu_n)_n\|_{B_p} = \sup_{k\geq 0} \left((k+1)^{p-1} \sum_{m=0}^{k} \|\lambda_{k,m}\|^p \right)^{1/p} \in [0,\infty].$$

(3) The B_∞-norm of $(\mu_n)_n$ is given by

$$\|(\mu_n)_n\|_{B_\infty} = \sup_{k\geq m\geq 0} (k+1)\|\lambda_{k,m}\| \in [0,\infty].$$

If the B_p-norm of (μ_n) is finite then it is called a B_p-sequence.

Definition 1.2. Let $I \subset \mathbb{R}$ be a closed interval with finite left end point and non empty interior, and let $1 \le q < \infty$ and $T : L_q(I, \mathbb{C}) \to X$ be a linear operator.

(1) The Dinculeanu-norm of T is given by

$$|||T||| = \sup_f \sum_{k=0}^{n-1} |c_k| \cdot \|T\mathbf{1}_{[a_k,b_k)}\| \in [0, \infty].$$

The supremum is taken over all functions $f = \sum_{k=0}^{n-1} c_k \cdot \mathbf{1}_{[a_k,b_k)}$ with $\|f\|_{L_q} \le 1$, where $n > 0$, $c_k \in \mathbb{C}$ and $[a_k, b_k) \subset I$.

(2) The order summing norm of T is given by

$$\|T\|_{\mathrm{os}} = \sup_{f_1,\dots,f_n} \sum_{k=1}^{n} \|Tf_k\| \in [0, \infty].$$

The supremum is taken over all real valued, positive functions $f_1, \dots, f_n : I \to \mathbb{R}$ with $\|\sum_k f_k\|_{L_q(I)} \le 1$, where $n > 0$.

We always have $|||T||| = \|T\|_{\mathrm{os}} \in [0, \infty]$, and the representing function[3] $\varphi \in PV_p(I, X)$ of T is of bounded p-variation if and only if $|||T||| < \infty$ or $\|T\|_{\mathrm{os}} < \infty$. In this case we have

$$\|\varphi\|_{V_p(I)} = |||T||| = \|T\|_{\mathrm{os}}.$$

Moreover, if $|||T||| < \infty$, there is some positive L_p-function $g : I \to \mathbb{R}$ with $\|g\|_{L_p} = |||T|||$ such that

$$\|Tf\| \le \int_I |f(t)| \cdot g(t)\, dt \quad \text{for all} \quad f \in L_q(I, \mathbb{C}). \tag{1}$$

On the other hand, if there is some $g \in L_p(I, \mathbb{R})$ such that (1) holds, then $|||T||| \le \|g\|_{L_p}$.

Definition 1.3. Let $T : Y \to X$ be a linear operator. T is called absolutely summing, if there is some $M \ge 0$ such that for any choice of finitely many $y_1, \dots, y_n \in Y$ we have

$$\sum_{k=1}^{n} \|Ty_k\| \le M \cdot \sup_{\|y^*\| \le 1} \sum_{k=1}^{n} |y^*(y_k)|\,. \tag{2}$$

The infimum over all $M \ge 0$ that fulfill (2) is called the absolutely summing norm $\|T\|_{\mathrm{as}}$ of T.

[3] i.e. $\varphi(t) = T\mathbf{1}_{[a,t]}$, where a is the left end point of I.

A bounded operator $T : C_0(I,\mathbb{C}) \to X$ is absolutely summing if and only if its representing function $\varphi \in PV_1(I, X^{**})$ takes its values in X and is of bounded variation. In this case we have

$$\|T\|_{\mathrm{as}} = \|\varphi\|_{V_1(I)}.$$

Moreover, if $T : C_0(I,\mathbb{C}) \to X$ is absolutely summing, there is some normalized function $\psi : I \to \mathbb{R}$ with $\|\psi\|_{V_1} = \|T\|_{\mathrm{as}}$ such that

$$\|Tf\| \leq \int_I |f(t)|\, d\psi(t) \quad \text{for all} \quad f \in C_0(I,\mathbb{C}). \tag{3}$$

On the other hand, if there is some $\psi \in V_1(I,\mathbb{R})$ such that (3) holds, then $\|T\|_{\mathrm{as}} \leq \|\psi\|_{V_1}$.

A proof of the previous assertions on order summing and absolutely summing operators can be found in [7], [12] or [9].

After this preliminaries we can state the announced characterizations of p-moment sequences and sketch the proofs.

Theorem 1.4. *Let $(\mu_n)_{n\geq 0}$ be a sequence in X and $1 < p < \infty$. Then the following assertions are equivalent:*

(1) *$(\mu_n)_n$ is a p-moment sequence;*
(2) *$(\mu_n)_n$ is a B_p-sequence;*
(3) *there is a linear operator $T : L_q([0,1],\mathbb{C}) \to X$ with bounded Dinculeanu-norm such that*

$$\mu_n = T(t^n) \quad \textit{for all } n \geq 0.$$

In this case we have

$$\|(\mu_n)_n\|_{B_p} = |||T||| = \|\varphi\|_{V_p[0,1]},$$

where φ is the representing function of $(\mu_n)_n$.

Proof: (2)$\Rightarrow$(1). With the same proof as in [13, Chapter III, Theorem 3] we see that

$$\mu_n = \lim_k \int_0^1 t^n L_{k,t}(\mu)\, dt \quad \text{for all} \quad n,$$

and with [13, Chapter III, Theorem 5] we see that

$$\left\| L_{k,\cdot}(\mu) \right\|_{L_p}^p \leq \frac{k+1}{k} \|\mu\|_{B_p}.$$

An argument concerning X-valued V_p-functions using the Banach Steinhaus Theorem and the reflexivity of $L_p([0,1],\mathbb{C})$ shows that there is some V_p-function $\varphi : [0,1] \to X$ with

$$\mu_n = \int_0^1 t^n \, d\varphi(t) \quad \text{for all} \quad n.$$

(1)$\Rightarrow$(3). It is not hard to see that the operator $T : L_q([0,1],\mathbb{C}) \to X$ given by $T\mathbf{1}_{[0,t)} := \varphi(t)$ has bounded Dinculeanu norm.

(3)$\Rightarrow$(2). The representing function $\varphi \in PV_p([0,1],X)$ is of bounded p-variation, and with

$$\lambda_{k,m}(t) := \binom{k}{m} t^m (1-t)^{k-m}$$

we obtain

$$\lambda_{k,m} = \int_0^1 \lambda_{k,m}(t) \, d\varphi(t).$$

Now the proof is the same as in [13, Chapter III, Theorem 5]. □

Theorem 1.5. *Let $(\mu_n)_{n \ge 0}$ be a sequence in X. Then the following assertions are equivalent:*

(1) *$(\mu_n)_n$ is a 1-moment sequence;*
(2) *$(\mu_n)_n$ is a B_1-sequence;*
(3) *there is an absolutely summing operator $T : C([0,1],\mathbb{C}) \to X$ such that*

$$\mu_n = T(t^n) \quad \textit{for all} \quad n \ge 0.$$

In this case we have

$$\|(\mu_n)_n\|_{B_1} = \|T\|_{\mathrm{as}} = \|\varphi\|_{V_1[0,1]},$$

where φ is the representing function of $(\mu_n)_n$.

Proof: (2)$\Rightarrow$(1). The only difference to the classical proof of Widder [13, Chapter III, Theorem 26] is that now we need a vector valued version of the Theorem of Helly ([13, Chapter I, Theorem 16.3]):

If $\varphi_n : [0,q] \to X$ is some uniformly bounded sequence of functions of bounded variation such that

$$\lim_n \int_0^1 t^k \, d\varphi_n(t)$$

exists for all $k \in \mathbb{N}_0$, then there is some function $\varphi : [0,1] \to X$ of bounded variation such that

$$\lim_n \int_0^1 f(t) \, d\varphi_n(t) = \int_0^1 f(t) \, d\varphi(t)$$

for all $f \in C([0,1],\mathbb{C})$.

This is proved with the Banach Steinhaus Theorem and the scalar version of Helly's Theorem.

(1)$\Rightarrow$(3). If $\mu_n = \int_0^1 t^n \, d\varphi(t)$, then it is not hard to see that $Tf := \int_0^1 f(t)\, d\varphi(t)$ is absolutely summing.

(3)$\Rightarrow$(2) If $\varphi \in V_1([0,1],X)$ is the representing function of T, then

$$\sum_{m=0}^{k} \|\lambda_{k,m}\| \le \sum_{m=0}^{k} \int_0^1 \lambda_{k,m}(t)\, d|\varphi|(t) \le \|\varphi\|_{V_1}.$$

□

Theorem 1.6. *Let $(\mu_n)_{n\ge 0}$ be a sequence in X. Then the following assertions are equivalent:*

(1) *$(\mu_n)_n$ is a ∞-moment sequence;*
(2) *$(\mu_n)_n$ is a B_∞-sequence;*
(3) *there is a bounded linear operator $T : L_1([0,1],\mathbb{C}) \to X$ such that*

$$\mu_n = T(t^n) \quad \textit{for all} \quad n \ge 0.$$

In this case we have

$$\|(\mu_n)_n\|_{B_\infty} = \|T\| = \|\varphi\|_{V_\infty[0,1]},$$

where φ is the representing function of $(\mu_n)_n$.

Proof: (2)$\Rightarrow$(3). From the classical Theorem [13, Chapter III, Theorem 6], it follows that for all $x^* \in X^*$

$$x^*\mu_n = \int_0^1 t^n \big(Sx^*\big)(t)\, dt,$$

where $S : X^* \to L_\infty([0,1],\mathbb{C})$ is bounded.

Define $T := S^t \Big|_{L_1([0,1],\mathbb{C})}$, the restriction of the dual S^t to L_1.

(3)$\Rightarrow$(1). The function $\varphi(t) := T\mathbf{1}_{[0,t)}$ is Lipschitz continuous, and

$$\mu_n = \int_0^1 t^n \, d\varphi(t).$$

(1)$\Rightarrow$(2). If $x^* \in X^*$ then $\|x^*\mu\|_{B_\infty} \le \|\varphi\|_{V_\infty}$, where φ is the representing function of μ.

Thus $\|\mu\|_{B_\infty} = \sup_{\|x^*\|\le 1} \|x^*\mu\|_{B_\infty} < \infty$. □

Proposition 1.7. *Let $U \subset \mathbb{C}$ be open and connected, $\mu \in U$, $1 \le p \le \infty$, $\tau > 0$ and $f : U \to X$ be analytic, that is, f can be represented locally by its Taylor series. Then the following assertions are equivalent:*

(1) *there is some $\varphi \in V_p([0,\tau], X)$ such that*

$$f(\lambda) = \int_0^\tau e^{-\lambda t}\, d\varphi(t) \quad \text{for all} \quad \lambda \in U;$$

(2) *the sequence*

$$\mu_n := \frac{(-1)^n f^{(n)}(\mu)}{\tau^n} \qquad (n \ge 0)$$

is a B_p-sequence.

Proof: (1)$\Rightarrow$(2). We define $\psi : [0,1] \to X$ by the Riemann-Stieltjes integral

$$\psi(s) = \int_0^s e^{-\mu\tau t}\, d_t\varphi(t\tau).$$

Then

$$\|\psi\|_{V_p} \le \left(\max_{0\le t\le 1} e^{-\mu\tau t}\right) \cdot \|\varphi(\cdot\colon \tau)\|_{V_p[0,1]} \le \|\varphi\|_{V_p[0,\tau]} \cdot \tau^{1-1/p},$$

where $1/\infty := 0$.

Further on,

$$\begin{aligned}\int_0^1 s^n\, d\psi(s) &= \int_0^1 s^n e^{-\mu\tau s}\, d_s\varphi(s\tau)\\ &= \int_0^\tau \left(\frac{u}{\tau}\right)^n e^{-\mu u}\, d\varphi(u) = \frac{(-1)^n f^{(n)}(\mu)}{\tau^n} = \mu_n.\end{aligned}$$

Thus $(\mu_n)_n$ is a B_p-sequence by the previous theorems, depending on p.

(2)$\Rightarrow$(1). Let $\mu_n = \int_0^1 t^n\, d\psi(t)$ for all $n \ge 0$, where $\psi \in V_p([0,1], X)$.

We define $\varphi : [0,\tau] \to X$ by

$$\varphi(s) = \int_0^{s/\tau} e^{\mu\tau t}\, d\psi(t).$$

Then

$$\|\varphi\|_{V_p[0,\tau]} \le e^{\mu\tau} \cdot \|\psi(\cdot/\tau)\|_{V_p[0,\tau]} \le \frac{e^{\mu\tau}}{\tau^{1-1/p}} \|\psi\|_{V_p[0,1]}.$$

Let $g(\lambda) = \int_0^\tau e^{-\lambda t}\, d\varphi(t)$. We show $f = g$. This is true since for $n \ge 0$ we have

$$\begin{aligned} g^{(n)}(\mu) &= (-1)^n \int_0^\tau t^n e^{-\mu t}\, d\varphi(t) = (-1)^n \int_0^1 \tau^n u^n e^{-\mu\tau u}\, d_u\varphi(u\tau)\\ &= (-1)^n \tau^n \int_0^1 u^n e^{-\mu\tau u} e^{\mu\tau u}\, d\psi(u) = f^{(n)}(\mu).\end{aligned}$$

Since both functions f and g are analytic we obtain $f = g$ by the uniqueness theorem for analytic functions. $\square$

2. Moment sequences and Cauchy problems

If $1 \le p \le \infty$, $\mu \in \mathbb{C}$ and $\tau > 0$ we say a function u belongs to $A_{\mu,p,\tau}(X)$ if $u \in C^1([0,\tau), X)$ and if the function $t \mapsto e^{-t}\left[\frac{d}{dt}\left(e^{-\mu t}u(t)\right)\right]_{t=\tau(1-e^{-t})}$ is of bounded p-variation on $[0,\infty)$.

Proposition 2.1. *Let $1 \le p \le \infty$, $\tau > 0$, $A : D(A) \to X$ be linear with $\mu \in \rho(A)$, $f \in C([0,\tau), X) \cap L_1([0,\tau], X)$ and $g(t) := e^{-\mu t} f(t)$.*
We define

$$c_k(y) := \frac{k!}{\tau^{k+1}}\left(\left(A-\mu\right)^{-k} y - \sum_{j=1}^{k} \left(A-\mu\right)^{-(k+1-j)} g^{[j+1]}(\tau)\right).$$

Then the following assertions are equivalent:

(1) *the abstract Cauchy problem*

$$\begin{cases} Au(t) + f(t) = u'(t) & \text{for } \ 0 \le t < \tau \\ u(0) = 0, \ u \in \mathcal{A}_{\mu,p,\tau}(X) \end{cases}$$

has a solution;

(2) *there is some $y \in X$ and some $v \in C^1([0,\infty), X)$ such that v' is of bounded p-variation on $[0,\infty)$ and with*

$$c_k(y) = \widehat{v}(k+1) \quad \text{for all } \ k \in \mathbb{N}_0;$$

(3) *there is some $y \in X$ and some $\varphi \in V_p([0,\infty), X)$ such that*

$$k^2 c_{k-1}(y) - \tau f(0) = \int_0^\infty e^{-kt}\, d\varphi(t) \quad \text{for all } \ k \ge 1.$$

Proof: (1)$\Rightarrow$(2). Let $w(t) := e^{-\mu t}u(t)$ if $0 \le t < \tau$ and $v(t) := w(\tau(1-e^{-t}))$ if $t \ge 0$. Then $v'(t) = \tau e^{-t} w'(\tau(1-e^{-t}))$ is of bounded p-variation on $[0,\infty)$, therefore $\widehat{v}(\lambda)$ and $\widehat{v'}(\lambda)$ exist if $\Re\lambda > 0$.

If $B := A - \mu$ we obtain

$$Bw(t) = -g(t) + w'(t) \quad \text{if } \ 0 \le t < \tau$$

and

$$Bv(t) = \frac{e^t}{\tau} v'(t) - g(\tau(1-e^{-t})) \quad \text{if } \ t \ge 0.$$

Thus, if $k \geq 2$,

$$
\begin{aligned}
B\widehat{v}(k) &= \int_0^\infty e^{-kt}\frac{e^t}{\tau}v'(t)\,dt - \int_0^\infty e^{-kt}g(\tau(1-e^{-t}))\,dt \\
&= \frac{1}{\tau}\left[e^{-(k-1)t}v(t)\Big|_0^\infty + (k-1)\int_0^\infty e^{-(k-1)t}v(t)\,dt\right] \\
&\quad - \int_0^\tau (1-s/\tau)^k g(s)\frac{ds}{\tau-s} \\
&= \frac{k-1}{\tau}\widehat{v}(k-1) - \frac{1}{\tau^k}\int_0^\tau (\tau-s)^{k-1}g(s)\,ds \\
&= \frac{k-1}{\tau}\widehat{v}(k-1) - \frac{(k-1)!}{\tau^k}g^{[k]}(\tau).
\end{aligned}
$$

This shows that

$$
\widehat{v}(k) = \frac{k-1}{\tau}B^{-1}\widehat{v}(k-1) - \frac{(k-1)!}{\tau^k}B^{-1}g^{[k]}(\tau) \quad \text{if} \quad k \geq 2.
$$

Let $y := \tau\widehat{v}(1)$. It follows by induction that $c_k(y) = \widehat{v}(k+1)$ if $k \geq 0$.

(2)$\Rightarrow$ (1). We have, if $k \geq 1$,

$$
\begin{aligned}
B\widehat{v}(k+1) &= \frac{k!}{\tau^{k+1}}\left(B^{-(k-1)}y - \sum_{j=1}^{k} B^{-(k-j)}g^{[j+1]}(\tau)\right) \\
&= \frac{k}{\tau}c_{k-1}(y) - \frac{k!}{\tau^{k+1}}g^{[k+1]}(\tau) \\
&= \frac{k}{\tau}\int_0^\infty e^{-kt}v(t)\,dt - \frac{k!}{\tau^{k+1}}g^{[k+1]}(\tau) \\
&= \frac{k}{\tau}\left[-\frac{1}{k}e^{-kt}v(t)\Big|_0^\infty + \frac{1}{k}\int_0^\infty e^{-kt}v'(t)\,dt\right] - \frac{k!}{\tau^{k+1}}g^{[k+1]}(\tau) \\
&= \frac{v(0)}{\tau} + \frac{1}{\tau}\int_0^\infty e^{-kt}v'(t)\,dt - \int_0^\infty e^{-(k+1)t}g(\tau(1-e^{-t}))\,dt.
\end{aligned}
$$

Applying B^{-1} to both sides of this equation and letting k tend to ∞, we obtain $B^{-1}\dfrac{v(0)}{\tau} = 0$ and therefore $v(0) = 0$. From

$$
\int_0^\infty e^{-kt}e^{-t}v(t)\,dt = \frac{1}{\tau}\int_0^\infty e^{-kt}B^{-1}v'(t)\,dt - \int_0^\infty e^{-kt}e^{-t}B^{-1}g(\tau(1-e^{-t}))\,dt
$$

and the uniqueness theorem for Laplace transforms it follows that

$$
e^{-t}v(t) = \frac{B^{-1}v'(t)}{\tau} - e^{-t}B^{-1}g(\tau(1-e^{-t})) \quad \text{for all} \quad t \geq 0,
$$

and, replacing t by $-\ln(1-t/\tau)$,

$$(\tau-t)Bv(-\ln(1-t/\tau)) = v'(-\ln(1-t/\tau)) - (\tau-t)g(t) \quad \text{for all} \quad t \in [0,\tau).$$

Let $w(t) := v(-\ln(1-t/\tau))$. Then $w'(t) = \dfrac{1}{\tau-t}v'(-\ln(1-t/\tau))$ and

$$Bw(t) = w'(t) - g(t) \quad \text{for all} \quad t \in [0,\tau).$$

Thus $u(t) := e^{\mu t}w(t)$ solves the abstract Cauchy problem. u is in $\mathcal{A}_{\mu,p,\tau}(X)$ since $t \mapsto e^{-t}w'(\tau(1-e^{-t})) = v'(t)/\tau$ is of bounded p-variation on $[0,\infty)$.

(2)$\Rightarrow$(3) We already saw that $v(0)=0$ and $v'(0) = \tau f(0)$. Let $\varphi(t) := v'(t) - v'(0)$. Then, if $k \geq 1$,

$$\begin{aligned}
\int_0^\infty e^{-kt}\,d\varphi(t) &= k\int_0^\infty e^{-kt}\varphi(t)\,dt \\
&= -\tau f(0) + k\int_0^\infty e^{-kt}v'(t)\,dt \\
&= -\tau f(0) + k\left[e^{-kt}v(t)\Big|_0^\infty + k\int_0^\infty e^{-kt}v(t)\,dt\right] \\
&= -\tau f(0) + k^2\widehat{v}(k) \\
&= -\tau f(0) + k^2 c_{k-1}(y).
\end{aligned}$$

(3)$\Rightarrow$(2). Let $v(t) := \int_0^t \varphi(s) + \tau f(0)\,ds$. Then, if $k \geq 0$,

$$\begin{aligned}
\widehat{v}(k+1) &= -\frac{1}{k+1}e^{-(k+1)t}v(t)\Big|_0^\infty + \frac{1}{k+1}\int_0^\infty e^{-(k+1)t}\big(\varphi(t)+\tau f(0)\big)\,dt \\
&= \frac{1}{(k+1)^2}\int_0^\infty e^{-(k+1)t}\,d\varphi(t) + \frac{\tau f(0)}{(k+1)^2} \\
&= c_k(y).
\end{aligned}$$

□

If $1 \leq p \leq \infty$, $\mu \in \mathbb{C}$ and $\tau > 0$ we say a function u belongs to $\mathcal{B}_{\mu,p,\tau}(X)$ if $u \in C^1([0,\tau),X)$ and if the function $t \mapsto t \cdot \left[\dfrac{d}{dt}\left(e^{-\mu t}u(t)\right)\right]_{t=\tau(1-t)}$ extends to a V_p-function on $[0,1]$.

We note that $\mathcal{B}_{\mu,p,\tau}(X) \subset \mathcal{A}_{\mu,p,\tau}(X)$, and if $u \in C^1([0,\tau],X)$ such that u' is of bounded p-variation on $[0,\tau]$ then $u \in \mathcal{B}_{\mu,p,\tau}(X)$ for all $\mu \in \mathbb{C}$.

The following theorem yields a connection to moment sequences, see Section 1.

Theorem 2.2. *Let $\tau > 0$, $1 \le p \le \infty$, let $A : D(A) \to X$ be linear with $\mu \in \rho(A)$ and let $f : [0,\tau] \to X$ be continuous. Define c_k as in Proposition 2.1.*

Then the following assertions are equivalent:

(1) *the abstract Cauchy problem*

$$\begin{cases} Au(t) + f(t) = u'(t) & \text{for } 0 \le t < \tau \\ u(0) = 0 \end{cases}$$

has a solution $u \in \mathcal{B}_{\mu,p,\tau}(X)$, i.e. it has a p-regular solution;

(2) *there is some $y \in X$ and some $\psi \in V_p([0,1],X)$ such that*[4]

$$k^2 c_{k-1}(y) - \tau f(0) = \int_0^1 t^k \, d\psi(t) \quad \text{for all } k \ge 0,$$

that is, $\big(k^2 c_{k-1}(y) - \tau f(0)\big)_k$ is a p-moment sequence.

Proof: (1)$\Rightarrow$(2). Since $u \in \mathcal{A}_{\mu,p,\tau}(X)$, Proposition 2.1 shows that there is some $y \in X$ such that

$$k^2 c_{k-1}(y) - \tau f(0) = \int_0^\infty e^{-kt} \, d\varphi(t) \quad \text{for all } k \ge 1,$$

where $\varphi(t) = \tau e^{-t} w'(\tau(1-e^{-t})) - \tau f(0)$ and $w(t) = e^{-\mu t} u(t)$.

Let $\psi(t) := -\tau t w'(\tau(1-t)) = -\varphi(-\ln t) - \tau f(0)$, which is a V_p-function on $[0,1]$. Then

$$\int_0^\infty e^{-kt} \, d\varphi(t) = \int_0^1 t^k \, d\psi(t) \quad \text{for all } k \ge 1.$$

Moreover $\psi(1) = -\tau w'(0) = -\tau u'(0) = -\tau f(0)$, thus $-\tau f(0) = \displaystyle\int_0^1 d\psi(t)$.

(2)$\Rightarrow$(1). Let $\varphi(t) := -\psi(e^{-t}) + \psi(1)$. Then

$$k^2 c_{k-1}(y) - \tau f(0) = \int_0^\infty e^{-kt} \, d\varphi(t) \quad \text{for all } k \ge 1.$$

Proposition 2.1 shows that

$$u(t) := e^{\mu t} \int_0^{-\ln(1-t/\tau)} \varphi(s) + \tau f(0) \, ds$$

solves the abstract Cauchy problem.

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[4] we define $k^2 c_{k-1}(y)\Big|_{k=0} := 0$.

Moreover

$$\begin{aligned} t \cdot \left[\frac{d}{dt}\left(e^{-\mu t}u(t)\right)\right]_{t=\tau(1-t)} &= t \cdot \left[\frac{\varphi(-\ln(1-t/\tau)) + \tau f(0)}{\tau - t}\right]_{t=\tau(1-t)} \\ &= \frac{-\psi(t) + \psi(1) + \tau f(0)}{\tau} \\ &= -\psi(t)/\tau, \end{aligned}$$

thus $u \in \mathcal{B}_{\mu,p,\tau}(X)$. □

If $\mu \in \mathbb{C}$ and $\tau > 0$ we say a function $u : [0,\tau) \to X$ belongs to $\mathcal{A}^*_{\mu,\tau}(X)$ if u is absolutely continuous on any interval $[0,\tau'] \subset [0,\tau)$, if u is differentiable almost everywhere and if the function

$$t \mapsto e^{-t} \cdot \left[\frac{d}{dt}\left(e^{-\mu t}u(t)\right)\right]_{t=\tau(1-e^{-t})}$$

coincides almost everywhere with a V_1-function on $[0,\infty)$.

Proposition 2.3. *Let $\tau > 0$, $A : D(A) \to X$ be linear with $\mu \in \rho(A)$ and let $f : [0,\tau] \to X$ be integrable. Define c_k as in Proposition 2.1.*

Then the following assertions are equivalent:

(1) *the abstract Cauchy problem*

$$\begin{cases} Au(t) + f(t) = u'(t) & \text{for almost all } \ 0 \le t < \tau \\ u(0) = 0, \ u \in \mathcal{A}^*_{\mu,\tau}(X) \end{cases}$$

has a solution;

(2) *there is some $y \in X$ and some $v \in C([0,\infty),X)$ which is locally absolutely continuous and differentiable almost everywhere such that v' coincides almost everywhere with a V_1-function on $[0,\infty)$ and such that*

$$c_k(y) = \widehat{v}(k+1) \quad \text{for all } \ k \in \mathbb{N}_0;$$

(3) *there is some $y \in X$ and some $\varphi \in V_1([0,\infty),X)$ such that*

$$k^2 c_{k-1}(y) = \int_0^\infty e^{-kt}\, d\varphi(t) \quad \text{for all } \ k \ge 1.$$

If f is continuous on $[0,\tau)$ then moreover we have the equivalence

(4) *the abstract Cauchy problem*

$$\begin{cases} Au(t) + f(t) = u'(t) & \text{for all } \ 0 \le t < \tau \\ u(0) = 0, \ u \in C^1([0,\tau),X) \cap \mathcal{A}^*_{\mu,\tau}(X) \end{cases}$$

has a solution.

Proof: (1)$\Rightarrow$(2). The function $w(t) := e^{-\mu t}u(t)$ is absolutely continuous on any interval $[0,\tau'] \subset [0,\tau)$, and the function $v(t) := w(\tau(1-e^{-t}))$ is absolutely continuous on any interval $[0,R] \subset [0,\infty)$.

Moreover its derivative $v'(t) = \tau e^{-t}w'(\tau(1-e^{-t}))$ coincides almost everywhere with a V_1-function on $[0,\infty)$.

Especially we obtain, if $k \geq 1$,

$$\int_0^\infty e^{-kt}v'(t)\,dt = k\int_0^\infty e^{-kt}v(t)\,dt.$$

Now the proof is the same as in Proposition 2.1.

(2)$\Rightarrow$(1). The proof is the same as in Proposition 2.1.

(2)$\Rightarrow$(3). Let $\varphi := v' \in V_1([0,\infty),X)$.

(3)$\Rightarrow$(2). Let $v(t) := \int_0^t \varphi(s)\,ds$.

(4)$\Rightarrow$(1) is trivial.

(1)$\Rightarrow$(4). We already saw that

$$Bw(t) + g(t) = w'(t) \ \text{ for almost all } \ t \in [0,\tau),$$

where $w(t) := e^{-\mu t}u(t)$, $g(t) := e^{-\mu t}f(t)$ and $B := A - \mu$. Thus

$$w(t) + B^{-1}g(t) = B^{-1}w'(t) \ \text{ almost everywhere on } \ [0,\tau).$$

The function $h(t) := w(\tau(1-e^{-t}))$ is differentiable almost everywhere on $[0,\infty)$, and

$$h'(t) = \tau e^{-t}w'(\tau(1-e^{-t})) \ \text{ almost everywhere on } \ [0,\infty).$$

From the assumption it follows that there is some $\sigma : [0,\infty) \to X$ of bounded variation which is everywhere continuous from the right with $\sigma = h'$ almost everywhere. The function $t \mapsto B^{-1}\sigma(t)$ is of bounded variation, continuous from the right and coincides almost everywhere with the continuous function

$$t \mapsto \tau e^{-t}h(t) + \tau e^{-t}B^{-1}g(\tau(1-e^{-t})),$$

therefore $t \mapsto B^{-1}\sigma(t)$ is continuous on $[0,\infty)$.

Since σ is of bounded variation the limit $\sigma(t+)$ exists for all $t \geq 0$, and we showed that

$$B^{-1}\big(\sigma(t+)\big) = \big(B^{-1}\sigma\big)(t+) = \big(B^{-1}\sigma\big)(t) = B^{-1}\big(\sigma(t)\big),$$

and in the same way for $\sigma(t-)$.

Since B^{-1} is injective we see that σ is continuous.

From

$$\tau e^{-t}w'(\tau(1-e^{-t})) = \sigma(t) \ \text{ almost everywhere on } \ [0,\infty)$$

it follows that w' and therefore u' coincides almost everywhere with a continuous function on $[0, \tau)$. Thus u is C^1, and the assertion follows since A is closed. $\square$

If $\mu \in \mathbb{C}$ and $\tau > 0$ we say a function $u : [0, \tau) \to X$ belongs to $\mathcal{B}^*_{\mu,\tau}(X)$ if u is absolutely continuous on any interval $[0, \tau'] \subset [0, \tau)$, if u is differentiabe almost everywhere and if the function

$$t \mapsto t \cdot \left[\frac{d}{dt}\left(e^{-\mu t} u(t)\right)\right]_{t=\tau(1-t)}$$

coincides almost everywhere with a V_1-function on $[0, 1]$.

Theorem 2.4. *Let $\tau > 0$, $A : D(A) \to X$ be linear with $\mu \in \rho(A)$ and let $f : [0, \tau] \to X$ be integrable. Define c_k as in Proposition* 2.1.

Then the following assertions are equivalent:

(1) *the abstract Cauchy problem*

$$\begin{cases} Au(t) + f(t) = u'(t) & \textit{for almost all } \ 0 \le t < \tau \\ u(0) = 0 \end{cases}$$

has a solution $u \in \mathcal{B}^*_{\mu,\tau}(X)$, i.e. it has a regular solution almost everywhere;

(2) *there is some $y \in X$ and some $\psi \in V_1([0, \tau], X)$ such that*[5]

$$k^2 c_{k-1}(y) = \int_0^1 t^k \, d\psi(t) \quad \textit{for all } \ k \ge 0.$$

Proof: (1)$\Rightarrow$(2). Since $u \in \mathcal{A}^*_{\mu,\tau}(X)$ Proposition 2.3 shows that there is some $y \in X$ and some $\varphi \in V_1([0, \infty), X)$ such that

$$k^2 c_{k-1}(y) = \int_0^\infty e^{-kt} \, d\varphi(t) \quad \text{for all } \ k \ge 1,$$

where φ coincides almost everywhere with the function

$$t \mapsto \tau e^{-t} \cdot \left[\frac{d}{dt}\left(e^{-\mu t} u(t)\right)\right]_{t=\tau(1-e^{-t})}.$$

Let $\psi(t) := -\varphi(-\ln t)$ if $t \in (0, 1]$ and $\psi(0) := 0$. Then $\psi \in V_1([0, \tau], X)$, and

$$\int_0^\infty e^{-kt} \, d\varphi(t) = \int_{0+}^1 t^k \, d\psi(t) = \int_0^1 t^k \, d\psi(t) \quad \text{for all } \ k \ge 1.$$

[5] again $k^2 c_{k-1}(y)\Big|_{k=0} := 0$.

Moreover $\int_0^1 d\psi(t) = 0$.

(2)$\Rightarrow$ (1). Let $\varphi(t) := -\psi(e^{-t})$. Then $\varphi \in V_1([0,\infty), X)$, and

$$k^2 c_{k-1}(y) = \int_0^\infty e^{-kt}\, d\varphi(t) \quad \text{for all} \quad k \geq 1.$$

Thus

$$u(t) := e^{\mu t} \int_0^{-\ln(1-t/\tau)} \varphi(s)\, ds$$

solves the abstract Cauchy problem. The solution $u \in C([0,\tau), X)$ is absolutely continuous on any interval $[0,\tau'] \subset [0,\tau)$, differentiable almost everywhere and

$$\begin{aligned} t \cdot \left[\frac{d}{dt}\left(e^{-\mu t} u(t)\right)\right]_{t=\tau(1-t)} &= t \left[\frac{\varphi(-\ln(1-t/\tau))}{\tau - t}\right]_{t=\tau(1-t)} \\ &= -\psi(t)/\tau. \end{aligned}$$

□

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