# On reflexive subobject lattices and reflexive endomorphism algebras

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*Abstract.* In this paper we study the reflexive subobject lattices and reflexive endomorphism algebras in a concrete category. For the category **Set** of sets and mappings, a complete characterization for both reflexive subobject lattices and reflexive endomorphism algebras is obtained. Some partial results are also proved for the category of abelian groups.

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Given a bounded operator $p : H \longrightarrow H$ on a Hilbert space $H$, a closed subspace $B$ of $H$ is called an invariant subspace of $p$ if $p(B) \subseteq B$. The invariant subspace problem in functional analysis asks whether the set of all invariant closed subspaces of an operator can consist of only the two trivial subspaces of $H$. But, as Halmos pointed out, what people really would like to know is what the set of all invariant subspaces of an operator, or of a set of operators can look like (see [2]). The set of all invariant subspaces of a collection of operators is also called a *reflexive subspace lattice* by Halmos. In [3] he proved that if a subspace lattice $\mathcal{A}$ is an atomic Boolean algebra, then it is reflexive. Halmos's pioneering investigation has since inspired lots of interesting studies of the lattice characterizations of such subspace lattices. In 1975, Longstaff generalized Halmos's result by proving that $\mathcal{A}$ is reflexive if it is a completely distributive lattice (see [6]). However, complete distributivity is far from being necessary for reflexivity. In fact, by Example 3.1 of [6], a reflexive subspace lattice may even fail to be distributive. It is unlikely that one can find a necessary and sufficient intrinsic order condition for reflexive subspace lattices. See [7], [8] for more recent work on reflexive subspace lattices.

On the other hand, one can also consider the algebra $\mathbf{B}(H)$ of all bounded operators on the Hilbert space $H$. A subalgebra $\mathcal{F}$ of $\mathbf{B}(H)$ is called *reflexive* if there is a collection $\mathcal{A}$ of subspaces of $H$ such that $\mathcal{F} = \{p \in \mathbf{B}(H) : \forall A \in \mathcal{A}, p(A) \subseteq A\}$. There have also been extensive studies of reflexive subalgebras for Hilbert spaces (see e.g. [8]).

In the present paper we study the reflexive families mainly in the category **Set** of sets and mappings. In this case, a complete characterization for both reflexive endomorphism algebras and reflexive subobject lattices is obtained.

In Section 1 we define reflexive subobject lattice and reflexive endomorphism algebra for a general concrete category and list some basic properties of such structures. The second and the third section are about reflexive subset lattices and reflexive endomorphism algebras in the category of sets and mappings. The last section is devoted to the study of reflexive subgroup lattices. For this case we are only able to prove that a subgroup lattice $\mathcal{A}$ of an abelian group $G$ is reflexive if and only if there is a module structure on $G$, such that $\mathcal{A}$ is the set of all submodules of $G$.

## 1. Reflexive subobject lattices and reflexive endomorphism algebras

By [4, Definition 1.1], a concrete category is a category $\mathcal{C}$ whose objects are structured sets, that is, pairs $(X, \xi)$, where $X$ is a set and $\xi$ is a $\mathcal{C}$-structure on $X$. Its morphisms are suitable mappings between sets $X$ and $Y$ and the composition law is the usual composition of mappings. In other words, a concrete category is a category $\mathcal{C}$ together with a faithful functor (forgetful functor) from $\mathcal{C}$ to the category **Set** of sets (see also [9]). The category **Set** of sets and mappings, the category **Hspace** of Hilbert spaces and bounded linear mappings, the category **Grp** of groups and group homomorphisms, and the category **Top** of topological spaces and continuous mappings are among the most important examples of concrete categories.

Let $(X, \xi)$ and $(Y, \eta)$ be two objects in a concrete category $\mathcal{C}$. A map $f : X \longrightarrow Y$ is called *admissible* from $(X, \xi)$ to $(Y, \eta)$ if $f$ is a $\mathcal{C}$-morphism from $(X, \xi)$ to $(Y, \eta)$.

An admissible map $f$ from $(X, \xi)$ to $(Y, \eta)$ is called *optimal* if for any object $(C, \omega)$ and any map $g : C \longrightarrow X$, $g$ is admissible from $(C, \omega)$ to $(X, \xi)$ whenever $f \circ g$ is admissible from $(C, \omega)$ to $(Y, \eta)$ (see [9, Chapter 2,3.14]).

In the following we shall assume the concrete category $\mathcal{C}$ has the following extra properties:

(I1) The identity map $\mathrm{id}_X$ is admissible from $(X, \xi)$ to itself for any object $(X, \xi)$ in $\mathcal{C}$.

(I2) If the identity map is admissible from $(X, \xi)$ to $(X, \tau)$ as well as from $(X, \tau)$ to $(X, \xi)$, then $\xi$ is the same as $\tau$.

**Remark 1.** Consider the concrete category **Set**$^*$ whose objects are the pairs $(X, p_X)$, where $X$ is a non-empty set and $p_X$ is a fixed point of the set $X$. There is a unique admissible map $f$ from $(X, p_X)$ to $(Y, p_Y)$ that sends every $x \in X$ to $p_Y$. In this case, the identity map $i_X : X \longrightarrow X$ is not admissible from $(X, p_X)$ to $(X, p_X)$.

However, most of the interesting concrete categories satisfy the condition (I1).

Condition (I2) is also satisfied by all the important concrete categories except some artificially constructed examples.

Let $(X, \xi)$ be an object of the concrete category $\mathcal{C}$. By [9, Chapter 2,3.20], a subset $A$ of $X$ is called an *optimal subset* of $(X, \xi)$ if the inclusion map $i_A : A \longrightarrow (X, \xi)$ has an optimal lift, that is if there is a $\mathcal{C}$-structure $\lambda$ on $A$ such that the map $i_A : A \longrightarrow X$ is an optimal admissible map from $(A, \lambda)$ to $(X, \xi)$ (from (I2) it follows that such a $\mathcal{C}$-structure $\lambda$ is unique if it exists).

We shall denote by $\mathrm{Sub}((X, \xi)))$, or just $\mathrm{Sub}(X)$, the set of all optimal subsets of $(X, \xi)$, and use $\mathbf{S}(X)$ to denote the set of all subsets of $\mathrm{Sub}(X)$. Then $(\mathbf{S}(X), \subseteq)$ is a poset (actually, a complete lattice).

The set $\mathrm{Hom}_{\mathcal{C}}(X, X)$ of all the $\mathcal{C}$-morphisms from $(X, \xi)$ to $(X, \xi)$ forms a semigroup with $\mathrm{id}_X$ as an identity, where the operation is the composition of morphisms. We shall use $\mathbf{H}(X)$ to denote the set of all subsets of $\mathrm{Hom}_{\mathcal{C}}(X, X)$. Again, $(\mathbf{H}(X), \subseteq)$ is a poset.

For any $\mathcal{A} \in \mathbf{S}(X)$, define

$$\mathrm{Alg}\ \mathcal{A} = \{f \in \mathrm{Hom}_{\mathcal{C}}(X, X) : f(A) \subseteq A,\ \forall A \in \mathcal{A}\}.$$

For any $\mathcal{H} \in \mathbf{H}(X)$, define

$$\mathrm{Lat}\ \mathcal{H} = \{A \in \mathrm{Sub}(X) : f(A) \subseteq A,\ \forall f \in \mathcal{H}\}.$$

Then the two mappings $\mathrm{Alg} : \mathbf{S}(X) \longrightarrow \mathbf{H}(X)$ and $\mathrm{Lat} : \mathbf{H}(X) \longrightarrow \mathbf{S}(X)$ form a Galois connection between $(\mathbf{S}(X), \subseteq)$ and $(\mathbf{H}(X), \subseteq)$ in the following sense:

$$\mathcal{A} \subseteq \mathrm{Lat}\ \mathcal{H} \Longleftrightarrow \mathcal{H} \subseteq \mathrm{Alg}\ \mathcal{A}$$

holds for all $\mathcal{A} \in \mathbf{S}(X)$ and $\mathcal{H} \in \mathbf{H}(X)$.

It is routine to verify the following lemma (see e.g. [1, Exercise 11.3]).

**Lemma 1.** *Let $\mathcal{A} \in \mathbf{S}(X)$ and $\mathcal{H} \in \mathbf{H}(X)$. Then,*

(1) $\mathcal{A} \subseteq \mathrm{Lat}(\mathrm{Alg}\ \mathcal{A})$*;*
(2) $\mathcal{H} \subseteq \mathrm{Alg}(\mathrm{Lat}\ \mathcal{H})$*;*
(3) $\mathrm{Alg}(\mathrm{Lat}(\mathrm{Alg}\ \mathcal{A})) = \mathrm{Alg}\ \mathcal{A}$, $\mathrm{Lat}(\mathrm{Alg}(\mathrm{Lat}\ \mathcal{H})) = \mathrm{Lat}\ \mathcal{H}$*; and*
(4) *the two mappings* $\mathrm{Lat} : \mathbf{H}(X) \longrightarrow \mathbf{S}(X)$ *and* $\mathrm{Alg} : \mathbf{S}(X) \longrightarrow \mathbf{H}(X)$ *are order reversing.*

**Definition 1.** (1) A set $\mathcal{A}$ of optimal subsets of an object $X$ in a concrete category $\mathcal{C}$ is called *reflexive* if

$$\mathcal{A} = \mathrm{Lat}(\mathrm{Alg}\ \mathcal{A}).$$

(2) A set $\mathcal{H}$ of endomorphisms on $X$ is called *reflexive* if

$$\mathcal{H} = \mathrm{Alg}(\mathrm{Lat}\ \mathcal{H}).$$

In most of the important concrete categories, the intersection of any collection of optimal subsets of $X$ is still an optimal subset of $X$, therefore $(\mathrm{Sub}(X), \subseteq)$ is a complete lattice with $X$ as the top element.

The following lemma can be verified directly.

**Lemma 2.** (1) *If* $\mathrm{Sub}(X)$ *is closed under arbitrary intersections, then every reflexive* $\mathcal{A} \in \mathbf{S}(X)$ *is also closed under arbitrary intersections.*

(2) *If* $\mathcal{H} \in \mathbf{H}(X)$ *is reflexive then it contains the identity morphism* $\mathrm{id}_X$ *and is closed under composition, that is,* $f, g \in \mathcal{H}$ *imply* $f \circ g \in \mathcal{H}$.

(3) $\mathcal{H} \in \mathbf{H}(X)$ *is reflexive if and only if there exists* $\mathcal{A} \in \mathbf{S}(X)$ *such that* $\mathcal{H} = \mathrm{Alg}\,\mathcal{A}$*; and* $\mathcal{A} \in \mathbf{S}(X)$ *is reflexive if and only if there is* $\mathcal{H} \in \mathbf{H}(X)$ *such that* $\mathcal{A} = \mathrm{Lat}\,\mathcal{H}$.

(4) *If* $\{\mathcal{A}_i\}_{i\in I}$ *is a collection of reflexive sets of optimal subsets of* $X$*, then* $\bigcap_{i\in I} \mathcal{A}_i$ *is reflexive.*

(5) *If* $\{\mathcal{H}_i\}_{i\in I}$ *is a collection of reflexive sets of endomorphisms on* $X$*, then* $\bigcap_{i\in I} \mathcal{H}_i$ *is reflexive.*

**Remark 2.** (1) From Lemma 2(1) it follows that if the same condition is satisfied, then every reflexive set of optimal subsets is a complete lattice.

(2) Part (2) of Lemma 2 indicates that every reflexive set of endomorphisms on $X$ is a subsemigroup of $\mathrm{Hom}_{\mathcal{C}}(X, X)$ with respect to the composition operation.

## 2. Reflexive subset lattices

In this section we consider reflexive sets of subobjects in the category **Set** of sets and mappings. For each set $X$, $\mathrm{Sub}(X)$ is now the set of all subsets of $X$ and $\mathrm{Hom}_{\mathbf{Set}}(X, X)$ is the set of all mappings from $X$ to $X$. Notice that in this case $\mathrm{Sub}(X)$ is closed under arbitrary intersection.

**Theorem 1.** *A set* $\mathcal{A} \subseteq \mathrm{Sub}(X)$ *is reflexive if and only if it satisfies the following conditions:*

(1) $\mathcal{A}$ *is closed under arbitrary intersections;*

(2) $\mathcal{A}$ *is closed under arbitrary unions.*

Proof: The necessity follows from the definition of reflexivity of $\mathcal{A}$ and Lemma 2(1). We now prove the sufficiency. Suppose $\mathcal{A}$ satisfies the two conditions. Then it follows that $\emptyset$ and $X$ are in $\mathcal{A}$. By Lemma 1(1), it only remains to show that $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{A}) \subseteq \mathcal{A}$.

Let $A \subseteq X$ and $A \notin \mathcal{A}$. We show that there exists $f \in \mathrm{Alg}\,\mathcal{A}$ such that $f(A) \not\subseteq A$, thus $A \notin \mathrm{Lat}(\mathrm{Alg}\,\mathcal{A})$. This will yield the required inclusion.

For any $C \in \mathrm{Sub}(X)$, define $\overline{C} = \bigcap\{B \in \mathcal{A} : C \subseteq B\}$. Then, by condition (1), $\overline{C}$ is the smallest element of $\mathcal{A}$ containing $C$.

Now $A \subseteq \overline{A}$, $\overline{A} \in \mathcal{A}$ and $A \neq \overline{A}$. Since $\mathcal{A}$ is closed under union it follows that

$$\overline{A} = \bigcup\{\overline{\{x\}} : x \in A\}.$$

Since $A \neq \overline{A}$, there exists $a \in A$ such that $\overline{\{a\}} \not\subseteq A$. Choose a point $b \in \overline{\{a\}} - A$, and define the map $f : X \longrightarrow X$ by

$$f(x) = \begin{cases} b, & \text{if } x = a, \\ x, & \text{otherwise.} \end{cases}$$

Now let $B \in \mathcal{A}$. For any $x \in B$, if $x \neq a$ then $f(x) = x \in B$. If $x = a$, then $a \in B$ implies $\overline{\{a\}} \subseteq B$, thus $f(x) = b \in \overline{\{a\}} \subseteq B$. Hence $f(B) \subseteq B$ for all $B \in \mathcal{A}$, therefore $f \in \mathrm{Alg}\, \mathcal{A}$. However $f(A) \nsubseteq A$ since $b \in f(A) - A$, so $A \notin \mathrm{Lat}(\mathrm{Alg}\, \mathcal{A})$. The proof is complete. $\square$

**Remark 3.** Notice that if a collection of subsets of $X$ satisfies conditions (1) and (2) in Theorem 1, then it contains $\emptyset$ and $X$, and such a collection is called an Alexandroff topology on $X$.

## 3. Reflexive subsemigroups of $\mathrm{Hom}_{\mathbf{Set}}(X, X)$

We now investigate the reflexive subsemigroups of $\mathrm{Hom}_{\mathbf{Set}}(X, X)$ for a set $X$. A complete characterization for such subsemigroups is obtained. To make the symbols simpler, in the following we shall use just $\mathrm{Hom}(X, X)$ to denote the set of all morphisms from $X$ to $X$ in the given category.

**Definition 2.** Let $f$ and $g$ be two mappings from $X$ to $X$. Define $f \leq g$ if for any $x \in X$, $f(x) \neq x$ implies $f(x) = g(x)$.

**Remark 4.** (1) The binary relation $\leq$ defined above is a partial order on $\mathrm{Hom}(X, X)$, that is, it satisfies the following conditions:
 (a) $f \leq f$ for each $f$;
 (b) $f \leq g$ and $g \leq h$ imply $f \leq h$; and
 (c) $f \leq g$ and $g \leq f$ imply $f = g$.

(2) The identity mapping $\mathrm{id}_X : X \longrightarrow X$ is the least element in the partially ordered set $(\mathrm{Hom}(X, X), \leq)$.

Recall from [1] that an element $a$ of a poset $P$ is called an atom if $0_P < a$ and there is no element $b$ satisfying $0_P < b < a$, where $0_P$ is the least element in $P$.

**Lemma 3.** (1) *For any two distinct elements $a, b$ in $X$, the following mapping $h(a, b) : X \longrightarrow X$ is an atom of $\mathrm{Hom}(X, X)$:*

$$h(a,b)(x) = \begin{cases} b, & \text{if } x = a, \\ x, & \text{otherwise.} \end{cases}$$

*Moreover, every atom of $\mathrm{Hom}(X, X)$ is in this form.*
 (2) *Every nonempty subset $\{f_i : i \in I\}$ of $\mathrm{Hom}(X, X)$ has an infimum.*
 (3) *If a subset $\{f_i : i \in I\}$ has an upper bound in $\mathrm{Hom}(X, X)$ then $\sup\{f_i : i \in I\}$ exists.*
 (4) *Every element $f$ of $\mathrm{Hom}(X, X)$ is the supremum of atoms below $f$.*
 (5) *Every up directed subset of $\mathrm{Hom}(X, X)$ has a supremum.*

Proof: (1) The mapping $h(a, b)$ fixes every point of $X$ except $a$, so it is easily seen to be an atom. Suppose $f$ is an atom, then $f \neq \mathrm{id}_X$, so there exists $a \in X$ such that $f(a) \neq a$. If $b = f(a)$, then $\mathrm{id}_X < h(a,b) \leq f$, this then yields $f = h(a,b)$ because $f$ is an atom.

(2) Given a nonempty collection $\{f_i : i \in I\}$ of elements of $\mathrm{Hom}(X,X)$, let $M$ be the set of all atoms which are below every $f_i (i \in I)$. If $M$ is empty, then $\inf\{f_i : i \in I\} = \mathrm{id}_X$. Now assume $M$ is nonempty. Define $g : X \longrightarrow X$ by

$$g(x) = \begin{cases} b, & \text{if there is an atom } h(x,b) \in M, \\ x, & \text{otherwise.} \end{cases}$$

First, $g$ is well defined. In fact, if $h(x,b)$ and $h(x,c)$ both are atoms in $M$. For any $f_i$, we have $h(x,b) \leq f_i$ and $h(x,c) \leq f_i$, hence $h(x,b)(x) = b = f_i(x) = h(x,c)(x) = c$. Second, for each $i \in I$ and any $x \in X$, if $g(x) \neq x$ then there is an atom $h(x,b) \leq f_i (i \in I)$ such that $g(x) = b$. Thus $b = f_i(x)$, so $g \leq f_i (i \in I)$. Now suppose $k \in \mathrm{Hom}(X,X)$ is below every $f_i (i \in I)$. If $k(x) \neq x$, then $k(x) = f_i(x)$ for every $i \in I$. Then $h(x,b) \in M$, where $b = k(x)$. Thus $k(x) = g(x)$, and hence $k \leq g$. So $g = \inf\{f_i : i \in I\}$.

(3) follows from (2) and a general fact on the existence of suprema of subsets that have an upper bound (see [1]).

(4) For each $f \in \mathrm{Hom}(X,X)$, let $M$ be the set of all the atoms below $f$, then from the proof of (2) it follows that $f = \sup\{h : h \in M\}$.

(5) If $\{f_i : i \in I\}$ is an up directed set, then the mapping $f$ defined below is well defined and is the supremum of $\{f_i : i \in I\}$:

$$f(x) = \begin{cases} f_i(x), & \text{if there is an } f_i \text{ such that } f_i(x) \neq x, \\ x, & \text{otherwise.} \end{cases}$$

The proof is complete. □

Notice that $(\mathrm{Hom}(X,X), \leq)$ is not a complete lattice unless $X$ is a singleton set because it does not have a top element if $X$ contains at least two distinct elements.

**Lemma 4.** *If $\mathcal{H} \subseteq \mathrm{Hom}(X,X)$ is reflexive, then it satisfies the following conditions:*

(1) *the identity mapping $\mathrm{id}_X$ is in $\mathcal{H}$, and if $f, g \in \mathcal{H}$ then the composition $g \circ f$ of $f$ and $g$ is also in $\mathcal{H}$;*
(2) *if $f \leq g$ and $g \in \mathcal{H}$ then $f \in \mathcal{H}$;*
(3) *if $\{f_i : i \in I\} \subseteq \mathcal{H}$ and $\sup\{f_i : i \in I\}$ exists, then $\sup\{f_i : i \in I\} \in \mathcal{H}$.*

Proof: (1) By Lemma 2(2).

(2) Suppose $f \leq g$ and $g \in \mathcal{H}$. If $A \in \mathrm{Lat}\,\mathcal{H}$, then for each $x \in A$, if $f(x) = x$ then $f(x) \in A$, if $f(x) \neq x$ then $f(x) = g(x)$ because $f \leq g$, so $f(x) = g(x) \in A$. Hence $f(A) \subseteq A$ holds for every $A \in \mathrm{Lat}\,\mathcal{H}$, thus $f \in \mathrm{Alg}(\mathrm{Lat}\,\mathcal{H}) = \mathcal{H}$.

(3) Put $h = \sup\{f_i : i \in I\}$. For every $A \in \mathrm{Lat}\, \mathcal{H}$ and $i \in I$ we have $f_i(A) \subseteq A$. Recall that $\{f_i : i \in I\}$ has an upper bound. Then in a similar way as in the proof of Lemma 3(5) we can show that $h$ is given by the following formula: for each $x \in X$,

$$h(x) = \begin{cases} f_i(x), & \text{if there is an } i \in I \text{ such that } f_i(x) \neq x, \\ x, & \text{otherwise.} \end{cases}$$

Now for any $A \in \mathrm{Lat}\, \mathcal{H}$ and any $x \in A$, if $h(x) \neq x$ then there exists $f_i$ such that $h(x) = f_i(x) \in A$. Thus $h(A) \subseteq A$, that is $h \in \mathrm{Alg}(\mathrm{Lat}\, \mathcal{H}) = \mathcal{H}$. □

**Definition 3.** A subset $\mathcal{H}$ of $\mathrm{Hom}(X,X)$ is called a *closed subalgebra* of $\mathrm{Hom}(X,X)$ if it satisfies the three conditions (1), (2) and (3) in Lemma 4.

A subset $A$ of $X$ is called *invariant* under a mapping $f : X \longrightarrow X$ if $f(A) \subseteq A$.

**Lemma 5.** *Suppose that $\mathcal{H}$ is a closed subalgebra of* $\mathrm{Hom}(X,X)$. *Then for any $a \in X$, the subset $B = \{a\} \cup \{h(a) : h \in \mathcal{H}\}$ is invariant under every $g \in \mathcal{H}$.*

Proof: For any $x \in B$ and $g \in \mathcal{H}$, if $x = a$ then obviously $g(x) = g(a) \in B$; if $x = h(a)$ for some $h \in \mathcal{H}$, then $g(x) = g \circ h(a) = f(a) \in B$, where $f = g \circ h$ is still in $\mathcal{H}$. Hence $B$ is invariant under $g$. □

**Lemma 6.** *If $\mathcal{H}$ is a closed subalgebra of* $\mathrm{Hom}(X,X)$ *and* $g \in \mathrm{Hom}(X,X)$ *but $g \notin \mathcal{H}$, then there exists $B \subseteq X$ such that $B$ is invariant under every $f \in \mathcal{H}$ but not invariant under $g$.*

Proof: Let

$$\underline{g} = \sup\{h \in \mathcal{H} : h \leq g\}.$$

Then by Lemma 3(3) and condition (3) of closed subalgebra, $\underline{g}$ is well defined and is in $\mathcal{H}$. In addition, $\underline{g} \leq g$ and $\underline{g} \neq g$. Hence $g \not\leq \underline{g}$, thus there exists $a \in X$ such that $g(a) \neq a$ and $g(a) \neq \underline{g}(a)$. But $\underline{g} \leq g$, so $\underline{g}(a) = a$. We claim that for every $f \in \mathcal{H}, f(a) \neq g(a)$. In fact, if there exists $f \in \mathcal{H}$ with $f(a) = g(a)$, then the atom $h(a, g(a))$ is below $f$, so is in $\mathcal{H}$. Also $h(a, g(a)) \leq g$, hence $h(a, g(a)) \leq \underline{g}$, which then implies $\underline{g}(a) = h(a, g(a))(a) = g(a)$. However, $\underline{g}(a) = a \neq g(a)$. This contradiction confirms our claim. Now put $B = \{a\} \cup \{f(a) : f \in \mathcal{H}\}$. By Lemma 5, it follows that $B$ is invariant under every $f \in \mathcal{H}$. However, $g(B) \not\subseteq B$ because $g(a) \notin B$. □

Combining the above lemmas, we obtain the following main result in this section.

**Theorem 2.** *A subset $\mathcal{H}$ of* $\mathrm{Hom}(X,X)$ *is reflexive if and only if it is a closed subalgebra.*

**Remark 5.** Now on the set $\mathrm{Hom}(X, X)$ there is a semigroup structure and a partial order. The reader may wonder whether the two structures make $\mathrm{Hom}(X, X)$ an ordered semigroup, that is whether $f \leq g$ and $h \leq k$ imply $f \circ h \leq g \circ k$. Unfortunately, as the following example shows, this is not always true.

Let $X = \{1, 2, 3\}$, $h(1) = 2$, $h(2) = 2$, $h(3) = 3$, $f(1) = 3$, $f(2) = 2$, $f(3) = 3$, $g(1) = 3$, $g(2) = 1$, $g(3) = 3$. Then $f < g$ but $f \circ h \not\leq g \circ h$, since $f \circ h(1) = 2$ and $g \circ h(1) = 1 \neq f \circ h(1)$.

## 4. Some remarks on reflexive subgroup lattices

Let **Ab** be the category of abelian groups and group homomorphisms. For any abelian group $G$, let $\mathrm{Sub}(G)$ denote the set of all subgroups of $G$ and $\mathrm{Hom}(G, G)$ be the set of all group homomorphisms $f : G \longrightarrow G$. There have been lots of studies of the lattice-theoretical characterization of groups based on the lattice $(\mathrm{Sub}(G), \subseteq)$ (see e.g. [12]). For example, $G$ is cyclic if and only if $\mathrm{Sub}(G)$ is distributive and satisfies the maximal conditions ([12, Theorem 1.2.5]). Also $(\mathrm{Sub}(G), \subseteq)$ is distributive if and only if $G$ is locally cyclic (see [10]).

Given a collection $\mathcal{A} \subseteq \mathrm{Sub}(G)$, it is easy to verify that if $\mathcal{A}$ is reflexive then (i) it is closed under arbitrary intersections, (ii) it is closed under arbitrary sums, and (iii) it contains the two trivial subgroups. However, unlike for the category of sets, these three conditions are not sufficient for reflexivity as the following example shows.

**Example 1.** Let $(Z, +)$ be the additive group of integers and $\mathcal{A} = \{\{0\}, Z, 2Z\}$. Then $\mathcal{A}$ satisfies the above three conditions. But $\mathrm{Alg}\ \mathcal{A} = \mathrm{Hom}(Z, Z)$ is the set of all homomorphisms from $Z$ to $Z$, and $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{A}) = \mathrm{Sub}(Z)$. Thus $\mathcal{A}$ is not reflexive.

**Example 2.** Let $G$ be a group and $N\,\mathrm{Sub}(G)$ be the set of all normal subgroups of $G$. Then $N\,\mathrm{Sub}(G) = \{A \in \mathrm{Sub}(G) : f_a(A) \subseteq A, \forall a \in G\}$, where for each $a \in G$, $f_a : G \longrightarrow G$ is the inner automorphism defined by

$$f_a(x) = a^{-1}xa,\ \ x \in G.$$

Thus $N\,\mathrm{Sub}(G) = \mathrm{Lat}(\{f_a : a \in G\})$, so it is reflexive.

The following result indicates that reflexive subgroup lattices are exactly the sets of submodules.

**Proposition 1.** *Let $G$ be an abelian group. Then $\mathcal{A} \subseteq \mathrm{Sub}(G)$ is reflexive if and only if there is a ring $R$ such that $G$ is an $R$-module and $\mathcal{A}$ is the set of all submodules of $G$.*

Proof: Suppose $\mathcal{A} = \mathrm{Lat}(\mathrm{Alg}\ \mathcal{A})$. Let $R = \mathrm{Alg}\,\mathcal{A}$. Then $R$ is a ring with an identity. The multiplication of $R$ is the composition and the addition is defined as the pointwise addition. The evaluation mapping

$$v_G : R \times G \longrightarrow G,$$

defined by $v_G(f, a) = f(a)$, makes $G$ an $R$-module. Obviously a subgroup $A$ is a submodule of $G$ if and only if it is invariant under every $f \in \operatorname{Alg} \mathcal{A}$, if and only if $A \in \mathcal{A}$.

Conversely, suppose $G$ is an $R$-module. Let $\mathcal{A}$ be the set of all submodules of $G$. Then $\mathcal{A} \subseteq \operatorname{Sub}(G)$. For each $r \in R$, the mapping $m_r : G \longrightarrow G$ is a group homomorphism, where $m_r(a) = r \cdot a$. In addition, $A \in \mathcal{A}$ if and only if it is a subgroup of $G$ and is invariant under every $m_r$. Thus $\mathcal{A} = \operatorname{Lat}(\{m_r : r \in R\})$, so it is reflexive. □

Compared with the cases of sets and Hilbert spaces, the characterization of reflexive subgroup lattices is far from being achieved. Finding a necessary and sufficient lattice condition for such reflexive families seems still remote.

**Remark 6.** (1) Consider the additive group $(\mathbb{R}, +)$ of real numbers. Let $\mathbb{Z}$ be the subgroup of all integers. It is easy to show that if $f \in \operatorname{Alg}(\{\mathbb{Z}\})$ then for each rational number $r$, $f(r) = f(1)r$ where $f(1)$ is an integer. From this it follows that every subgroup of $(\mathbb{Q}, +)$ is in $\operatorname{Lat}(\operatorname{Alg}(\{\mathbb{Z}\}))$, where $\mathbb{Q}$ is the additive group of all rational numbers. Now let $A$ be a subgroup of $(\mathbb{R}, +)$ such that $A \not\subseteq \mathbb{Q}$ and $A \neq \mathbb{R}$. Choose a number $b \in A - \mathbb{Q}$ and extend $\{b, 1\}$ into a basis of $\mathbb{R}$, regarded as a $\mathbb{Q}$-vector space. Define a linear transformation $T : \mathbb{R} \longrightarrow \mathbb{R}$ that takes $b$ to a number in $\mathbb{R} - A$ and sends 1 to 1. Then $T \in \operatorname{Alg}(\{\mathbb{Z}\})$ but $T(A) \not\subseteq A$. Hence $\operatorname{Lat}(\operatorname{Alg}(\{\mathbb{Z}\})) = \operatorname{Sub}(\mathbb{Q}) \cup \{\mathbb{R}\}$, where $\operatorname{Sub}(\mathbb{Q})$ denotes the collection of all subgroups of $(\mathbb{Q}, +)$.

(2) For the additive group $(\mathbb{Z}, +)$ of integers, $\operatorname{Sub}(\mathbb{Z})$ is the only reflexive subgroup lattice. It is easy to show that if $G$ is a cyclic group, then $\operatorname{Sub}(G)$ is the only reflexive subgroup lattice. But we do not know whether the converse implication is also true.

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