Estimation functions and uniformly most powerful tests for inverse Gaussian distribution

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Abstract. The aim of this article is to develop estimation functions by confidence regions for the inverse Gaussian distribution with two parameters and to construct tests for hypotheses testing concerning the parameter λ when the mean parameter μ is known. The tests constructed are uniformly most powerful tests and for testing the point null hypothesis it is also unbiased.

Keywords: inverse Gaussian distribution, estimation functions, uniformly most powerful test, unbiased test

Classification: 62F03, 62F25

1. Introduction

The *inverse Gaussian* distribution was first derived by Schrodinger (1915) in connection to the first hitting time in Brownian motion. In statistics it was derived by Wald (1947) for sequential testing, by Hadwiger (1940) and Tweedie (1957). Some monographs dedicated to this subject are Chhikara and Folks (1989), Seshadri (1994) and Seshadri (1999).

A bivariate inverse Gaussian distribution is investigated in Essam and Nagi (1981). Although in the literature there are several goodness-of-fit tests, see Edgeman et al. (1988), O'Reilly and Rueda (1992), Pavur et al. (1992), Mergel (1999) and Henze and Klar (2001), and some other empirical distribution function tests such as *Kolmogorov-Smirnov* test, the *Cramer-von Mises* test, the *Anderson-Darling* test and the *Watson* test have been investigated in Gunes et al. (1997), there are no uniformly most powerful tests developed for testing in the inverse Gaussian context.

The inverse Gaussian distribution has many applications in actuarial statistics (for example Ter Berg 1980, 1984) and it has been also used lately in mathematical finance due to its useful properties such as closure under convolution and flexibility in modeling positively-skewed and leptokurtic sets of data.

The main aim of this paper is to propose estimation functions by confidence regions and some uniformly most powerful tests for the λ parameter when the mean parameter μ is known. Various point, unidirectional and bidirectional tests are considered for testing hypotheses.

All probability density functions in this paper are considered with respect to the Lebesgue measure on the relevant metric space, most often this is $\Re$ the set of real numbers. For any random variable g we denote by F_g the cumulative distribution function of g and by ρ_g the probability density function of g.

The inverse Gaussian distribution $G_{\lambda,\mu}$ has the following probability density function

$$\rho(x:\lambda,\mu)=\left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}}\exp\left[-\frac{\lambda}{2\mu^2}\frac{(x-\mu)^2}{x}\right]1_{(0,\infty)}(x).$$

Under this parameterization, the inverse Gaussian distribution has mean μ and variance μ^3/λ. Its shape is modeled by the value of λ/μ. The cumulant generating function is the inverse of that of the normal or Gaussian distribution and this is the reason for the name of this distribution, the *inverse Gaussian*.

The next results are useful to prove the main results of this paper.

Theorem 1. *Let $(\Omega,\mathcal{F},P)$ be a probability space and $f:\Omega\longrightarrow\Re$ be a random variable having the distribution $G_{\lambda,\mu}$. Then*

(a) *$cf\sim G_{c\lambda,c\mu}$ for any $c>0$;*

(b) *$\frac{\lambda}{\mu^2}\frac{(f-\mu)^2}{f}\sim\chi^2(1)$, where $\chi^2(1)$ is the chi-square distribution with one degree of freedom.*

The first point was proved in Tweedie (1957) while the second can be found in Shuster (1968).

For any positive integer n the cumulative distribution function of the chi-square distribution $\chi^2(n)$ is denoted by $F_n(\cdot)$.

Using the characteristic function it can be easily shown that if $f_1,\ldots,f_n$ are random variables independent and identically distributed with distribution $G_{\lambda,\mu}$ then

$$\frac{1}{n}\sum_{i=1}^{n}f_i\sim G_{n\lambda,\mu}. \tag{1}$$

Consider the statistical model given by

$$\Big((0,\infty),\mathcal{B}_{(0,\infty)},\{G_{\lambda,\mu}\mid\lambda,\mu>0\}\Big)^{(n)}. \tag{2}$$

The mappings

$$\mathrm{pr}_i:(0,\infty)^n\longrightarrow(0,\infty)$$

defined by $\mathrm{pr}_i(x^{(n)})=x_i$ for any $x^{(n)}=(x_1,\ldots,x_n)\in(0,\infty)^n$ and any $i=1,\ldots,n$ are independent, identically distributed with distribution $G_{\lambda,\mu}$.

Theorem 1 above implies that

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{pr}_i \sim G_{n\lambda,\mu}. \tag{3}$$

Applying then the second point of Theorem 1 we get that

$$n\lambda \frac{\left(\frac{1}{n\mu}\sum_{i=1}^{n} \mathrm{pr}_i - 1\right)^2}{\frac{1}{n}\sum_{i=1}^{n} \mathrm{pr}_i} \sim \chi^2(1). \tag{4}$$

Next we need to define the functions $\pi_n(\cdot;\cdot) : (0,\infty)^n \times (0,\infty) \to (0,\infty)$ by

$$\pi_n(x^{(n)};\lambda) = \frac{n\lambda}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2,$$

where $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$. In other words,

$$\pi_n(\cdot;\lambda) = n\lambda \frac{\left(\frac{1}{n\mu}\sum_{i=1}^{n} \mathrm{pr}_i - 1\right)^2}{\frac{1}{n}\sum_{i=1}^{n} \mathrm{pr}_i} \tag{5}$$

for any $\lambda > 0$.

2. Main results for estimation functions

In this section we are preparing the way to the main results providing confidence regions and uniformly most powerful tests.

Lemma 1. *Let n be a positive integer and μ be a positive real number. Then*

$$\pi_n(\cdot;\lambda) > 0, \quad G^n_{\lambda,\mu} - \text{ a.e.} \tag{6}$$

for any $\lambda > 0$.

Proof: Obviously $\pi_n(x^{(n)};\lambda) \geq 0$ for any $x^{(n)} \in (0,\infty)^n$ and $\lambda > 0$. In addition

$$\begin{aligned} G^n_{\lambda,\mu}(\{x^{(n)} \in (0,\infty)^n \mid \pi_n(x^{(n)};\lambda) = 0\}) &= \left(G^n_{\lambda,\mu} \circ (\pi_n(\cdot;\lambda))^{-1}\right)(\{0\}) \\ &= \chi^2(1)(\{0\}) = 0. \end{aligned}$$

Similarly with the construction above, if n is a positive integer and λ is a positive real number we can define $\tilde{\pi}_n(\cdot;\cdot) : (0,\infty)^n \times (0,\infty) \to (0,\infty)$ by

$$\tilde{\pi}_n(x^{(n)};\mu) = \frac{n\lambda}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2$$

where $\bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i$. Once again

$$\tilde{\pi}_n(\cdot;\mu) = n\lambda \frac{\left(\frac{1}{n\mu}\sum_{i=1}^n \mathrm{pr}_i - 1\right)^2}{\frac{1}{n}\sum_{i=1}^n \mathrm{pr}_i} \sim \chi^2(1) \tag{7}$$

for any $\mu > 0$. □

Theorem 2. *Let* $\Big((0,\infty), \mathcal{B}_{(0,\infty)}, \{G_{\lambda,\mu} \mid \lambda,\mu > 0\}\Big)^{(n)}$ *be a statistical model.*

(a) *If* $\mu > 0$ *and* $\alpha \in (0,1)$ *are known and* $0 < u < v$ *such that* $\chi^2(1)([u,v]) = 1-\alpha$ *then the mapping* $\delta_n : (0,\infty)^n \longrightarrow 2^{(0,\infty)}$ *defined as*

$$\delta_n(x^{(n)}) = \{\lambda > 0 \mid u \le \pi_n(x^{(n)};\lambda) \le v\} \tag{8}$$

is an estimation function by confidence regions at the level of significance $1-\alpha$ *for the parameter* λ.

(b) *If* $\lambda > 0$ *and* $\alpha \in (0,1)$ *are known and* $0 < u < v$ *are real numbers such that* $\chi^2(1)([u,v]) = 1-\alpha$ *then the mapping* $\tilde{\delta}_n : (0,\infty)^n \longrightarrow 2^{(0,\infty)}$ *defined as*

$$\tilde{\delta}_n(x^{(n)}) = \{\mu > 0 \mid u \le \tilde{\pi}_n(x^{(n)};\mu) \le v\} \tag{9}$$

is an estimation function by confidence regions at the level of significance $1-\alpha$ *for the parameter* μ.

Proof: (a) From (5) it follows that $\pi_n(\cdot;\lambda)$ is $(\mathcal{B}_{(0,\infty)^n}, \mathcal{B}_{(0,\infty)})$-measurable while (4) implies that $G^n_{\lambda,\mu} \circ (\pi_n(\cdot;\lambda)^{-1} = \chi^2(1)$ for any $\lambda > 0$. Thus $\pi_n(\cdot;\cdot)$ is a pivotal function for the parameter λ.

Taking into account that $\chi^2([u,v]) = 1-\alpha$ we conclude that $\delta_n(\cdot)$ is an estimation function by confidence regions at level of significance $1-\alpha$ for the parameter λ.

(b) The proof is similar with that for (a) replacing $\pi_n(\cdot;\lambda)$ by $\tilde{\pi}_n(\cdot;\mu)$.

Combining the above theorem with Lemma 1 we get that

$$\delta_n(x^{(n)}) = \left[\frac{\bar{x}_n}{n(\frac{1}{\mu}\bar{x}_n - 1)^2}u, \frac{\bar{x}_n}{n(\frac{1}{\mu}\bar{x}_n - 1)^2}v\right], \quad G^n_{\lambda,\mu} - \text{ a.e.}$$

Let $h_{n;\beta}$ be the quantile of order β for the $\chi^2(n)$ distribution and $\alpha_1, \alpha_2 \in (0,1)$ such that $\alpha_1 + \alpha_2 = \alpha$. Then

$$\bar{\delta}_n(x^{(n)}) = \left[\frac{\bar{x}_n}{n(\frac{1}{\mu}\bar{x}_n - 1)^2} h_{1;\alpha_1}, \frac{\bar{x}_n}{n(\frac{1}{\mu}\bar{x}_n - 1)^2} h_{1;1-\alpha_2} \right], \quad G^n_{\lambda,\mu} - \text{ a.e.}$$

provides an estimation method by confidence regions at the level of confidence α for the parameter λ.

Moreover, the above theorem may be used to conclude that the mapping $\bar{\delta}^*_n : (0,\infty)^n \longrightarrow 2^{(0,\infty)}$ defined as

$$\begin{aligned} \bar{\delta}^*_n = \quad & \left[\frac{1}{\bar{x}_n}\left(1 - \sqrt{\frac{v\bar{x}_n}{n\lambda}}\right), \frac{1}{\bar{x}_n}\left(1 - \sqrt{\frac{u\bar{x}_n}{n\lambda}}\right) \right] \cup \\ & \left[\frac{1}{\bar{x}_n}\left(1 + \sqrt{\frac{u\bar{x}_n}{n\lambda}}\right), \frac{1}{\bar{x}_n}\left(1 + \sqrt{\frac{v\bar{x}_n}{n\lambda}}\right) \right] \end{aligned}$$

provides an estimation method by confidence regions at level of significance $1-\alpha$ for the parameter $\frac{1}{\mu}$. $\square$

3. Preliminary results for testing hypotheses

Lemma 2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, λ a positive real number and $f : \Omega \to \Re$ a random variable such that $\lambda f \sim \chi^2(1)$. Then, the probability density function of the random variable f is*

$$\rho(x;\lambda) = \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} \exp\left(-\frac{\lambda x}{2}\right) 1_{(0,\infty)}(x). \tag{10}$$

Proof:

$$\begin{aligned} F_f(x) &= P(f < x) = P(\lambda f < \lambda x) 1_{(0,\infty)}(x) \\ &= \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} \exp\left(-\frac{\lambda x}{2}\right) 1_{(0,\infty)}(x). \end{aligned}$$

Consider the probability measure ν_λ on the set of real numbers $\Re$ having the probability density $\rho(\cdot;\lambda)$. For any positive parameter λ it is obvious then that $\mathrm{supp}(\nu_\lambda) = [0,\infty)$ and, if $T_\lambda : \Re \to \Re$ is a function defined as $T_\lambda(x) = \lambda x$, then $\nu_\lambda \circ T_\lambda^{-1} = \chi^2(1)$. Hence, if the random variable h has the distribution ν_λ then the random variable λh has the distribution $\chi^2(1)$. $\square$

Lemma 3. *Let ν_λ be a probability distribution having the probability density*

$$\rho(x;\lambda)=\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}}x^{-\frac{1}{2}}\exp\left(-\frac{\lambda x}{2}\right). \tag{11}$$

The statistical model $\Big((0,\infty),\mathcal{B}_{(0,\infty)},\{\nu_\lambda \mid \lambda>0\}\Big)$ *has a monotone likelihood ratio.*

PROOF: Consider $0<\lambda_1<\lambda_2$. Then

$$\frac{\rho(x;\lambda_2)}{\rho(x;\lambda_1)}=\left(\frac{\lambda_2}{\lambda_1}\right)^{1/2}\exp\left(-\frac{\lambda_2-\lambda_1}{2}x\right)=h_{\lambda_1,\lambda_2}(T(x)),$$

where $T:(0,\infty)\to\Re$, $\;T(x)=-x$ is a $(\mathcal{B}_{(0,\infty)},\mathcal{B}_{\Re})$-measurable function and the function $h_{\lambda_1,\lambda_2}:\Re\to\Re$ defined by

$$h_{\lambda_1,\lambda_2}(x)=\left(\frac{\lambda_2}{\lambda_1}\right)^{1/2}\exp\left(\frac{\lambda_2-\lambda_1}{2}x\right)$$

is increasing. □

4. Main results for testing hypotheses

Theorem 3. *Consider the statistical model*

$$\Big((0,\infty),\mathcal{B}_{(0,\infty)},\{G_{\lambda,\mu}\mid\lambda>0\}\Big)^{(n)} \tag{12}$$

with $\mu>0$ and n known. Let $\lambda_0>0$ be a fixed value of parameter λ and α a level of significance.

(a) *For testing the null hypothesis $H_0:\lambda\in(0,\lambda_0]$ versus the alternative $H_1:\lambda\in(\lambda_0,\infty)$, the pure test $\varphi_n=1_{C_n}$ with*

$$C_n=\left\{x^{(n)}\in(0,\infty)^n \mid \frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n-1\right)^2<\frac{h_{1;\alpha}}{\lambda_0}\right\} \tag{13}$$

is uniformly most powerful.

(b) *For testing the null hypothesis $H_0:\lambda\in[\lambda_0,\infty)$ versus the alternative $H_1:\lambda\in(0,\lambda_0)$, the pure test $\varphi_n=1_{C_n}$ with*

$$C_n=\left\{x^{(n)}\in(0,\infty)^n \mid \frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n-1\right)^2>\frac{h_{1;1-\alpha}}{\lambda_0}\right\} \tag{14}$$

is uniformly most powerful.

(c) *Let* $0 < \lambda_1 < \lambda_2$ *be some known values. For testing the hypothesis* $H_0 : \lambda \in (0, \lambda_1] \cup [\lambda_2, \infty)$ *versus* $H_1 : \lambda \in (\lambda_1, \lambda_2)$ *at the level of significance* α, *a uniformly most powerful test is the pure test* $\varphi_n = 1_{C_n}$ *where*

$$C_n = \left\{ x^{(n)} \in (0,\infty)^n \mid -\frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2 \in [c_1, c_2] \right\} \tag{15}$$

and $c_1 < 0$ *and* c_2 *are determined from the conditions*

$$F_1(-c_1\lambda_1) - F_1(-c_2\lambda_1) = \alpha,$$
$$F_1(-c_1\lambda_2) - F_1(-c_2\lambda_2) = \alpha.$$

(d) *Let* $0 < \lambda_1 < \lambda_2$ *be some known values. For testing the hypothesis* $H_0 : \lambda \in [\lambda_1, \lambda_2]$ *versus* $H_1 : \lambda \in (0, \lambda_1) \cup (\lambda_2, \infty)$ *at the level of significance* α, *a uniformly most powerful test is the pure test* $\varphi_n = 1_{C_n}$ *where*

$$\begin{aligned} C_n = \ & \left\{ x^{(n)} \in (0,\infty)^n \mid -\frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2 < c_1 \right\} \cup \\ & \left\{ x^{(n)} \in (0,\infty)^n \mid -\frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2 > c_2 \right\} \end{aligned} \tag{16}$$

and $c_1 < c_2 < 0$ *are determined from the conditions*

$$F_1(-c_1\lambda_1) - F_1(-c_2\lambda_1) = 1 - \alpha,$$
$$F_1(-c_1\lambda_2) - F_1(-c_2\lambda_2) = 1 - \alpha.$$

(e) *Let* $\lambda > 0$ *be some known value. For testing the hypothesis* $H_0 : \lambda = \lambda_0$ *versus* $H_1 : \lambda > \lambda_0$ *at the level of significance* α, *an unbiased, uniformly most powerful test is the pure test* $\varphi_n = 1_{C_n}$ *where*

$$\begin{aligned} C_n = \ & \left\{ x^{(n)} \in (0,\infty)^n \mid -\frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2 < c_1 \right\} \cup \\ & \left\{ x^{(n)} \in (0,\infty)^n \mid -\frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2 > c_2 \right\} \end{aligned} \tag{17}$$

and $c_1 < c_2 < 0$ *are determined from the conditions*

$$F_1(-c_1\lambda_0) - F_1(-c_2\lambda_0) = 1 - \alpha,$$
$$\frac{\partial}{\partial\lambda}\left(F_1(-c_1\lambda) - F_1(-c_2\lambda)\right)|_{\lambda=\lambda_0} = 0.$$

Proof: (a) Consider the probability density function ν_λ given in (11) and the statistical model

$$\left((0,\infty), \mathcal{B}_{(0,\infty)}, \{\nu_\lambda \mid \lambda > 0\}\right). \tag{18}$$

If $v_n : (0,\infty)^n \to (0,\infty)$ is an application defined by

$$v_n(x^{(n)}) = \frac{n}{\bar{x}_n}\left(\frac{1}{\mu}\bar{x}_n - 1\right)^2$$

then the above statistical model can be rewritten as

$$\left((0,\infty), \mathcal{B}_{(0,\infty)}, \{G^n_{\lambda,\mu} \circ v_n^{-1} \mid \lambda > 0\}\right).$$

Using Lemma 3 this statistical model has a monotone likelihood ratio with respect to the statistic $T(x) = -x$. Applying Lehmann's theorem (see Lehmann, 1959, for further details), we get that the pure test $\varphi_0 = 1_{C_0}$ with

$$C_0 = \{x > 0 \mid \pi(x) > c\}$$

where $c < 0$ is determined from the condition $\nu_{\lambda_0}(C_0) = \alpha$, is uniformly most powerful at the level of significance α for testing $H_0 : \lambda \in (0, \lambda_0]$ against the alternative $H_1 : \lambda \in (\lambda_0, \infty)$.

Observe that

$$\begin{aligned}
\alpha = \nu_{\lambda_0}(C_0) &= \nu_{\lambda_0}(\{x > 0 \mid -x > c\}) = \nu_{\lambda_0}(\{x > 0 \mid \lambda_0 x < -c\lambda_0\}) \\
&= \chi^2(1)(\{x > 0 \mid \lambda_0 x < -c\lambda_0\}) \\
&= F_1(-c\lambda_0).
\end{aligned}$$

Now, $-c\lambda_0 = h_{1;\alpha}$ or $c = -\frac{h_{1;\alpha}}{\lambda_0}$ and therefore

$$C_0 = \{x > 0 \mid x < \frac{h_{1;\alpha}}{\lambda_0}\}.$$

At the same time

$$\begin{aligned}
\alpha = \nu_{\lambda_0}(C_0) &= (G_{\lambda_0,\mu} \circ v_n^{-1})\left(\{x > 0 \mid x < \frac{h_{1;\alpha}}{\lambda_0}\}\right) \\
&= G^n_{\lambda_0,\mu}\left(\{x^{(n)} \in (0,\infty) \mid v_n(x^{(n)}) < \frac{h_{1;\alpha}}{\lambda_0}\}\right).
\end{aligned}$$

Thus, the uniformly most powerful critical region at the level of significance α, for testing the null hypothesis $H_0 : \lambda \in (0, \lambda_0]$ versus $H_1 : \lambda \in (\lambda_0, \infty)$ is given by (13).

(b) Starting with the statistical model (18) the pure test $\varphi_0 = 1_{C_0}$ where $C_0 = \{x > 0 \mid T(x) < c\}$ and c being determined from the condition $\nu_{\lambda_0}(C_0) = \alpha$ is uniformly most powerful at the level of significance α for testing $H_0 : \lambda \in [\lambda_0, \infty)$ versus $H_1 : \lambda \in (0, \lambda_0)$.

First remark that

$$\begin{aligned}\alpha &= \nu_{\lambda_0}(\{x > 0 \mid -x < c\}) = \nu_{\lambda_0}(\{x > 0 \mid \lambda_0 x > -c\lambda_0\})\\ &= \chi^2(1)(\{x > 0 \mid \lambda_0 x > -c\lambda_0\})\\ &= 1 - F_1(-c\lambda_0).\end{aligned}$$

This means that $-\lambda_0 c = h_{1;1-\alpha}$ or $c = -\frac{h_{1;1-\alpha}}{\lambda_0}$. Thus,

$$C_0 = \left\{ x > 0 \mid x > \frac{h_{1;1-\alpha}}{\lambda_0} \right\}.$$

From this point the proof continues similarly as in (a).

(c) The statistical model (18) is of exponential type since

$$\rho(x; \lambda) = c(\lambda) d(x) \exp\left(Q(\lambda) T(x)\right), \tag{19}$$

where $c(\lambda) = \left(\frac{\lambda}{2\pi}\right)^{1/2}$; $d(x) = x^{-\frac{1}{2}}$; $T(x) = -x$; $Q(\lambda) = \frac{\lambda}{2}$ for any $\lambda > 0$, $x > 0$. It is obvious that d and T are measurable and that Q is increasing, so using a theorem from Lehmann, 1959, p. 128, gives a uniformly most powerful test at the level of significance α for testing $H_0 : \lambda \in (0, \lambda_1] \cup [\lambda_2, \infty)$ versus $H_1 : \lambda \in [\lambda_1, \lambda_2]$. The test is $\varphi_0 = 1_{C_0}$ where $C_0 = \{x > 0 \mid c_1 < T(x) < c_2\}$ with c_1, c_2 being calculated from the conditions

$$\nu_{\lambda_1}(C_0) = \alpha, \quad \nu_{\lambda_2}(C_0) = \alpha. \tag{20}$$

Proceeding as in the proof of points (a), (b) we get that $c_1 < 0$ and that the equations (20) are equivalent to

$$\begin{aligned}F_1(-c_1\lambda_1) - F_1(-c_1\lambda_1) = \alpha,\\ F_1(-c_1\lambda_2) - F_1(-c_2\lambda_1) = \alpha.\end{aligned}$$

In addition

$$\begin{aligned}\alpha &= \nu_{\lambda_1}(C_0) = (G^n_{\lambda_1,\mu} \circ v_n^{-1})(\{x > 0 \mid -c_2 < x < -c_1\})\\ &= G^n_{\lambda_1,\mu}(\{x^{(n)} \in (0,\infty)^n \mid -c_2 < v_n(x(n)) < -c_1\})\\ &= G^n_{\lambda_1,\mu}(\{x^{(n)} \in (0,\infty)^n \mid c_1 < -v_n(x(n)) < c_2\})\end{aligned}$$

and analogously

$$\alpha = G^n_{\lambda_2,\mu}(\{x^{(n)} \in (0,\infty)^n \mid c_1 < -v_n(x(n)) < c_2\}).$$

For the statistical model $((0,\infty), \mathcal{B}_{(0,\infty)}, \{G_{\lambda,\mu} \mid \lambda > 0\})^{(n)}$ the result now follows.

(d) The proof is very similar to the above for (c) observing that Q is a continuous and increasing function and applying a well-known theorem from Lehmann, 1959.

(e) Starting again from the theorem from Lehmann, 1959, we can say that the pure test $\varphi_0 = 1_{C_0}$, where

$$C_0 = \{x > 0 \mid T(x) < c_1\} \cup \{x > 0 \mid T(x) > c_2\}$$

and c_1, c_2 are calculated from the conditions

$$\nu_{\lambda_0}(C_0) = \alpha \tag{21}$$

$$\frac{\partial}{\partial \lambda}\left(\nu_\lambda(C_0)\right)|_{\lambda=\lambda_0} = 0 \tag{22}$$

is uniformly most powerful and unbiased at the level of significance α for testing the null hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda > 0$. The first equation from (21) is equivalent to

$$F_1(-c\lambda_0) - F_1(-c_2\lambda_0) = 1 - \alpha.$$

Take into account that

$$\begin{aligned}
\nu_\lambda(C_0) &= \nu_\lambda(\{x > 0 \mid x > -c_1\} \cup \{x > 0 \mid x < -c_2\}) \\
&= \nu_\lambda(\{x > 0 \mid \lambda x > -\lambda c_1\}) + \nu_\lambda(\{x > 0 \mid \lambda x < -\lambda c_2\}) \\
&= \chi^2(1)(\{x > 0 \mid x > -\lambda c_1\}) + \chi^2(1)(\{x > 0 \mid x < -\lambda c_2\}) \\
&= F_1(-\lambda c_2) + 1 - F_1(-\lambda c_2).
\end{aligned}$$

Thus, the second equation from (21) is equivalent to

$$\frac{\partial}{\partial \lambda}\left(F_1(-c_2\lambda) - F_1(-c_1\lambda)\right)|_{\lambda=\lambda_0} = 0.$$

Similarly to the proof detailed for (a) above, we get the stated result. $\square$

5. Conclusion

The inverse Gaussian distribution has been used for many decades in actuarial statistics and it makes its way through mathematical finance. This distribution is a flexible positive-support probabilistic model with two parameters.

Although in the literature there are several goodness-of-fit tests and some other empirical distribution function tests such as *Kolmogorov-Smirnov* test, the *Cramer-von Mises* test, the *Anderson-Darling* test and the *Watson* test, there are no uniformly most powerful tests developed for testing in the inverse Gaussian context.

The theorems proved in this paper fill a gap in the literature about uniformly most powerful tests. The theoretical results proved here may be used for model selection, so making a useful link to the practical world of actuary and finance.

In addition, in the first part of the paper, estimation functions through confidence regions are constructed for the parameters of the inverse Gaussian distribution.

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(*Received* January 29, 2002)