# Korn's First Inequality with variable coefficients and its generalization

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*Abstract.* If $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $\Gamma$ is an open subset of $\partial\Omega$, we prove that the following inequality

$$\left(\int_\Omega |A(x)\nabla u(x)|^p\,dx\right)^{1/p} + \left(\int_\Gamma |u(x)|^p\,d\mathcal{H}^{n-1}(x)\right)^{1/p} \geq c\,\|u\|_{W^{1,p}(\Omega)}$$

holds for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $1 < p < \infty$, where

$$(A(x)\nabla u(x))_k = \sum_{i=1}^{m}\sum_{j=1}^{n} a_k^{ij}(x)\,\frac{\partial u_i}{\partial x_j}(x) \quad (k = 1, 2, \dots, r;\ r \geq m)$$

defines an elliptic differential operator of first order with continuous coefficients on $\overline{\Omega}$. As a special case we obtain

$$(*) \qquad \int_\Omega \left|\nabla u(x)F(x) + (\nabla u(x)F(x))^T\right|^p dx \geq c\int_\Omega |\nabla u(x)|^p\,dx\,,$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ vanishing on $\Gamma$, where $F : \overline{\Omega} \to M^{n\times n}(\mathbb{R})$ is a continuous mapping with $\det F(x) \geq \mu > 0$. Next we show that $(*)$ is not valid if $n \geq 3$, $F \in L^\infty(\Omega)$ and $\det F(x) = 1$, but does hold if $p = 2$, $\Gamma = \partial\Omega$ and $F(x)$ is symmetric and positive definite in $\Omega$.

*Keywords:* Korn's Inequality, coercive inequalities

*Classification:* 35F15, 35J55

## 1. Introduction

In the recent paper [10] Neff proves that if $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary and if the mapping $F : \Omega \to M^{3\times 3}(\mathbb{R})$ is of class $C^2(\overline{\Omega})$ with $\det F(x) \geq \mu > 0$, then the following inequality

$$(1.1) \qquad \int_\Omega \left|\nabla u(x)F(x) + (\nabla u(x)F(x))^T\right|^2 dx \geq c\int_\Omega |\nabla u(x)|^2\,dx\,,$$

holds for all $u \in W^{1,2}(\Omega;\mathbb{R}^3)$ vanishing on some open, fixed subset $\Gamma$ of $\partial\Omega$. If $F(x)$ is constant and equal to the identity matrix, the above inequality is well-known and called (First) Korn's Inequality (see cf. Ciarlet [4, p. 292], or Nečas-Hlaváček [9, p. 85]). Recently, Korn's Inequality was also generalized to hold on Riemann manifolds (see Chen-Jost [3] for details).

Neff [12] uses inequality (1.1) to obtain an existence result in the nonlinear theory of elasto-viscoplasticity. The coefficients $F(x)$, denoted in Neff's papers by $F_p(x)$, represent the plasticity part of a model (see Neff [12] for details). They also satisfy $\det F(x) = 1$ and appear as a solution of some evolution problem, which gives few information about the smoothness of $F(x)$. Therefore the natural task is to minimize the smoothness assumptions on the coefficients $F(x)$ in (1.1). Neff [11] was later able to relax the assumption in (1.1) from $F \in C^2(\overline{\Omega})$ to $F \in C(\overline{\Omega})$, $\operatorname{rot} F \in L^3(\Omega)$. The proof, very similar to that one in [10], is complicated and the method applies only to the case $n = 3$.

If $F(x) = F$ does not depend on $x$, inequality (1.1) is quite easy to obtain: after suitable affine transformation it reduces to the classical Korn's Inequality. The situation changes diametrally, if one deals with variable coefficients: Trying to follow the method from Ciarlet [4], or Nečas-Hlaváček [9] for the case $F(x) = \mathrm{Id}$, one encounters unpleasant technical difficulties, which seem to be hard to overcome, even having some extra (superfluous) regularity assumptions on the coefficients. On the other hand, the standard way to pass from constant coefficients to variable ones by localization, like in the coercive inequalities (Theorem 2.2 below), does not work, because of the lack of the term $\|u\|^2_{L^2(\Omega)}$ on the left hand side of (1.1).

In the present paper we propose another, simpler approach to inequality (1.1) obtaining at the same time generalization to any elliptic operator $A$ of degree 1 (see Definition 2.1 and inequality (2.4)) in any dimension $n \geq 2$. We will require only that the coefficients $a^{ij}_k(x)$ are continuous. In particular, if we choose

$$A(x)\nabla u(x) = \nabla u(x)F(x) + (\nabla u(x)F(x))^T ,$$

we strengthen the result of Neff obtaining inequality (1.1) for $F \in C(\overline{\Omega})$. For this particular choice of $A$ our proof will turn out to be extremally short and simple.

In the next part of the present paper we concentrate on inequality (1.1) and show that the continuity of $F$ is essential, in the sense that (1.1) does not hold if $n \geq 3$, $F \in L^\infty(\Omega)$ and $\det F(x) = 1$. The case $n = 2$ is of a quite different nature (see remarks after Corollary 4.1).

Moreover, we prove that (1.1) does hold (at least when $\Gamma = \partial\Omega$) if $F$ is not continuous but possesses some algebraic structure, instead.

We remark that taking in our inequality as $A(x)$ the identity mapping we obtain Friedrich's Inequality:

$$\left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p} + \left(\int_\Gamma |u(x)|^p \, d\mathcal{H}^{n-1}(x)\right)^{1/p} \geq c \, \|u\|_{W^{1,p}(\Omega)} .$$

From this point of view, inequality (2.4) obtained below is a common generalization of Korn's and Friedrich's Inequalities, but of course the main point is to explain how to overcome the difficulties caused by the variable coefficients keeping at the same time minimal assumptions on their regularity.

## 2. Preliminaries and the general inequality

Let $\Omega$ be an open, bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with Lipschitz boundary $\partial\Omega$. Let $\Gamma$ be an open subset of $\partial\Omega$. We consider the space $W^{1,p}(\Omega) = W^{1,p}(\Omega;\mathbb{R}^m)$ with $1 < p < \infty$ of the (vector-valued) Sobolev functions $u\colon \Omega \to \mathbb{R}^m$ $(m \geq 1)$, equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p} + \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p}.$$

For a subset $S$ of $\overline{\Omega}$ denote by $W_0^{1,p}(\Omega, S)$ the set of those functions from $W^{1,p}(\Omega)$, which vanish on $S$. In the sequel $S$ will be either an open subset of the boundary $\partial\Omega$ or an open subset of $\Omega$ itself.

Moreover, let $A\colon M^{m\times n}(\mathbb{C}) \to \mathbb{C}^r$ be a linear mapping represented by the matrix $(a_k^{ij})$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq r)$, equipped with the Euclidean norm:

$$|A| = \left(\sum_{i,j,k} |a_k^{ij}|^2\right)^{1/2}.$$

**Definition 2.1.** We shall say that the mapping $A\colon M^{m\times n}(\mathbb{C}) \to \mathbb{C}^r$ with $r \geq m$ is *elliptic* if the condition $A(\eta \otimes \xi) \neq 0$ holds for all $\eta \in \mathbb{C}^m$, $\xi \in \mathbb{C}^n$ with $\eta \neq 0$, $\xi \neq 0$.

We denote by $\mathcal{E} = \mathcal{E}(m,n,r)$ the set of all elliptic mappings.

Having defined the linear mapping $A$, we can introduce the $r \times m$ matrix $(c_{ki}(\xi))$ of linear homogeneous polynomials given by

$$c_{ki}(\xi) = \sum_{j=1}^n a_k^{ij} \xi_j \quad (\xi \in \mathbb{C}^n).$$

Obviously, $A$ is elliptic if and only if $\operatorname{rank}(c_{ki}(\xi)) = m$ for every $\xi \in \mathbb{C}^n$ with $\xi \neq 0$.

The importance of the above definition lies in the following coercive inequality, due to Nečas [8]. It was later generalized by Besov [1] to anisotropic Sobolev spaces. More recently, the paper of Kałamajska [5] contains a version with Muckenhoupt weights.

**Theorem 2.2.** *Let $A(x)$ $(x \in \overline{\Omega})$ be a family of elliptic mappings, whose coefficients $a_k^{ij}(x)$ are continuous on $\overline{\Omega}$. Then the family $(A(x)\nabla u(x))_k$ $(1 \le k \le r)$ of differential operators, given by*

$$(A(x)\nabla u(x))_k = \sum_{i=1}^{m}\sum_{j=1}^{n} a_k^{ij}(x)\,\frac{\partial u_i}{\partial x_j}(x) \tag{2.1}$$

*is* coercive, *i.e. there is a constant $c > 0$ such that the following inequality*

$$\left(\int_\Omega |A(x)\nabla u(x)|^p\,dx\right)^{1/p} + \|u\|_{L^p(\Omega)} \ge c\,\|u\|_{W^{1,p}(\Omega)} \tag{2.2}$$

*holds for all $u \in W^{1,p}(\Omega)$.* □

Our goal is to modify inequality (2.2) by replacing the term $\|u\|_{L^p(\Omega)}$ with

$$\left(\int_\Gamma |u(x)|^p\,d\mathcal{H}^{n-1}(x)\right)^{1/p}.$$

We achieve this using Theorem 2.2 and the following theorem, which reflects the typical method of obtaining inequalities similar to (2.4).

**Theorem 2.3.** *If the family $(A(x)\nabla u(x))_k$ of differential operators given by (2.1) with variable coefficients $a_k^{ij}(x)$ is coercive and if the following implication holds*

$$A(x)\nabla u(x) = 0,\ u \in W_0^{1,p}(\Omega,\Gamma) \quad \Rightarrow \quad u = 0, \tag{2.3}$$

*then there is a constant $c > 0$ such that the inequality*

$$\left(\int_\Omega |A(x)\nabla u(x)|^p\,dx\right)^{1/p} + \left(\int_\Gamma |u(x)|^p\,d\mathcal{H}^{n-1}(x)\right)^{1/p} \ge c\,\|u\|_{W^{1,p}(\Omega)} \tag{2.4}$$

*holds for all $u \in W^{1,p}(\Omega)$.*

The proof of Theorem 2.3 uses standard compactness argument, used already by many authors, for example Nečas-Hlaváček [9, p. 85], or Neff [10, Theorem 3]. Since it is neither long nor difficult, we represent it here for the convenience of the reader.

Proof of Theorem 2.3: Suppose (2.4) does not hold.

Then there exists a sequence $u_k \in W^{1,p}(\Omega)$ with $\|u_k\|_{W^{1,p}(\Omega)} = 1$ such that

$$\left(\int_\Omega |A(x)\nabla u_k(x)|^p\,dx\right)^{1/p} + \left(\int_\Gamma |u_k(x)|^p\,d\mathcal{H}^{n-1}(x)\right)^{1/p} \le \frac{1}{k}\,. \tag{2.5}$$

Therefore there is a subsequence of $(u_k)$ (still denoted by $(u_k)$) and a function $u \in W^{1,p}(\Omega)$ such that $u_k \to u$ strongly in $L^p(\Omega)$ and $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. From (2.2) we obtain

$$c\|u_k - u_l\|_{W^{1,p}(\Omega)} \le \|u_k - u_l\|_{L^p(\Omega)} + \left(\int_\Omega |A(x)\nabla u_k(x)|^p\,dx\right)^{1/p} + \left(\int_\Omega |A(x)\nabla u_l(x)|^p\,dx\right)^{1/p},$$

which by (2.5) implies that $(u_k)$ is a Cauchy sequence in $W^{1,p}(\Omega)$, so $u_k \to u$ strongly in $W^{1,p}(\Omega)$. This, together with (2.5) implies that $u$ vanishes on $\Gamma$, i.e. $u \in W_0^{1,p}(\Omega,\Gamma)$ and $A(x)\nabla u(x) = 0$ a.e. on $\Omega$. Finally, using (2.3) we obtain $u = 0$, which provides a contradiction, since $\|u_k\|_{W^{1,p}(\Omega)} = 1$ and $u_k \to 0$ strongly in $W^{1,p}(\Omega)$. □

Assuming that $A(x)$ is continuous, Theorem 2.2 implies that the differential operators (2.1) are coercive. So in order to prove inequality (2.4) it remains to check if (2.3) holds. In the applications condition (2.3) seems to be difficult to verify, even if the coefficients are smooth enough. It turns out, however, that dealing with continuous coefficients $A(x)$ this unpleasant implication can be removed from the assumptions. To prove this assertion is our goal in the next section. Namely, we prove the following

**Theorem 2.4.** *Let $A(x)$ $(x \in \overline{\Omega})$ be a family of elliptic mappings, whose coefficients $a_k^{ij}(x)$ are continuous on $\overline{\Omega}$. Then the implication* (2.3) *holds. In particular, there is a constant $c > 0$ such that the inequality* (2.4) *holds for all $u \in W^{1,p}(\Omega)$.*

## 3. Proof of Theorem 2.4

We start with the following

**Lemma 3.1.** *Let $B$ be a ball in $\mathbb{R}^n$. Denote by $B_\lambda$ an open cone whose vertex coincides with the center of the ball $B$ and such that the surface measure $\mathcal{H}^{n-1}$ of $(\partial B) \cap B_\lambda$ is equal to $\lambda\mathcal{H}^{n-1}(\partial B)$. Moreover, let $A$ be elliptic (with constant coefficients). Then there exists a constant $c > 0$ such that the inequality*

$$\left(\int_B |A(\nabla u(x))|^p\,dx\right)^{1/p} \ge c\,\|u\|_{W^{1,p}(B)} \tag{3.1}$$

*holds for all $u \in W_0^{1,p}(B, B_\lambda)$.*

**Remark.** Readers interested in the special case $m = n$ and

$$A(x)\nabla u(x) = \nabla u(x)F(x) + (\nabla u(x)F(x))^T\,,$$

where $\det F(x) \geq \mu > 0$, may omit the proof of Lemma 3.1 and replace it by the following short reasoning: Inequality (3.1), which in this case reads

$$\left(\int_B |\nabla u(x)F + (\nabla u(x)F)^T|^p\,dx\right)^{1/p} \geq c\,\|u\|_{W^{1,p}(B)}\,,$$

($F$ is constant here) is after the affine coordinate transformation $x \mapsto F^{-1}x$ equivalent to

$$\left(\int_E |\nabla v(y) + (\nabla v(y))^T|^p\,dy\right)^{1/p} \geq c'\,\|v\|_{W^{1,p}(E)}\,,$$

where $c'$ is a positive constant and $v(y) = u(Fy)$ vanishes on some fixed part of the boundary of $E = F^{-1}(B)$. The last displayed inequality is just Korn's Inequality, so it is valid.

Proof of Lemma 3.1: Since the proof uses standard and well-known methods, we only outline it briefly indicating the main steps.

Fix a function $\omega \in C_0^\infty(B')$, where $B'$ is a fixed ball, whose closure lies in $B_\lambda$.

Assume that $u$ is a smooth (i.e. $C^\infty(\overline{B})$) function vanishing on $B'$.

Using the Hilbert Nullstellensatz and the method from [8] or [1] we find a positive integer $N$ and homogeneous polynomials $p_k^{i\alpha}(\xi)$ (with complex coefficients) of degree $N-1$ such that

$$D^\alpha u_i = \sum_{k=1}^{r} p_k^{i\alpha}(D)(\mathcal{A}_k u) \tag{3.2}$$

holds for all $1 \leq i \leq m$ and all multiindices $\alpha$ with $|\alpha| = N$, where

$$(\mathcal{A}_k u)(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_k^{ij}\,\frac{\partial u_i}{\partial x_j}(x) \quad (1 \leq k \leq r)\,.$$

Using the integral representation of Sobolev (see Maz'ya [6]) and observing that $\omega$ and $u$ have disjoint supports, we find that

$$u_i(x) = \sum_{|\alpha|=N}\int_B K_\alpha(x,y)\,D^\alpha u_i(y)\,dy \quad (1 \leq i \leq m)\,, \tag{3.3}$$

where

$$K_\alpha(x,y) = \frac{(-1)^N \cdot N}{\alpha!} \cdot \frac{\theta^\alpha}{r^{n-N}} \int_r^\infty \omega(x+t\theta)t^{n-1}\,dt.$$

In the last formula we have substituted $r = |y-x|$ and $\theta = \dfrac{y-x}{|y-x|}$. Then for every fixed $x \in B$, the function $K_\alpha(x,\ \cdot\ )$ is smooth on $B \setminus \{x\}$ and vanishes near the boundary $\partial B$. Write

$$\frac{\theta^\alpha}{r^{n-N}} \int_r^\infty \omega(x+t\theta)t^{n-1}\,dt = \frac{\theta^\alpha}{r^{n-N}}\, c(x,\theta) - \frac{\theta^\alpha}{r^{n-N}}\, d(x,\theta,r),$$

where

$$c(x,\theta) = \int_0^\infty \omega(x+t\theta)t^{n-1}\,dt \quad \text{and} \quad d(x,\theta,r) = \int_0^r \omega(x+t\theta)t^{n-1}\,dt\,.$$

The function $c(x,\theta)$ is smooth on $B \times (\mathbb{R}^n \setminus \{0\})$. Then for any $y \neq x$ and any multiindex $\beta$ with $|\beta| = N-1$ we obtain that

$$D_y^\beta\left(\frac{\theta^\alpha}{r^{n-N}}\, c(x,\theta)\right)$$

is the sum of the terms of the form

$$\frac{p(\theta)}{r^{n-1}} \int_0^\infty w(x+t\theta)t^k\,dt, \tag{3.4}$$

where $k \geq n-1$, $w \in C_0^\infty(B')$ and $p$ is a polynomial. Therefore

$$\frac{\partial}{\partial x_j} D_y^\beta\left(\frac{\theta^\alpha}{r^{n-N}}\, c(x,\theta)\right)$$

is the sum of the terms of the form

$$\frac{p_1(\theta)}{r^n} \int_0^\infty w_1(x+t\theta)t^k\,dt \ + \ \frac{p_2(\theta)}{r^{n-1}} \int_0^\infty w_2(x+t\theta)t^l\,dt, \tag{3.5}$$

where $k, l \geq n-1$, $w_1, w_2 \in C_0^\infty(B')$ and $p_1, p_2$ are polynomials.

Similarly, the function $d(x,\theta,r)$ is smooth on $B \times (\mathbb{R}^n \setminus \{0\}) \times (0,\infty)$. Differentiating with respect to the multiindex $\beta$ with $|\beta| = N-1$ we obtain that

$$D_y^\beta\left(\frac{\theta^\alpha}{r^{n-N}}\, d(x,\theta,r)\right)$$

is the sum of the terms of the following form

$$\frac{p_1(\theta)}{r^{n-1}} \int_0^r w_1(x+t\theta)t^k\,dt \;+\; w_2(y)r^l p_2(\theta)\,, \tag{3.6}$$

where $k \geq n-1$, $l \geq 1$, $w_1, w_2 \in C_0^\infty(B')$ and $p_1, p_2$ are polynomials.

From the above computations and expressions (3.4) and (3.6) we see that for a fixed $x \in B$ and for a multiindex $\beta$ with $|\beta| \leq N-1$, the function $D_y^\beta K_\alpha(x,y)$ is integrable with respect to $y$ on $B$. Thus using (3.2), (3.3) and integrating by parts we obtain

$$u_i(x) = \sum_{k=1}^{r} \sum_{|\alpha|=N} \int_B (-1)^{N-1} p_k^{i\alpha}(D_y)(K_\alpha(x,y))\,\mathcal{A}_k u(y)\,dy\,,$$

where the kernels $p_k^{i\alpha}(D_y)K_\alpha(x,y)$ are of the form (3.4) plus the terms of the form (3.6).

Differentiating the terms (3.6) with respect to $x_j$ and using that

$$\frac{1}{r}\int_0^r w(x+t\theta)t^k\,dt \leq C\,,$$

for all $w \in C_0^\infty(B')$ appearing in (3.6) and all $|\theta| = 1$, $r > 0$, $k \geq 1$, we see that

$$D_y^\beta\Big(\frac{\theta^\alpha}{r^{n-N}}\,d(x,\theta,r)\Big) \leq \frac{C}{r^{n-2}} \quad \text{and} \quad \frac{\partial}{\partial x_j} D_y^\beta\Big(\frac{\theta^\alpha}{r^{n-N}}\,d(x,\theta,r)\Big) \leq \frac{C}{r^{n-1}}\,.$$

Therefore for any $f \in L^1(B)$,

$$\frac{\partial}{\partial x_j}\int_B D_y^\beta\Big(\frac{\theta^\alpha}{r^{n-N}}\,d(x,\theta,r)\Big)\,f(y)\,dy = \int_B \frac{\partial}{\partial x_j} D_y^\beta\Big(\frac{\theta^\alpha}{r^{n-N}}\,d(x,\theta,r)\Big)\,f(y)\,dy.$$

Now using Theorem 1.29 from [7], the fact that the terms (3.5) are bounded on $B\times S^{n-1}$ ($S^{n-1}$ is the unit sphere in $\mathbb{R}^n$) and the result of Calderón and Zygmund (see [2, Theorem 2] or [7, Theorem 2.1]), we obtain

$$\Big\|\frac{\partial u_i}{\partial x_j}\Big\|_{L^p(B)}^p \leq c\sum_{k=1}^{r}\|\mathcal{A}_k u\|_{L^p(B)}^p \quad (1 \leq i \leq m,\ 1 \leq j \leq n), \tag{3.7}$$

for some constant $c > 0$ independent of $u$ (vanishing on $B'$).

Since every $u \in W_0^{1,p}(B, B_\lambda)$ can be approximated in the norm $\|\cdot\|_{W^{1,p}(B)}$ by $C^\infty(\overline{B})$ functions vanishing on $B'$, we get from (3.7) and from the Poincaré inequality, inequality (3.1) for all $u \in W_0^{1,p}(B, B_\lambda)$. □

We shall also need the following lemma, which states that there is a common positive constant $c$ in (3.1) for all elliptic mappings $A$ lying in some compact subset of $\mathcal{E}$.

**Lemma 3.2.** *Let $B$ and $B_\lambda$ be like in Lemma* 3.1. *Let moreover $\mathcal{K}$ be a compact subset of $\mathcal{E}$. Then there exists a constant $c > 0$ such that the inequality* (3.1) *holds for all mappings $A \in \mathcal{K}$ and all $u \in W_0^{1,p}(B, B_\lambda)$.*

Proof: For a fixed mapping $A$, denote by $c_A$ the best constant in (3.1). It is enough to show that $c = \inf\{c_A : A \in \mathcal{K}\} > 0$. Suppose, to the contrary, that $c = 0$. Then there exist a sequence $A_n \in \mathcal{K}$ and a sequence $u_n \in W_0^{1,p}(B, B_\lambda)$ such that $\|u_n\|_{W^{1,p}(B)} = 1$ and

$$\|A_n(\nabla u_n)\|_{L^p(B)} \to 0.$$

Since $\mathcal{K}$ is compact, we can choose a subsequence from $(A_n)$, still denoted by $(A_n)$ and $A_\infty \in \mathcal{K}$, such that $|A_n - A_\infty| \to 0$. Then

$$\begin{aligned}\|A_\infty(\nabla u_n)\|_{L^p(B)} &\le \|(A_\infty - A_n)(\nabla u_n)\|_{L^p(B)} + \|A_n(\nabla u_n)\|_{L^p(B)} \\ &\le |A_\infty - A_n| \cdot \|u_n\|_{W^{1,p}(B)} + \|A_n(\nabla u_n)\|_{L^p(B)}\,,\end{aligned}$$

which implies that $\|A_\infty(\nabla u_n)\|_{L^p(B)} \to 0$. On the other hand, since $A_\infty$ is elliptic, we apply inequality (3.1) to obtain

$$\|A_\infty(\nabla u_n)\|_{L^p(B)} \ge c_\infty \|u_n\|_{W^{1,p}(B)} = c_\infty\,.$$

Letting $n \to \infty$ we get $c_\infty \le 0$, a contradiction. □

Using Lemma 3.2 and scaling we arrive at the following

**Corollary 3.3.** *Let $B$, $B_\lambda$ and $\mathcal{K}$ be like in Lemma* 3.2. *Then there is a constant $c > 0$ such that the inequality*

$$\|A(\nabla u)\|_{L^p(B)} \ge c\,\|\nabla u\|_{L^p(B)} \tag{3.8}$$

*holds for all $A \in \mathcal{K}$ and all $u \in W_0^{1,p}(B, B_\lambda)$ and the constant $c$ does not depend on the radius of the ball $B$.* □

Now we are ready to prove Theorem 2.4:

Assume that $A(x)(\nabla u(x)) = 0$ a.e. in $\Omega$ and $u \in W_0^{1,p}(\Omega, \Gamma)$.

Denote by $B(x, \rho)$ the ball with center $x$ and radius $\rho$ and by $B_\lambda(x)$ the corresponding cone, like in Lemma 3.1.

Fix $x_0 \in \Gamma$. Choose $\rho > 0$, such that $B(x_0, \rho) \cap (\partial\Omega) \subset \Gamma$. Extend $u$ by 0 on $B(x_0, \rho) \setminus \Omega$. Then for some $\lambda > 0$ we have $u \in W_0^{1,p}(B(x_0, \rho), B_\lambda(x_0))$. By inequality (3.8) we obtain

$$\begin{aligned}c\int_{B(x_0,\rho)} |\nabla u(x)|^p\,dx &\le \int_{B(x_0,\rho)} |A(x_0)(\nabla u(x))|^p\,dx\\ &= \int_{B(x_0,\rho)} |(A(x_0)-A(x))(\nabla u(x))|^p\,dx\\ &\le \int_{B(x_0,\rho)} |A(x_0)-A(x)|^p\cdot|\nabla u(x)|^p\,dx\\ &\le \varepsilon^p\int_{B(x_0,\rho)} |\nabla u(x)|^p\,dx.\end{aligned}$$

Since the above constant $c$ does not depend on $\rho$, and since $A(x)$ is continuous on $\overline{\Omega}$, we use the above inequality to find a number $\rho>0$ such that

$$\int_{B(x_0,\rho)} |\nabla u(x)|^p\,dx = 0.$$

This implies that $u(x)=0$ for $x\in B(x_0,\rho)$.

Now fix $x\in\Omega$. Take any curve $\gamma$ lying within $\Omega$ and connecting $x_0$ with $x$. We repeat the above argument with $x_1=\gamma\cap\partial B(x_0,\rho)$ in place of $x_0$. The continuity of the coefficients $a_k^{ij}(x)$ implies that $\mathcal{K}=\{A(x)\mid x\in\gamma\}$ is a compact subset of $\mathcal{E}$. It follows therefore (by Corollary 3.3 and by repeating the above reasoning) that we can cover $\gamma$ with a finite sequence of the balls $B(x_k,\rho)$ with $x_k\in\gamma$ and equal radii $\rho$ ($\lambda$ can be chosen the same is each step and the coefficients $a_k^{ij}(x)$ are uniformly continuous on $\overline{\Omega}$), proving in each step that $u=0$ on $B(x_k,\rho)$. This shows that $u=0$ on $\Omega$. □

## 4. Special case: First Korn's Inequality with variable coefficients

From now on assume that $m=n$. Directly from Theorem 2.4 we obtain the following

**Corollary 4.1.** *Let $F:\overline{\Omega}\to M^{n\times n}(\mathbb{R})$ ($n\ge 2$) be a continuous mapping with $\det F(x)\ge\mu>0$. Then there is a constant $c>0$ such that the following inequality*

$$\int_\Omega \left|\nabla u(x)F(x)+(\nabla u(x)F(x))^T\right|^p dx \ge c\int_\Omega |\nabla u(x)|^p\,dx \tag{4.1}$$

*holds for all $u\in W_0^{1,p}(\Omega,\Gamma)$.*

Proof: It is enough to check that an $n\times n$ (real) matrix $F$ with $\det F\neq 0$ verifies

$$(\eta\otimes\xi)F+((\eta\otimes\xi)F)^T\neq 0$$

whenever $\eta \in \mathbb{C}^n$, $\xi \in \mathbb{C}^n$ and $\eta \neq 0$, $\xi \neq 0$.

Indeed, if the matrix $B = (\eta \otimes \xi)F$ were nonzero and antisymmetric, then we would get $\operatorname{rank} B \geq 2$, which would imply that $\operatorname{rank}(\eta \otimes \xi) \geq 2$, a contradiction. Therefore $B = 0$, which gives $\eta \otimes \xi = 0$, whence $\eta = 0$ or $\xi = 0$. □

**Remarks.** (a) If $n = 2$, $p = 2$ and $\Gamma = \partial\Omega$, then inequality (4.1) holds with $F \in L^\infty(\Omega)$ instead of being continuous on $\overline{\Omega}$. This was shown by Neff [10], who assumed additionally that $\det F(x)$ is constant and positive. His proof can be easily modified to the much more general case $\det F(x) \geq \mu > 0$. Indeed, in the inequality (see Neff [10, Theorem 4.14])

$$|\nabla u(x)F(x) + (\nabla u(x)F(x))^T|^2 \geq 2\,|\nabla u(x)F(x)|^2 - 4\,\det(\nabla u(x))\,\det F(x)$$

we first divide both sides by $\det F(x)$ and then integrate.

(b) The class of the mappings $F(x)$ for which (4.1) holds is larger than the class of continuous mappings $F$ with $\det F \geq \mu > 0$, also when $n \geq 3$. Indeed, if inequality (4.1) holds for a mapping $F_0 \in L^\infty(\Omega)$, then it also holds (perhaps with another constant $c > 0$) for all mappings $F$ lying in some $L^\infty(\Omega)$-neighborhood of $F_0$.

(c) Inequality (4.1) is also valid if we assume that $F(x) = \nabla G(x)$, where $G\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bi-Lipschitz mapping — just make the coordinate transformation like in the remark after Lemma 3.1, or see Neff [10, Theorem 4.13] for details.

The above remarks suggest the following question: Does (4.1) hold if $n \geq 3$, $F \in L^\infty(\Omega)$ and $\det F(x) \geq \mu > 0$?

The answer turns out to be negative, even if $\Gamma = \partial\Omega$ and $\det F(x) = 1$ in $\Omega$. So for $n \geq 3$ the class of the mappings $F(x)$ for which (4.1) holds lies somewhere *strictly* between $C(\overline{\Omega})$ and $L^\infty(\Omega)$. Theorem 4.3 below shows that this class contains also the symmetric, positive definite a.e. mappings $F(x)$, at least when $p = 2$ and $\Gamma = \partial\Omega$.

**Theorem 4.2.** *Assume that $n \geq 3$. Then there exist a nonzero function $u \in W_0^{1,\infty}(\Omega;\mathbb{R}^n)$ and a mapping $F \in L^\infty(\Omega; M^{n\times n}(\mathbb{R}))$ with $\det F(x) = 1$ in $\Omega$ such that*

$$\nabla u(x)F(x) + (\nabla u(x)F(x))^T = 0 \quad \text{a.e. on } \Omega.$$

Proof: Denote by $e_1, e_2, \dots, e_n$ the standard orthonormal basis of $\mathbb{R}^n$ and let $R\colon \mathbb{R}^n \to \mathbb{R}^n$ be any fixed rotation satisfying $R(e_i) \neq \pm e_j$ for all $i, j = 1, 2, \dots, n$.

Let $\{Q_i\}$ (where $i \in \mathbb{N}$) be a Vitali covering of $\Omega$, such that the cubes $Q_i$ have pairwise disjoint entries and their edges are parallel to the coordinate axes (i.e. to the vectors $e_1, e_2, \dots, e_n$). Let moreover $\{P_j\}$ (where $j \in \mathbb{N}$) be another

Vitali covering of $\Omega$ with the cubes $P_j$, whose edges are parallel to the vectors $R(e_1), R(e_2), \dots, R(e_n)$. Define

$$\begin{aligned} u_1(x) &= \operatorname{dist}(x, \partial Q_i) \quad \text{if } x \in Q_i\,, \\ u_2(x) &= \operatorname{dist}(x, \partial P_j) \quad \text{if } x \in P_j\,, \\ u_3(x) &= \ldots = u_n(x) = 0\,. \end{aligned}$$

Then the function $u = (u_1, u_2, \dots, u_n)$ is nonzero, Lipschitz continuous and $u = 0$ on $\partial\Omega$. Moreover $|\nabla u_1(x)| = |\nabla u_2(x)| = 1$ a.e. in $\Omega$. The vectors $\nabla u_1$ and $\nabla u_2$ attain only a finite number of values for $x \in \Omega$ and $v$, $w$ are linearly independent for all $v \in \{\nabla u_1(x) \mid x \in \Omega\}$ and $w \in \{\nabla u_2(x) \mid x \in \Omega\}$. Therefore there exist functions $g_3, g_4, \dots, g_n \in L^\infty(\Omega; \mathbb{R}^n)$ such that the determinant of the $n \times n$ matrix

$$G(x) = \begin{bmatrix} \nabla u_2(x) \\ -\nabla u_1(x) \\ g_3(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

is equal to 1 for every $x \in \Omega$. Take $F(x) = (G(x))^{-1}$. Then

$$(\nabla u(x) F(x))_{ij} = \begin{cases} 1 & \text{if } \ i = 2,\ j = 1 \\ -1 & \text{if } \ i = 1,\ j = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\nabla u(x)F(x) + (\nabla u(x)F(x))^T = 0$ a.e. in $\Omega$. □

**Theorem 4.3.** *Let $n \geq 2$ and $F \in L^\infty(\Omega; M^{n\times n}(\mathbb{R}))$ be such that $F(x)$ is symmetric and positive definite a.e. in $\Omega$ and $\det F(x) \geq \mu > 0$. Then there is a constant $c > 0$ such that the inequality*

$$\int_\Omega \big|\nabla u(x)F(x) + (\nabla u(x)F(x))^T\big|^2\,dx \geq c\int_\Omega |\nabla u(x)|^2\,dx \tag{4.2}$$

*holds for all $u \in W_0^{1,2}(\Omega)$.*

Proof: From the assumptions it follows that the eigenvalues $\lambda_1(x), \lambda_2(x), \dots,$ $\lambda_n(x)$ of $F(x)$ lie in the interval $[a, b]$, where $0 < a < b$. Writing $F^{-1}(x)$ in the orthonormal basis composed of its eigenvectors and using that $\lambda_i(x) \geq a$, we obtain for any real $n \times n$ matrix $C$,

$$\langle F^{-1}(x)C, CF^{-1}(x)\rangle \leq \frac{1}{a^2}|C|^2$$

for any $x \in \Omega$. Similarly we get

$$\langle CF(x), F^{-1}(x)C\rangle \geq \frac{a}{b}\,|C|^2\,.$$

Setting $F = F(x)$ and using the above inequalities we get

$$\begin{aligned}\frac{1}{a^2}\,|CF + FC^T|^2 &\geq \langle F^{-1}(CF + FC^T), (CF + FC^T)F^{-1}\rangle \\ &= \langle F^{-1}CF + C^T, C + FC^T F^{-1}\rangle \\ &= \langle CF, F^{-1}C\rangle + 2\langle C, C^T\rangle + \langle C^T F^{-1}, FC^T\rangle \\ &\geq \frac{2a}{b}\,|C|^2 + 2\langle C, C^T\rangle\,.\end{aligned}$$

Substituting $C = \nabla u(x)$, where $u \in W_0^{1,2}(\Omega)$ and integrating yields

$$\int_\Omega \left|\nabla u(x)F(x) + (\nabla u(x)F(x))^T\right|^2 dx \geq \frac{2a^3}{b}\int_\Omega |\nabla u(x)|^2\,dx,$$

which proves (4.2). □

**Acknowledgments.** The author thanks Dr. A. Kałamajska and Dr. P. Neff for valuable suggestions.

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(*Received* May 20, 2002)