Decay of solutions of some degenerate hyperbolic equations of Kirchhoff type

Barbara Szomolay

Abstract. In this paper we study the asymptotic behavior of solutions to the damped, nonlinear vibration equation with self-interaction

$$\ddot{u} = -\gamma\dot{u} + m(\|\nabla u\|^2)\Delta u - \delta|u|^\alpha u + f,$$

which is known as degenerate if $m(\cdot) \geq 0$, and non-degenerate if $m(\cdot) \geq m_0 > 0$. We would like to point out that, to the author's knowledge, exponential decay for this type of equations has been studied just for the special cases of α. Our aim is to extend the validity of previous results in [5] to $\alpha \geq 0$ both to the degenerate and non-degenerate cases of m. We extend our results to equations with Δ^2.

Keywords: asymptotic behavior of solutions, hyperbolic PDE of degenerate type

Classification: 35B40, 35L80

1. Introduction

This article presents a study of the asymptotic behavior of solutions to the initial value problem

$$\begin{gathered} \ddot{u} + \gamma\dot{u} - m(\|\nabla u\|^2)\Delta u + \delta|u|^\alpha u = f(x,t) \quad \text{for} \quad x \in \Omega, \quad t \geq 0, \\ u(x,0) = g(x), \quad \dot{u}(x,0) = h(x) \quad \text{for} \quad x \in \Omega, \\ u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega, \quad t \geq 0, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$; $\gamma > 0$, $\delta > 0$, $\alpha \geq 0$ are given constants, m is a non-negative and continuous function on $[0,\infty)$.

This equation appears in mathematical physics as a specific type of the damped Kirchhoff-Carrier equation, when modeling planar vibrations. For constant m and $\alpha = 0$ we have the so-called telegraph equation; in which for arbitrary small tension and resistance ($\gamma \sim 0$, $\delta \sim 0$) we obtain the well-known wave equation $u_{tt} = a^2\Delta u$. In what follows we take $\delta = 1$ without loss of generality.

Asymptotic behavior of solutions to equations without the self-restoring term $|u|^\alpha u$ have been studied by many authors (see [5] and the references therein).

Of these, most striking are perhaps the results of J.G. Dix, who studied exponential decay of solutions to the damped, nonlinear wave equation

$$(*) \qquad \ddot{u} + \gamma \dot{u} - m(\|\nabla u\|^2)\Delta u = f(x,t),$$

which is known as degenerate if the greatest lower bound for m is zero, and non-degenerate if the greatest lower bound for m is positive. The non-degenerate case has been the subject of many publications and it is already known that solutions decay exponentially, but for the degenerate case exponential decay has remained an open question and it has been proven just for the special cases of m. J.G. Dix has shown that in general degenerate-problem solutions cannot decay exponentially, but that $\|\dot{u}\|$ is square integrable on $[0,\infty)$.

When $\alpha = 2$, $f \equiv 0$ and the function m is linear, exponential decay of global strong solutions has been shown by M. Aassila (see [3]).

Inspired by the articles [1] and [5], we have added the self-restoring term $|u|^\alpha u$ and followed the steps of the proofs of theorems in these papers with little modifications. Our aim was to generalize the results in [5] to the mixed problem (1.1) and to weaken some of the assumptions imposed on m.

The outline is as follows: In Section 2 we state the theorem of existence of global solutions. Section 3 proves exponential decay for the non-degenerate case. Section 4 shows that if $m \equiv 0$ and $\alpha = 0$, the decay is exponential and if $m(\cdot) \geq 0$, $f \equiv 0$, the energy $E(t) \to 0$ as $t \to \infty$. In this section we also indicate with an example that in case $\alpha = 0$ solutions cannot decay with exponential order. Section 5 extends our results to equations with Δ^2.

Notation

$\|\cdot\|$ and $(\cdot,\cdot)$ will denote the norm and inner product in the Hilbert space $L^2(\Omega)$. $\|\cdot\|_p \quad (p \geq 1)$ will denote the norm in the $L^p(\Omega)$ space. We define the positive, self-adjoint operator A as the negative Laplacian, with domain

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega),$$

where H^2, H^1 are the usual Sobolev spaces. The operator $-\Delta$, with zero boundary conditions, has eigenvectors and corresponding eigenvalues denoted as follows

$$Aw_i = \lambda_i^2 w_i, \quad \text{with} \quad 0 < \lambda_1 \leq \lambda_2 \ldots, \quad \lim_{i\to\infty} \lambda_i = \infty.$$

Moreover, these eigenvectors can be chosen to form an orthogonal basis for $L^2(\Omega)$, in which functions have Fourier expansion of the form

$$u(x) = \sum_{i=1}^{\infty} u_i w_i(x), \quad \text{with} \quad u_i = (u, w_i).$$

Using this spectral decomposition, powers of A are defined by

$$Au(x) = \sum_{i=1}^{\infty} \lambda_i^2 u_i w_i, \; A^2u(x) = \sum_{i=1}^{\infty} \lambda_i^4 u_i w_i, \; \ldots \; A^k u(x) = \sum_{i=1}^{\infty} \lambda_i^{2k} u_i w_i,$$

provided that $\sum_{i=1}^{\infty} \lambda_i^{4k}|u_i|^2 < \infty$. Parseval's equality yields

$$\lambda_1^k \|u\|^2 = \lambda_1^k \sum_{i=1}^{\infty} u_i^2 \le \sum_{i=1}^{\infty} \lambda_i^k u_i^2 = \|A^{\frac{k}{4}} u\|^2,$$

where $k = \{4, 2\}$. Therefore,

$$\lambda_1{}^2 \|u\| \le \|Au\|, \quad \lambda_1 \|u\| \le \|\nabla u\| = \|A^{\frac{1}{2}} u\|. \tag{1.2}$$

Now, the mixed problem (1.1) can be rewritten as

$$\begin{gathered} \ddot{u} + \gamma \dot{u} + m(\|A^{\frac{1}{2}} u\|^2) A u + |u|^{\alpha} u = f(x,t), \\ u(x,0) = g(x), \quad \dot{u}(x,0) = h(x), \end{gathered} \tag{1.3}$$

the solutions of which are to be in the class of functions determined by Theorem 1.

2. Existence of global solutions

T. Matsuyama and R. Ikehata [6] have shown the following theorem:

Theorem 1. *Let m, α, u^0, u^1 and f satisfy the following assumptions*

(A.1) $\quad m \in C^1[0,\infty) \quad$ *and* $\quad m(r) \ge m_0 > 0 \quad$ *for* $\quad r \ge 0$,

(A.2) $\quad g \in H_0^1(\Omega) \cap H^2(\Omega) \quad$ *and* $\quad h \in H_0^1(\Omega)$,

(A.3) $\quad 0 \le \alpha \le \dfrac{2}{n-4} \quad$ *if* $\quad n \ge 5 \quad$ *and* $\quad \alpha \ge 0 \quad$ *if* $\quad n = 1,2,3,4$,

(A.4) $\quad f \in L^1([0,\infty), H_0^1(\Omega)) \cap L^\infty([0,\infty), L^2(\Omega))$.

Under assumptions (A.1)–(A.4)*, there exists a positive constant* ϵ_0 (*depending on* $\|\nabla u^0\|$, $\|u^0\|_{\alpha+2}$, $\|u^1\|$ *and* $\|f\|_{L^1(0,\infty;L^2(\Omega))}$) *such that, if*

$$\|\Delta u^0\| + \|\nabla u^1\| + \int_0^\infty \|\nabla f(t)\|\,dt < \epsilon_0, \tag{A.5}$$

then there exists a unique solution for (1.1) *in class*

$$\begin{gathered} u \in BC([0,\infty), H_0^1(\Omega)) \cap BC_w([0,\infty), H^2(\Omega)) \cap L^2((0,\infty), H^2(\Omega)), \\ u_t \in BC([0,\infty), L^2(\Omega)) \cap BC_w([0,\infty), H_0^1(\Omega)) \cap L^2((0,\infty), H_0^1(\Omega)), \\ u_{tt} \in L^2((0,\infty), L^2(\Omega)). \end{gathered}$$

3. Decay in the non-degenerate case

The following statement is already known for $(*)$ (see [5]). To prove Theorem 2, we will consider in addition to (A.1)–(A.4) that

$$\text{(A}^*\text{)} \qquad g \in L^{\alpha+2}(\Omega) \quad \text{and} \quad u \in BC((0,\infty), H_0^1(\Omega) \cap L^{\alpha+2}(\Omega)).$$

Now we can state our result:

Theorem 2. *Assume that $0 < m_0 \leq m(\cdot) \leq M_0$ and that $\|f(\cdot,t)\|$ decays exponentially to zero as $t \to \infty$. Then for every solution of (1.1), $\|\dot{u}\|$, $\|\nabla u\|$ and $\|u\|_p \quad (1 \leq p \leq \alpha+2)$ decay exponentially to zero.*

Proof: We shall find bounds for $\dot{u}, A^{\frac{1}{2}}u$ and u by estimating the energy functional

$$E(t) = \|\dot{u}\|^2 + \int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr + \delta(\dot{u},u) + \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}, \tag{3.4}$$

where $\delta = \min\{2\lambda_1\sqrt{m_0}, \frac{\gamma}{2}\}$. This choice of δ ensures that E is non-negative. Indeed,

$$\begin{aligned} E(t) - \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} &\geq \|\dot{u}\|^2 + m_0\|A^{\frac{1}{2}}u\|^2 - \|\dot{u}\|^2 - \frac{\delta^2}{4}\|u\|^2 \\ &\geq (m_0 - \frac{\delta^2}{4{\lambda_1}^2})\|A^{\frac{1}{2}}u\|^2 \geq 0. \end{aligned}$$

Here we used the inequality

$$2|(u,v)| \leq 2\|u\|\|v\| \leq \beta\|u\|^2 + \frac{1}{\beta}\|v\|^2 \quad \text{for} \quad \beta > 0. \tag{3.5}$$

Now, we take the inner product of each term in (1.3) with $2\dot{u}$, and with u, respectively.

$$\frac{d}{dt}\|\dot{u}\|^2 + 2\gamma\|\dot{u}\|^2 + \frac{d}{dt}\int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr + \frac{2}{\alpha+2}\frac{d}{dt}\|u\|_{\alpha+2}^{\alpha+2} = 2(f,\dot{u}), \tag{3.6}$$

$$(\ddot{u},u) + \gamma(\dot{u},u) + m(\|A^{\frac{1}{2}}u\|^2)\|A^{\frac{1}{2}}u\|^2 + \|u\|_{\alpha+2}^{\alpha+2} = (f,u). \tag{3.7}$$

We differentiate E in order to build a first-order linear inequality.

$$\begin{aligned} E'(t) &= \frac{d}{dt}\|\dot{u}\|^2 + \frac{d}{dt}\int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr + \delta(\ddot{u},u) + \delta\|\dot{u}\|^2 + \frac{2}{\alpha+2}\frac{d}{dt}\|u\|_{\alpha+2}^{\alpha+2} \\ &= -(2\gamma-\delta)\|\dot{u}\|^2 + 2(f,\dot{u}) - \gamma\delta(\dot{u},u) \\ &\quad - \delta m(\|A^{\frac{1}{2}}u\|^2)\|A^{\frac{1}{2}}u\|^2 - \delta\|u\|_{\alpha+2}^{\alpha+2} + \delta(f,u). \end{aligned}$$

From (1.2) and (3.5) we obtain

$$2(f,\dot{u}) \leq \frac{2}{\gamma}\|f\|^2 + \frac{\gamma}{2}\|\dot{u}\|^2 \tag{3.8}$$

and

$$2\delta(f,u) \leq 2\frac{\delta}{\lambda_1}\|f\|\|A^{\frac{1}{2}}u\| \leq \frac{\delta}{m_0{\lambda_1}^2}\|f\|^2 + \delta m_0\|A^{\frac{1}{2}}u\|^2. \tag{3.9}$$

Using (3.8) and (3.9) and the boundedness of m, we obtain

$$\begin{aligned}
E^{'}(t) &\leq (-2\gamma + \delta + \frac{\gamma}{2})\|\dot{u}\|^2 - \gamma\delta(\dot{u},u) + (\frac{2}{\gamma} + \frac{\delta}{2m_0{\lambda_1}^2})\|f\|^2 \\
&\quad - \frac{\delta m_0}{2}\|A^{\frac{1}{2}}u\|^2 - \delta\|u\|_{\alpha+2}^{\alpha+2} \\
&\leq -\gamma\|\dot{u}\|^2 - \gamma\delta(\dot{u},u) + (\frac{2}{\gamma} + \frac{\delta}{2m_0{\lambda_1}^2})\|f\|^2 \\
&\quad - \frac{\delta m_0}{2M_0}\int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr - \delta\|u\|_{\alpha+2}^{\alpha+2} \\
&\leq -\sigma E(t) + (\frac{2}{\gamma} + \frac{\delta}{2m_0{\lambda_1}^2})\|f\|^2,
\end{aligned}$$

where $\sigma = \min\{\gamma, \frac{\delta m_0}{2M_0}, \frac{\delta(\alpha+2)}{2}\}$. It is easy to see that

$$\frac{d}{dt}\{E(t)e^{\sigma t}\} \leq (\frac{2}{\gamma} + \frac{\delta}{2m_0{\lambda_1}^2})e^{\sigma t}\|f\|^2.$$

From this inequality and the Fundamental Theorem of Calculus, it follows that

$$E(t) \leq e^{-\sigma t}\Big\{E(0) + (\frac{2}{\gamma} + \frac{\delta}{2m_0{\lambda_1}^2})\int_0^t e^{\sigma s}\|f(\cdot,s)\|^2\,ds\Big\}.$$

Also, (A^*) implies that $E(0) \leq C$. From the assumption that $\|f(\cdot,s)\|^2 \leq Ce^{-as}$ $(a > 0)$ we obtain

$$0 \leq E(t) \leq Ce^{-\sigma t}\Big\{1 + \int_0^t e^{(\sigma-a)s}\,ds\Big\} \leq Ke^{-at},$$

where $a < \sigma$. Therefore,

$$\|\dot{u}\|^2 \leq Ke^{-at}, \quad \|A^{\frac{1}{2}}u\|^2 \leq \frac{K}{m_o}e^{-at}, \quad \|u\|_p \leq \sqrt{[}\alpha+2]\frac{K(\alpha+2)}{2}e^{-at}, \quad \forall\, t \geq 0,$$

with $1 \leq p \leq \alpha + 2$, which concludes this proof. $\square$

Remark 1. Theorem 2 also holds if $0 < m_0 \le m(\cdot)$ is non-decreasing on $[0, \infty)$. In this case

$$-\delta m(\|A^{\frac{1}{2}}u\|^2)\|A^{\frac{1}{2}}u\|^2 + \frac{\delta m_0}{2}\|A^{\frac{1}{2}}u\|^2 \le -\frac{\delta}{2}\int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr,$$

and so

$$E'(t) \le -\sigma E(t) + (\frac{2}{\gamma} + \frac{\delta}{2m_0{\lambda_1}^2})\|f\|^2,$$

where $\sigma = \min\{\gamma, \frac{\delta}{2}, \frac{\delta(\alpha+2)}{2}\}$.

Remark 2. The order of exponential decay approaches zero as the lower bound m_0 approaches zero. This is so because the constant a satisfies

$$a < \sigma \le \frac{\lambda_1 m_0^{\frac{3}{2}}}{M_0}\,.$$

Also, the smaller the effect of the damping term is, the faster the order of exponential decay approaches zero.

4. Decay in the degenerate case

Exponential decay has been shown in [5] for equation of the type $\ddot{u} + \gamma\dot{u} = f$. We will show exponential decay for $\alpha = 0$, $m \equiv 0$ and asymptotic stability for $\alpha \ge 0$, $f \equiv 0$ (for $\alpha = 2$, $f \equiv 0$ a stronger result is known — exponential decay has been shown in [2]). As far as we know, there is no result concerning asymptotic behavior of the global strong solution when $f \not\equiv 0$ and α is more general.

Theorem 3. *Assume that $m \equiv 0$ and that $\|f(\cdot,t)\|$ decays exponentially to zero as $t \to \infty$. Then for any solution of* (1.1), *$\|\dot{u}\|$ and $\|u\|$ decay exponentially to zero.*

Proof: Since $\alpha = 0$, $m \equiv 0$, equation (1.1) reduces to $\ddot{u} + \gamma\dot{u} + u = f$, which is a very simple case of type of equations studied in [4]. Here existence of a strong global solution is shown under condition that $g \in H_0^1(\Omega)$ and $h \in L^2(\Omega)$. We shall consider the energy functional

$$E(t) = \|\dot{u}\|^2 + \delta(\dot{u}, u) + \|u\|^2, \tag{4.10}$$

where $\delta = \min\{\frac{\gamma}{2}, \frac{1}{2}\}$. Just as in Theorem 2, this choice of δ ensures that E is non-negative. In fact,

$$E(t) \ge \|\dot{u}\|^2 - \|\dot{u}\|^2 - \frac{\delta^2}{4}\|u\|^2 + \|u\|^2 = (1 - \frac{\delta^2}{4})\|u\|^2 \ge 0.$$

Also, we take the inner product of each term in (1.3) with $2\dot{u}$, and with u, respectively.

$$\frac{d}{dt}\|\dot{u}\|^2 + 2\gamma\|\dot{u}\|^2 + \frac{d}{dt}\|u\|^2 = 2(f,\dot{u}), \tag{4.11}$$

$$(\ddot{u}, u) + \gamma(\dot{u}, u) + \|u\|^2 = (f, u). \tag{4.12}$$

Now, we differentiate E and make use of (4.11) and (4.12).

$$\begin{aligned} E^{'}(t) &= \frac{d}{dt}\{\|\dot{u}\|^2 + \|u\|^2\} + \delta(\ddot{u}, u) + \delta\|\dot{u}\|^2 \\ &= (-2\gamma + \delta)\|\dot{u}\|^2 + 2(f,\dot{u}) - \gamma\delta(\dot{u}, u) - \delta\|u\|^2 + \delta(f, u). \end{aligned}$$

From (3.5) we obtain

$$\delta(f, u) \le \frac{\delta}{2}\|f\|^2 + \frac{\delta}{2}\|u\|^2. \tag{4.13}$$

Using (3.8) and (4.13), one obtains

$$\begin{aligned} E^{'}(t) &\le -\gamma\|\dot{u}\|^2 - \frac{\delta}{2}\|u\|^2 - \gamma\delta(\dot{u}, u) + (\frac{\delta}{2} + \frac{2}{\gamma})\|f\|^2 \\ &\le -\sigma E(t) + (\frac{2}{\gamma} + \frac{\delta}{2})\|f\|^2, \end{aligned}$$

where $\sigma = \min\{\gamma, \frac{\delta}{2}\}$. From this first-order linear inequality and the assumption that $\|f(\cdot, s)\|$ decays exponentially, follows the existence of positive constants C, b, such that $E(t) \le Ce^{-bt}$, with $b < \sigma$. Hence,

$$\|\dot{u}\|^2 \le Ce^{-bt}, \quad \|u\|^2 \le Ce^{-bt}, \quad \forall\, t \ge 0,$$

and the proof is complete. □

Remark 3. Notice that the order of exponential decay $b < \frac{1}{4}$. From the physics point of view, Theorems 2 and 3 state that the energy $\|\dot{u}\|^2 + \|\nabla u\|^2 + \|u\|^2$ decays as time goes by.

Theorem 4. *Consider the mixed problem*

$$\begin{gathered} \ddot{u} + \gamma\dot{u} - m(\|\nabla u\|^2)\Delta u + \delta|u|^\alpha u = f(x,t) \quad \text{for} \quad x \in \Omega, \quad t \ge 0, \\ u(x, 0) = g(x), \quad \dot{u}(x, 0) = h(x) \quad \text{for} \quad x \in \Omega, \\ u(x, t) = 0 \quad \text{for} \quad x \in \partial\Omega, \quad t \ge 0, \end{gathered} \tag{4.14}$$

where Ω, γ, α *are as in* (1.1) *and* m *is a continuous function on* $[0,\infty)$ *satisfying* $0 \le m(\cdot) \le M_0$. *We take initial data such that*

$$g \in H_0^1(\Omega) \cap L^{\alpha+2}(\Omega) \quad \text{and} \quad h \in L^2(\Omega).$$

Assume that there exists a solution in class

$$u \in C((0,\infty), H_0^1(\Omega) \cap L^{\alpha+2}(\Omega)) \cap L^2((0,\infty), H^2(\Omega)) \tag{4.15}$$

$$\dot{u} \in C((0,\infty), L^2(\Omega)) \cap L^2((0,\infty), H_0^1(\Omega)) \tag{4.16}$$

$$\ddot{u} \in L^2((0,\infty), L^2(\Omega)). \tag{4.17}$$

Define the energy associated to the solution of (4.14) *by the following formula*

$$E(t) = \frac{1}{2}\|\dot{u}\|^2 + \int_0^{\|\nabla u\|^2} m(r)\,dr + \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}.$$

It holds that

$$E(t) \to 0, \quad \text{as} \;\; t \to \infty,$$

for every solution u *satisfying* (4.15)–(4.17).

Proof: It is easy to see that

$$\dot{E}(t) = (\ddot{u}, \dot{u}) + m(\|\nabla u\|^2)(\nabla u, \nabla \dot{u}) + (|u|^\alpha u, \dot{u}) = -\gamma\|\dot{u}\|^2 \le 0.$$

Hence, $E(t)$ is a non-increasing continuous function on $[0,\infty)$ and so

$$E(t) \le E(0) \;\; \text{for} \;\; t \ge 0.$$

For the proof we need two lemmas.

Lemma 1. *It holds that*

$$\int_0^t \|\dot{u}(s)u(s)\|_1\,ds = O(t^{\frac{1}{2}}), \quad t \to \infty.$$

Proof of Lemma 1: We have

$$\int_0^t \|\dot{u}(s)u(s)\|_1 \le \int_0^t \|u(s)\|\|\dot{u}(s)\|\,ds \le (\int_0^t \|\dot{u}(s)\|^2\,ds)^{\frac{1}{2}}\sqrt{t}\sup_{[0,t]}\|u(s)\|.$$

It also holds that

$$\int_0^t \|\dot{u}(s)\|^2\,ds = -\frac{1}{\gamma}(E(t) - E(0)) \le \frac{1}{\gamma}E(0)$$

and

$$\|u(s)\| \le C\|u(s)\|_{\alpha+2} \le C\sqrt{[\alpha+2]E(0)}.$$

From these inequalities it follows that

$$\int_0^t \|\dot u(s)u(s)\|_1\, ds \le C\sqrt{t} = O(t^{\frac{1}{2}}), \quad t \to \infty.$$

□

Lemma 2. *It holds that*

$$\int_0^t \|\dot u(s)\|^2\, ds = O(t^{\frac{1}{2}}), \quad t \to \infty.$$

Proof of Lemma 2: We have

$$\int_0^t \|\dot u(s)\|^2\, ds = \int_0^t \|\dot u(s)\|\|\dot u(s)\|\, ds \le \left(\int_0^t \|\dot u(s)\|^2\, ds\right)^{\frac{1}{2}} \sqrt{t} \sup_{[0,t]} \|\dot u(s)\|.$$

Since $\|\dot u(s)\| \le \sqrt{2E(0)}$, it follows that

$$\int_0^t \|\dot u(s)\|^2\, ds \le C\sqrt{t} = O(t^{\frac{1}{2}}), \quad t \to \infty.$$

□

Proof of Theorem 4: Assume on the contrary that $l = \lim_{t\to\infty} E(t) > 0$. We shall prove that

$$\|\dot u\|^2 + \int_0^{\|\nabla u\|^2} m(r)\, dr + \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} \ge \bar C > 0. \tag{4.18}$$

We have

$$\|\dot u\|^2 + 2\int_0^{\|\nabla u\|^2} m(r)\, dr + \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} \ge l.$$

If $\|\dot u\|^2 \ge \frac{l}{2}$, then (4.18) is satisfied with $\bar C = \frac{l}{2}$. Conversely, if $\|\dot u\|^2 \le \frac{l}{2}$, then

$$2\int_0^{\|\nabla u\|^2} m(r)\, dr + \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2} \ge \frac{l}{2},$$

and so (4.18) is satisfied with $\bar{C} = \frac{l}{4}$. Now,

$$B(t) = (u, \dot{u}) = \int_0^t \|\dot{u}\|^2 - \gamma(\dot{u}, u) - m(\|\nabla u\|^2)\|\nabla u\|^2 - \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}\, ds$$
$$\leq \int_0^t \{2\|\dot{u}\|^2 - \gamma(\dot{u}, u) - \bar{C}\}\, ds \to -\infty \quad \text{as} \quad t \to \infty.$$

This is a contradiction to the fact that $|B(t)| \leq C\{E(0) + \sqrt{[}\alpha + 2]E^2(0)\}$. Therefore,

$$\lim_{t\to\infty} E(t) = 0.$$

□

Remark 4. The proof of Theorem 4 was inspired by the paper of M. Aassila (see [1]). He has shown asymptotic stability for equations of the type $\ddot{u} - \operatorname{div}((1 + |\nabla u|^a)^b |\nabla u|^{c-2}\nabla u) + g(\dot{u}) = 0$, where the nonlinear function g satisfies certain conditions.

Example. This example shows that in case $\alpha = 0$ decay of solutions is not necessarily exponential. J.G. Dix has shown a similar statement for the equation $\ddot{u} + \dot{u} - m(\|u_x\|^2)u_{xx} = 0$ (see [5]). For $\alpha > 0$ we are unable to give a counter-example. Consider the initial value problem

$$\ddot{u} + 3\dot{u} - m(\|u_x\|^2)u_{xx} + u = 0, \quad \text{for} \quad 0 \leq x \leq \pi, \quad t \geq 1,$$
$$u(x, 1) = \sqrt{\frac{2}{\pi}}\sin x, \quad \dot{u}(x, 1) = -\sqrt{\frac{2}{\pi}}\sin x,$$
$$u(0, t) = 0, \quad u(\pi, t) = 0 \quad \text{for} \quad t \geq 1,$$

where

$$m(r) = -2r + 3\sqrt{r} - 1 \quad \text{if} \quad \frac{1}{4} \leq r \leq 1$$
$$\text{and} \quad m(r) = 0 \quad \text{otherwise.}$$

Then $u(x, t) = \sqrt{\frac{2}{\pi}}\sin x$ is a solution. Since

$$\dot{u} = -\frac{u}{t}, \quad \ddot{u} = \frac{2u}{t^2}, \quad u_x = \sqrt{\frac{2}{\pi}}\frac{\cos x}{t}, \quad u_{xx} = -u,$$

$\|u_x\|^2 = \frac{1}{t^2}$ and $m(\frac{1}{t^2}) = -\frac{2}{t^2} + 3\frac{1}{t} - 1$ for $t \geq 1$, it is easy to check that u satisfies the initial-value problem. Notice that $\|\dot{u}\|$ decays polynomially rather than exponentially as $t \to \infty$. Indeed,

$$\|\dot{u}\|^2 = \| - \frac{1}{t}u\|^2 = \frac{\pi}{2t^2} = O(t^{-2}).$$

5. Extension of results to equations with Δ^2

In this section we extend the previous analysis to equations that involve the bi-harmonic operator which appears in modeling non-planar vibrations, for example when studying the swaying of bridges. Consider the problem

$$\begin{gathered} \ddot{u} + \gamma\dot{u} + \epsilon\Delta^2 u - m(\|\nabla u\|^2)\Delta u = f(x,t), \quad \text{for} \quad x \in \Omega, \quad t \geq 0, \\ u(x,0) = g(x), \quad \dot{u}(x,0) = h(x) \quad \text{for} \quad x \in \Omega, \end{gathered} \tag{5.19}$$

with boundary conditions corresponding to hinged ends

$$u(x,t) = \Delta u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega, \quad t \geq 0.$$

Here Ω, γ, m are as in (1.1) and ϵ is a positive constant.

When $f \equiv 0$, asymptotic behavior of solutions to a more general problem has been studied by M. Aassila (see [1]). He considered a similar problem with a nonlinearity $g(\dot{u})$ and that $m(r) \geq \alpha + \beta r \quad (\alpha, \beta > 0)$.

Our aim was to make use of the remark in [4, Section 5] and to show on a particular example that the previous theorems can be extended to equations with higher powers of A. In Theorem 5 we show that $\|\dot{u}\|$ is square integrable on $[0,\infty)$ and in Theorem 6 we introduce an analogous statement with Theorem 2.

Define the positive, self-adjoint bi-harmonic operator as the second power of the negative Laplacian, with domain

$$D(A^2) = \{u \in H^2(\Omega) : u, Au \in D(A)\}.$$

In fact, $u \in D(A)$ and $Au \in D(A)$ imply that $u = 0$ and $\Delta u = 0$ on $\partial\Omega \times [0,\infty)$. Notice that the spectrum for A^2 is given by

$$\sigma(A^2) = \{\lambda^4; \lambda^2 \in \sigma(A)\}$$

and that (5.19) can be written as a nonlinear evolution equation in $L^2(\Omega)$:

$$\begin{gathered} \ddot{u} + \gamma\dot{u} + \epsilon A^2 u + m(\|A^{\frac{1}{2}}u\|^2)Au = f(x,t), \\ u(x,0) = g(x), \quad \dot{u}(x,0) = h(x). \end{gathered} \tag{5.20}$$

In what follows we take initial data such that

$$g \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{and} \quad h \in L^2(\Omega).$$

Under solution on $[0,T)$ we mean a function u satisfying (5.20) and

$$u \in C((0,T), D(A^2)) \cap C^1((0,T), L^2(\Omega)).$$

Now we are ready to state our results:

Theorem 5. *If* $\|f(\cdot,t)\|$ *is square integrable on* $[0,\infty)$, *and* u *is a solution to* (5.19), *then* $\|\dot{u}\|$ *is square integrable on* $[0,\infty)$.

PROOF: We will estimate the growth of the energy functional

$$E(t) = \|\dot{u}\|^2 + \epsilon\|Au\|^2 + \int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr. \tag{5.21}$$

Taking the inner product of each term in (5.20) with $2\dot{u}$ implies

$$\frac{d}{dt}\left(\|\dot{u}\|^2 + \epsilon\|Au\|^2 + \int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr\right) + 2\gamma\|\dot{u}\|^2 = 2(f,\dot{u}).$$

Using (3.5), it follows that the derivative of E satisfies

$$E^{'}(t) = -2\gamma\|\dot{u}\|^2 + 2(f,\dot{u}) \le -\gamma\|\dot{u}\|^2 + \frac{1}{\gamma}\|f\|^2.$$

From this inequality, we obtain

$$E(t) + \gamma\int_0^t \|\dot{u}\|^2\,ds \le E(0) + \frac{1}{\gamma}\int_0^t \|f(\cdot,s)\|^2\,ds.$$

Since $\int_0^\infty \|f\|^2 < \infty$, it follows that $\int_0^\infty \|\dot{u}\|^2 < \infty$, which concludes the proof. □

Remark 5. In Theorem 5 there is no assumption regarding the degenerate and non-degenerate cases of m. The previous statement also remains valid for problem (1.1). In this case the energy functional is just as in Theorem 4.

Theorem 6. *Assume that* $0 < m_0 \le m(\cdot) \le M_0$, *and that* $\|f(\cdot,t)\|$ *decays exponentially to zero as* $t \to \infty$. *Then for any solution of* (5.19), $\|\dot{u}\|$, $\|\nabla u\|$, $\|\Delta u|$, $\|u\|$ *decay exponentially to zero.*

PROOF: When proving this statement, the energy functional (5.21) is redefined to be

$$E(t) = \|\dot{u}\|^2 + \epsilon\|Au\|^2 + \int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\,dr + \delta(\dot{u},u), \tag{5.22}$$

where $\delta = \min\{2\lambda_1^2\sqrt{\epsilon}, \frac{\gamma}{2}\}$. When choosing δ, we used (1.2) to ensure that E is non-negative. Also, we take the inner-product of each term in (5.20) with $2\dot{u}$, and with $2u$, respectively. Substituting the two equations into $E^{'}(t)$ yields

$$E^{'}(t) = 2(f,\dot{u}) + (-2\gamma+\delta)\|\dot{u}\|^2 - \delta\gamma(\dot{u},u) - \delta\epsilon\|Au\|^2 - \delta m(A^{\frac{1}{2}}u)\|A^{\frac{1}{2}}u\|^2 + \delta(f,u).$$

Using (3.8) and (3.9) and the boundedness of m, we obtain

$$E^{'}(t) \le -\sigma E(t) + (\frac{2}{\gamma} + \frac{\delta}{2m_0\lambda_1^2})\|f\|^2,$$

where $\sigma = \min\{\gamma, \frac{\delta m_0}{2M_0}, \delta\}$. The rest of the proof is the same as in Theorem 2. Hence, there exist positive constants C, c, such that

$$\|\dot{u}\|^2 \le Ce^{-ct}, \quad \|Au\|^2 \le \frac{C}{\epsilon}e^{-ct} \quad \forall\, t \ge 0.$$

Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $m_0 > 0$, $\|u\|$ and $\|\nabla u\|$ also decay exponentially to zero. □

Remark 6. Theorem 6 also holds if m is non-decreasing and $m(\cdot) \ge 0$ on $[0, \infty)$. In fact, in this case

$$-\delta m(\|A^{\frac{1}{2}}u\|^2)\|A^{\frac{1}{2}}u\|^2 \le -\delta \int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\, dr.$$

Instead of (3.9) we take

$$2\delta(f, u) \le 2\frac{\delta}{\lambda_1^2}\|f\|\|Au\| \le \frac{\delta}{\epsilon{\lambda_1}^4}\|f\|^2 + \delta\epsilon\|Au\|^2.$$

Using (3.8) and the two inequalities above, we obtain

$$E^{'}(t) \le -\sigma E(t) + \left(\frac{2}{\gamma} + \frac{\delta}{2\epsilon{\lambda_1}^4}\right)\|f\|^2,$$

where $\sigma = \min\{\gamma, \delta, \frac{\delta\epsilon}{2}\}$. Note that the difference between Remark 1 and Remark 6 is that here m is allowed attain zero values. It evokes the idea that adding powers of A into $(*)$ “improves” the asymptotic stability. Since $m(\cdot) \ge 0$, exponential decay of $\|\nabla u\|$ and $\|u\|$ follows from the inequality

$$\|u\|_{H^2(\Omega)} \le C\|\Delta u\| \quad \text{for} \quad u \in H_0^1(\Omega) \cap H^2(\Omega).$$

Remark 7. Notice that if $\epsilon \to 0_+$, then the order of exponential decay approaches zero. An analogous remark with Remark 2 also holds.

Remark 8. Theorems 5 and 6 can be applied to equations of the type

$$\ddot{u} + \gamma\dot{u} + \kappa u + \epsilon\Delta^2 u - m(\|\nabla u\|^2)\Delta u = f(x, t),$$

where γ, κ, ϵ are positive constants. In this case the energy functional (5.21) is redefined to be

$$E(t) = \|\dot{u}\|^2 + \kappa\|u\|^2 + \epsilon\|\Delta u\|^2 + \int_0^{\|A^{\frac{1}{2}}u\|^2} m(r)\, dr.$$

We obtain the same results.

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Department of Mathematical Analysis, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovak Republic

E-mail: Szomolay@fmph.uniba.sk

(*Received* July 1, 2002, *revised* December 3, 2002)