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Completeness properties of function rings in pointfree topology

Bernhard Banaschewski, Sung Sa Hong

Abstract. This note establishes that the familiar internal characterizations of the Tychonoff spaces whose rings of continuous real-valued functions are complete, or σ-complete, as lattice ordered rings already hold in the larger setting of pointfree topology. In addition, we prove the corresponding results for rings of integer-valued functions.

Keywords: frame of reals, lattice ordered rings of real valued continuous functions and integer valued continuous functions, extremally disconnected frame, basically disconnected frame, cozero map

Classification: 06D22, 54C30, 54G05

1. Background

Recall that pointfree topology deals with frames and their homomorphisms where a *frame* is a complete lattice L in which

$$a \wedge \bigvee S = \bigvee \{a \wedge t | t \in S\}$$

for all $a \in L$ and $S \subseteq L$ and a *frame homomorphism* is a map $h : L \to M$ between frames which preserves finitary meets, including the unit (= top) e, and arbitrary joins, including the zero (= bottom) 0. The category thus determined will be denoted **Frm**.

For general notions and results concerning frames we refer to Johnstone [8] or Vickers [11]. In the present context, the following will be of particular importance.

The *frame* $\mathfrak{L}(\mathbb{R})$ *of reals* is the frame generated by all ordered pairs (p, q), p and q in $\mathbb{Q}$, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \le r < q \le s$
(R3) $(p, q) = \bigvee \{(r, s) \mid p < r < s < q\}$
(R4) $\bigvee \{(p, q) \mid p, q \in \mathbb{Q}\} = e$,

and for any frame L *the real valued continuous functions* on L are the homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$. Further, the operations of $\mathbb{Q}$ as lattice-ordered ring (= ℓ-ring) induce operations on the latter as follows.

For $\diamond = +, \cdot, \wedge, \vee$ and $\alpha, \beta : \mathfrak{L}(\mathbb{R}) \to L$,

$$\alpha \diamond \beta(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(t,u) \mid \langle r,s\rangle \diamond \langle t,u\rangle \subseteq \langle p,q\rangle\}$$

where $\langle \cdot, \cdot \rangle$ signifies open intervals in $\mathbb{Q}$ and the condition for the join says that

$$p < x \diamond y < q \quad \text{whenever} \quad r < x < s \quad \text{and} \quad t < y < u,$$

for any $\alpha : \mathfrak{L}(\mathbb{R}) \to L$, $(-\alpha)(p,q) = \alpha(-q,-p)$, and
for each $r \in \mathbb{Q}$, $\mathbf{r}(p,q) = e$ if $p < r < q$ and $\mathbf{r}(p,q) = 0$ otherwise.

These operations satisfy all the identities which hold for their counterparts in $\mathbb{Q}$ and hence they determine a commutative lattice-ordered ring with unit, denoted $\mathcal{R}L$. Moreover, the correspondence $L \mapsto \mathcal{R}L$ is functorial such that the ℓ-ring homomorphism $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ determined by a frame homomorphism $h : L \to M$ takes each $\varphi \in \mathcal{R}L$ to $h\varphi \in \mathcal{R}M$.

Concerning the relation between the present and the classical notion of ring of continuous functions, we note that, for any topological space X, if $\mathfrak{O}X$ is its frame of open sets and $C(X)$ its usual ring of real-valued continuous functions then there is an ℓ-ring isomorphism $C(X) \to \mathcal{R}(\mathfrak{O}X)$ taking $a \in C(X)$ to $\alpha \in \mathcal{R}(\mathfrak{O}X)$ such that

$$\alpha(p,q) = a^{-1}[\{\lambda \in \mathbb{R} \mid p < \lambda < q\}].$$

Further, this isomorphism is natural in X, providing an isomorphism of functors which shows that the $\mathcal{R}L$ indeed constitute the correct generalization of the usual $C(X)$.

For a detailed treatment of this particular topic we refer to Banaschewski [4].

Regarding the partial order in $\mathcal{R}L$ we note that $\alpha \leq \beta$ iff $\alpha(r,-) \leq \beta(r,-)$ for all $r \in \mathbb{Q}$ iff $\beta(-,r) \leq \alpha(-,r)$ for all $r \in \mathbb{Q}$, where

$$(r,-) = \bigvee\{(r,q) \mid r < q \in \mathbb{Q}\} \quad \text{and} \quad (-,r) = \bigvee\{(p,r) \mid p < r \in \mathbb{Q}\}$$

in $\mathfrak{L}(\mathbb{R})$. In particular, for $\alpha = \mathbf{p}$ this reduces to $\mathbf{p} \leq \beta$ iff $\beta(r,-) = e$ whenever $r < p$ iff $\beta(-,r) = 0$ whenever $r < p$, and analogous conditions hold for $\beta = \mathbf{q}$.

An important ℓ-subring of $\mathcal{R}L$ is its *bounded part* $\mathcal{R}^*L$ consisting of all $\varphi \in \mathcal{R}L$ such that $-\mathbf{n} \leq \varphi \leq \mathbf{n}$ for some natural n or, equivalently, $\varphi(p,q) = e$ for some $p, q \in \mathbb{Q}$.

For any element a of a frame L we have the homomorphism $L \to \downarrow a$ taking x to $x \wedge a$, and the associated $\mathcal{R}L \to \mathcal{R}(\downarrow a)$ will be denoted $\varphi \mapsto \varphi|a$. Evidently, this is the counterpart of restricting a continuous function on a space to some open subspace, and this process has the same properties here as in the latter situation: for any $\alpha, \beta \in \mathcal{R}L$, if $\alpha|s = \beta|s$ for all $s \in C$ where $\bigvee C = e$ then $\alpha = \beta$.

Finally an important feature of $\mathcal{R}L$ is its *cozero map* coz : $\mathcal{R}L \to L$ taking each $\alpha \in \mathcal{R}L$ to

$$\mathrm{coz}(\alpha) = \alpha((-,0) \vee (0,-))$$

which has the following important properties:

$$\begin{gathered}\operatorname{coz}(\mathbf{0}) = 0, \operatorname{coz}(\mathbf{1}) = e,\\ \operatorname{coz}(\alpha\beta) = \operatorname{coz}(\alpha) \wedge \operatorname{coz}(\beta),\\ \operatorname{coz}(\alpha+\beta) \leq \operatorname{coz}(\alpha) \vee \operatorname{coz}(\beta).\end{gathered}$$

We note that this is the exact counterpart to the notion of cozero set for ordinary continuous functions: for any space X, if $a \in C(X)$ and $\alpha \in \mathcal{R}(\mathfrak{O}X)$ correspond to each other by the isomorphism described earlier then

$$\operatorname{coz}(\alpha) = a^{-1}[\{\lambda \in \mathbb{R} \mid \lambda < 0 \text{ or } 0 < \lambda\}],$$

the cozero set of a. Moreover, as in the spatial case, if $\operatorname{coz}(\alpha) = e$ for some $\alpha \in \mathcal{R}L$ then α is invertible, that is, $\alpha\beta = \mathbf{1}$ for some $\beta \in \mathcal{R}L$. On the other hand $\operatorname{coz}(\alpha) = 0$ obviously implies $\alpha = \mathbf{0}$.

In the following, $\operatorname{Coz} L$ will be the set of all cozero elements of L, that is, of all $\operatorname{coz}(\alpha)$, $\alpha \in \mathcal{R}L$.

The particular frame properties we shall be concerned with here are as follows: a frame L is called

completely regular if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ where $x \prec\!\!\prec a$ (x is completely below, or: really inside, a) means there exists a family $(x_{n,k})_{n=0,1,\cdots;k=0,1,\cdots,2^n}$ in L such that $x_{0,0} = x$, $x_{0,1} = a$, $x_{n,k} = x_{n+1,2k}$, and $x_{n,k} \prec x_{n,k+1}$ where $y \prec z$ says that $z \vee y^* = e$ for the pseudocomplement

$$y^* = \bigvee\{t \in L \mid y \wedge t = 0\}$$

of y;

zero-dimensional if each element of L is a join of complemented elements, that is, of elements c for which $c \vee c^* = e$;

extremally disconnected if it satisfies the *Stone identity* which says that $a^* \vee a^{**} = e$ for all $a \in L$; and

basically disconnected if $c^* \vee c^{**} = e$ for all $c \in \operatorname{Coz} L$.

All these notions have the important feature that, for any space X, $\mathfrak{O}X$ has one of these properties iff X has the same-named property in the usual sense — a fact which ensures that the results which we shall prove here will be the exact extensions of the relevant classical results to the pointfree setting.

Finally, the terminology adopted here is that an ℓ-ring will be called *order complete* whenever each non-void subset S which is bounded above has a join (= supremum) $\bigvee S$; similarly, it will be called *σ-complete* if $\bigvee S$ exists for any countable subset of this type.

2. Completeness properties of $\mathcal{R}L$

To begin with, we note that it is natural in the present context to consider only *completely regular* frames, these being exactly the frames L which have enough real-valued continuous functions in the sense that they are generated by all $\alpha(p,q)$, $\alpha \in \mathcal{R}L$ and $p, q \in \mathbb{Q}$, or alternatively by $\mathrm{Coz}\, L$. In general, any frame L has a largest completely regular subframe $\mathrm{C\,Reg}\, L$, and the identical embedding $\mathrm{C\,Reg}\, L \to L$ then induces an isomorphism $\mathcal{R}(\mathrm{C\,Reg}\, L) \to \mathcal{R}L$ by the complete regularity of $\mathfrak{L}(\mathbb{R})$ so that any result concerning L and $\mathcal{R}L$ for completely regular L immediately implies a corresponding result for arbitrary L. In view of this, all frames will be taken as completely regular here, as an assumption of convenience which involves no essential loss of generality.

We first establish a couple of auxiliary results.

Lemma 1. *If $a \prec\!\!\prec b$ in L then there exist $\varphi \in \mathcal{R}L$ such that $\mathbf{0} \leq \varphi \leq \mathbf{1}$, $\varphi|a = \mathbf{1}$, and $\varphi|b^* = \mathbf{0}$.*

Proof: By Proposition 6 of Banaschewski [4] there exist $\psi \in \mathcal{R}L$ for which $a \leq \psi(-, \frac{1}{2})$ and $\psi(-,1) \leq b$. Hence $\psi|a \leq \frac{\mathbf{1}}{\mathbf{2}}$ and $\psi|b^* \geq \mathbf{1}$ by the rules for inequalities in $\mathcal{R}(\downarrow a)$ and $\mathcal{R}(\downarrow b^*)$ respectively, and then $\varphi = ((\mathbf{2}(\mathbf{1} - \psi)) \wedge \mathbf{1}) \vee \mathbf{0}$ will have the desired properties. □

Lemma 2. *If $\gamma = \bigvee S$ in $\mathcal{R}L$ and $a \in L$ such that $\varphi|a = \mathbf{0}$ for all $\varphi \in S$ then $\gamma|a = \mathbf{0}$.*

Proof: For any $x \prec\!\!\prec a$, let $x \prec\!\!\prec z \prec\!\!\prec a$ and take $\alpha \in \mathcal{R}L$ such that

$$\mathbf{0} \leq \alpha \leq \mathbf{1},\ \alpha|x = \mathbf{0}, \quad \text{and} \quad \alpha|z^* = \mathbf{1},$$

using $\mathbf{1} - (\cdot)$ on Lemma 1. Then

$$\varphi|a = \mathbf{0} \leq ((\alpha|\gamma|) \vee \frac{\mathbf{1}}{\mathbf{2}}\gamma)|a \text{ and } \varphi|z^* \leq \gamma|z^* \leq ((\alpha|\gamma|) \vee \frac{\mathbf{1}}{\mathbf{2}}\gamma)|z^*$$

for each $\varphi \in S$, and since $a \vee z^* = e$ it follows that $\varphi \leq (\alpha|\gamma|) \vee \frac{\mathbf{1}}{\mathbf{2}}\gamma$. Consequently, $\gamma \leq (\alpha|\gamma|) \vee \frac{\mathbf{1}}{\mathbf{2}}\gamma$, hence $\gamma|x \leq \frac{\mathbf{1}}{\mathbf{2}}\gamma|x$ and therefore $\gamma|x = \mathbf{0}$, showing $\gamma|a = \mathbf{0}$ by complete regularity. □

As a convenient method of defining a function on a frame to be used below we recall the following. For any frame L, a *trail in* L is a map $t : \mathbb{Q} \to L$ such that $t(r) \prec t(s)$ whenever $r < s$ and

$$\bigvee\{t(r) \mid r \in \mathbb{Q}\} = e = \bigvee\{t(r)^* \mid r \in \mathbb{Q}\},$$

and any such t determines a $\varphi \in \mathcal{R}L$ by the definition

$$\varphi(p,q) = \bigvee\{t(r)^* \wedge t(s) \mid p < r < s < q\}.$$

Our first result is now,

Proposition 1. *$\mathcal{R}L$ is order complete iff L is extremally disconnected.*

Proof: ($\Rightarrow$) For any $a \in L$, let S be the set of all $\varphi \in \mathcal{R}L$ such that $\mathbf{0} \leq \varphi \leq \mathbf{1}$, $\varphi|x = \mathbf{1}$ for some $x \prec\prec a$, and $\varphi|a^* = \mathbf{0}$. Then S is non-void by Lemma 1 and trivially bounded above so that $\gamma = \bigvee S$ exists in $\mathcal{R}L$. Now, again by Lemma 1, $\gamma|x = \mathbf{1}$ for each $x \prec\prec a$ and hence $\gamma|a = \mathbf{1}$ (complete regularity) while $\gamma|a^* = \mathbf{0}$ by Lemma 2. It follows that

$$\operatorname{coz}(\gamma) \wedge a^* = \operatorname{coz}(\gamma|a^*) = 0$$

and

$$\operatorname{coz}(\mathbf{1} - \gamma) \wedge a = \operatorname{coz}((\mathbf{1} - \gamma)|a) = 0$$

showing that

$$\operatorname{coz}(\gamma) \leq a^{**} \quad \text{and} \quad \operatorname{coz}(\mathbf{1} - \gamma) \leq a^*.$$

Further, by the properties of the cozero map,

$$e = \operatorname{coz}(\mathbf{1}) = \operatorname{coz}(\gamma + \mathbf{1} - \gamma) \leq \operatorname{coz}(\gamma) \vee \operatorname{coz}(\mathbf{1} - \gamma) \leq a^{**} \vee a^*,$$

which proves the Stone identity.

($\Leftarrow$) Given any non-void $S \subseteq \mathcal{R}L$ which is bounded above, define $t : \mathbb{Q} \to L$ by

$$t(r) = \left(\bigvee \{\alpha(r, -) \mid \alpha \in S\}\right)^*.$$

Then $r \leq s$ in $\mathbb{Q}$ implies $\alpha(s, -) \leq \alpha(r, -)$ for each $\alpha \in S$ so that $t(r) \leq t(s)$, and since these are complemented elements by the present hypothesis it follows that $t(r) \prec t(s)$. Further, for any upper bound β of S, $\alpha(r, -) \leq \beta(r, -)$ for all $\alpha \in S$ and $r \in \mathbb{Q}$, hence $\beta(-, r) \leq \beta(r, -)^* \leq t(r)$, and therefore $\bigvee\{t(r) \mid r \in \mathbb{Q}\} = e$. On the other hand, if $\alpha \in S$ then $\alpha(r, -) \leq t(r)^*$ for each $r \in \mathbb{Q}$ and consequently also $\bigvee\{t(r^*) \mid r \in \mathbb{Q}\} = e$.

This shows t is a trail in L, and as noted earlier there exists $\gamma \in \mathcal{R}L$ such that

$$\gamma(p, q) = \bigvee\{t(r)^* \wedge t(s) \mid p < r < s < q\}$$

for any $p, q \in \mathbb{Q}$. Further,

$$(*) \qquad \begin{aligned} \gamma(p, -) &= \bigvee\{t(r)^* \wedge t(s) \mid p < r < s\} \\ &= \bigvee\{t(r)^* \mid p < r\} \wedge \bigvee\{t(s) \mid p < s\} = \bigvee\{t(r)^* \mid p < r\} \end{aligned}$$

by the properties of t. Now $t(r)^* \geq \alpha(r, -)^{**} \geq \alpha(r, -)$ for each $\alpha \in S$ and $r \in \mathbb{Q}$ and hence by $(*)$

$$\gamma(p, -) \geq \bigvee\{\alpha(r, -) \mid p < r\} = \alpha(p, -),$$

showing γ is an upper bound of S. On the other hand, if β is any upper bound of S then

$$t(r)^* \leq \beta(r, -)^{**} \leq \beta(p, -)$$

for any $p < r$, the second step since $\beta(r,-) \prec\!\prec \beta(p,-)$ and hence $\beta(p,-) \vee \beta(r,-)^* = e$; it follows that $\gamma(p,-) \leq \beta(p,-)$ for any $p \in \mathbb{Q}$ by $(*)$, showing that $\gamma \leq \beta$. In all, then, $\gamma = \bigvee S$. □

The following links the order completeness of $\mathcal{R}L$ with a completeness condition *inside* L where BL, the Boolean part of L, is the Boolean algebra of complemented elements of L.

Corollary 1. *$\mathcal{R}L$ is order complete iff L is zero dimensional and BL is complete.*

PROOF: Any completely regular frame which is extremally disconnected is zero-dimensional because $x \prec\!\prec a$ implies $x^{**} \leq a$ in any frame. Hence the result follows from the known fact (Johnstone [8, III, 3.5]) that a zero-dimensional frame L is extremally disconnected iff BL is complete. □

Next, recall that the compact completely regular frames are coreflective in the category **Frm** of all frames, and that for completely regular L the corresponding coreflection map $\beta L \to L$ is dense onto, dense meaning that only 0 is mapped to 0.

Corollary 2. *L is extremally disconnected iff this holds for βL.*

PROOF: $(\Rightarrow)$ If $\mathcal{R}L$ is order complete then the same evidently holds for $\mathcal{R}^*L$. On the other hand, for any bounded $\varphi : \mathfrak{L}(\mathbb{R}) \to L$, if $\varphi(p,q) = e$ then φ factors through the homomorphism $\mathfrak{L}(\mathbb{R}) \to \uparrow((-,p)\vee(q,-))$ and since the latter frame is known to be compact completely regular φ lifts to a homomorphism $\mathfrak{L}(\mathbb{R}) \to \beta L$. As a result $\mathcal{R}(\beta L) \cong \mathcal{R}^*L$, showing βL is extremally disconnected whenever L is.

$(\Leftarrow)$ Much more generally, if $h : M \to L$ is dense onto and M is extremally disconnected then so is L, an immediate consequence of the fact that $h(a^*) = h(a)^*$ for dense onto h: $h(b) = h(a)^*$ implies $b \leq a^*$ by denseness and hence $h(a)^* \leq h(a^*)$, the nontrivial part of this identity. □

Remark 1. The above proposition could also be obtained as a consequence of the following special case of a result by Johnstone [7] concerning an arbitrary topos with a natural numbers object.

For any frame L, the real numbers object $\mathbb{R}_{\mathrm{Sh}\,L}$ in the topos of sheaves on L is internally order complete iff $\mathrm{Sh}\,L$ *is a De Morgan topos.*

Generally, the latter means that the subobject classifier Ω, always a Heyting algebra in the topos and hence pseudocomplemented, satisfies the Stone identity. Specifically in $\mathrm{Sh}\,L$, Ω is the sheaf assigning $\downarrow a$ to each $a \in L$, and $\mathrm{Sh}\,L$ is De Morgan iff L is extremally disconnected because for such L each $\downarrow a$ is extremally disconnected. On the other hand, $\mathbb{R}_{\mathrm{Sh}\,L}$ is the sheaf $a \mapsto \mathcal{R}(\downarrow a)$ and hence the present proposition could be obtained by showing that, for completely regular L, $\mathbb{R}_{\mathrm{Sh}\,L}$ is internally order complete iff $\mathcal{R}L$ is order complete. Nonetheless, it seemed of some merit to derive this proposition entirely within the setting of pointfree topology which then has the latter assertion as corollary.

Next we turn to σ-completeness. We note that the proof of the following proposition involves the Axiom of Countable Choice (ACC) which one uses to show that any countable join of cozero elements in a frame is a cozero element.

Proposition 2. *$\mathcal{R}L$ is σ-complete iff L is basically disconnected.*

PROOF: ($\Rightarrow$) For any cozero element $c \in L$, let $c = \operatorname{coz}(\varphi)$ where $\mathbf{0} \leq \varphi \leq \mathbf{1}$ and put $\gamma = \bigvee\{(\mathbf{n}\varphi) \wedge \mathbf{1} \mid n = 1, 2, \cdots\}$. Then

$$\mathbf{1} \geq \gamma|\varphi(\frac{1}{n}, -) \geq ((\mathbf{n}\varphi) \wedge \mathbf{1})|\varphi(\frac{1}{n}, -) \geq \mathbf{1},$$

the third step since $\varphi|\varphi(\frac{1}{n}, -) \geq \frac{\mathbf{1}}{\mathbf{n}}$ by the rules concerning the partial order in our function rings, and hence $\gamma|c = \mathbf{1}$ because $c = \varphi(0, -) = \bigvee\{\varphi(\frac{1}{n}, -) \mid n = 1, 2, \cdots\}$. On the other hand, $\gamma|c^* = \mathbf{0}$ by Lemma 2 because

$$(\mathbf{n}\varphi \wedge \mathbf{1})|c^* = \mathbf{n}\varphi|c^* \wedge \mathbf{1} = \mathbf{0},$$

and by the proof of Proposition 1 we conclude that $c^* \vee c^{**} = e$.

($\Leftarrow$) If $S \subseteq \mathcal{R}L$ is non-void, bounded above, and *countable* then the join in the definition of the trail in the proof of Proposition 1 is a countable join of cozero elements and therefore a cozero element (using ACC). Hence the present hypothesis makes it complemented so that we still have a trail, and the remaining arguments in the above proof then apply, showing that $\gamma = \bigvee S$. □

Corollary 3. *L is basically disconnected iff this holds for βL.*

PROOF: ($\Rightarrow$) $\mathcal{R}^*L \cong \mathcal{R}(\beta L)$, as noted earlier, and $\mathcal{R}^*L$ is σ-complete whenever $\mathcal{R}L$ is.

($\Leftarrow$) The coreflection map $\beta L \to L$ takes $\operatorname{Coz}(\beta L)$ onto $\operatorname{Coz}(L)$ because any cozero element of L is $\operatorname{coz}(\varphi)$ for some $\varphi \in \mathcal{R}^*L$, and consequently the argument in the proof of Corollary 2 applies. □

Remark 2. The proof of Proposition 2 lends itself to the following generalization. For any cardinal κ, call a set a *κ-set* if there exists a map from κ onto it, and call a frame L *κ-disconnected* whenever $a^* \vee a^{**} = e$ for all $a \in L$ which are joins of κ-sets of cozero elements of L. Then, using the choice principle C^κ, that is, the existence of choice functions for any κ-indexed family of non-void sets, we can obtain for each element c of a given κ-set $K \subseteq \operatorname{Coz} L$ a $\gamma_c \in \mathcal{R}L$ such that $\mathbf{0} \leq \gamma_c \leq \mathbf{1}$, $\gamma_c|c = \mathbf{1}$, and $\gamma_c|c^* = \mathbf{0}$, and if $\gamma = \bigvee\{\gamma_c \mid c \in K\}$ exists then $\gamma|a = \mathbf{1}$ and $\gamma|a^* = \mathbf{0}$ for $a = \bigvee K$. Calling an ℓ-ring *κ-complete* if every non-void κ-set which is bounded above has a join we then obtain, using C^κ, for each $\kappa > \omega$:

$\mathcal{R}L$ is κ-complete iff L is κ-disconnected.

Remark 3. An obvious strengthening of basic disconnectedness is that each cozero element is complemented. This, somewhat interestingly, is characterized by a purely ring-theoretic condition on $\mathcal{R}L$, just as it is in the spatial case (where the property in question is that every cozero set is closed):

Every cozero element of L is complemented iff $\mathcal{R}L$ is regular.

Proof: ($\Rightarrow$) For any $\alpha \in \mathcal{R}L$, since $a = \mathrm{coz}(\alpha)$ is complemented we can define $\beta \in \mathcal{R}L$ by the condition

$$\beta|a = \frac{1}{\alpha|a} \quad \text{and} \quad \beta|a^* = \mathbf{0},$$

given the isomorphism $L \to (\downarrow a) \times (\downarrow a^*)$ and the fact that $\mathrm{coz}(\alpha|a) = a$, and then clearly $\alpha = \alpha^2\beta$ because this holds for the restrictions to a and a^* while $a \vee a^* = e$.

($\Leftarrow$) For any $\alpha \in \mathcal{R}L$, if $\alpha = \alpha^2\beta$ then $\mathrm{coz}(\alpha) = \mathrm{coz}(\gamma)$ for the idempotent $\gamma = \alpha\beta$, and the obvious calculation shows that $\mathrm{coz}(\gamma)$ is complemented. $\square$

By Corollary 3, a natural *weakening* of basic disconnectedness is strong zero-dimensionality, meaning that βL is zero-dimensional. Again this is characterized in purely ring theoretic terms: it holds iff $\mathcal{R}L$ is an *exchange ring*, saying that each element of $\mathcal{R}L$ is the sum of an idempotent and an invertible element. We omit the details.

Next, we describe some non-spatial examples of extremally and basically disconnected completely regular frames in order to demonstrate that these notions considerably transcend the spatial situation.

Example 1. Any Boolean frame (= complete Boolean algebra) is trivially extremally disconnected but it is spatial iff it is atomic. Hence any non-atomic complete Boolean algebra is an example of this kind.

Example 2. For any Boolean σ-algebra A, the frame $\mathfrak{H}A$ of its σ-ideals has the property (assuming ACC) that

$$\mathrm{Coz}(\mathfrak{H}A) = B(\mathfrak{H}A) = \{\downarrow a \mid a \in A\}$$

by the observation that $\mathfrak{H}A$ is Lindelöf and its cozero elements are exactly its Lindelöf elements, and the latter are just the principal ideals (Banaschewski [3]). Hence any such $\mathfrak{H}A$ is basically disconnected but it is spatial iff the Boolean σ-homomorphism $A \to \mathbf{2}$ separate the elements of A. In particular, $\mathfrak{H}A$ is certainly non-spatial whenever A is non-trivial (zero $\neq$ unit) but has no Boolean σ-homomorphisms $A \to \mathbf{2}$ at all. Note that, for any second countable Tychonoff space without isolated points, the Boolean algebra of regular open sets considered as a σ-algebra is of this kind.

As an additional aspect we note that $A \simeq B(\mathfrak{H}A)$ for any Boolean σ-algebra and hence $\mathfrak{H}A$ is extremally disconnected iff A is complete — which provides convenient examples of completely regular frames which are basically but not extremally disconnected.

We close with some comments regarding the relation of the results in this section to classical topology. To begin with, as indicated at the outset, the two propositions as well as their corollaries extend familiar facts concerning topological spaces: the latter are exactly the special cases of these for spatial completely regular frames, in view of the fact, noted earlier, that the usual function ring $C(X)$ for a space X is isomorphic, as a lattice-ordered ring, to the function ring $\mathcal{R}(\mathfrak{O}X)$ in the pointfree sense for the frame $\mathfrak{O}X$ of open subsets of X. Regarding the history of these results, the basic original work is Stone [10] although there is some overlap with Nakano [9]. [10] provides the full treatment, and updated versions, of results briefly announced several years earlier; in particular, it presents the space form of Proposition 1 and Corollary 1 (which seems less well known), and a forerunner of that of Proposition 2 (whose final formulation appears in Gillman–Jerison [6]). It also provides a result on κ-completeness but this is different from, and not as conclusive as, that given here in Remark 2, the reason being that instead of unions of at most κ *cozero sets* it deals with unions of at most κ *open-closed* sets.

An important device used in [10], and specifically developed for this, is what is called the *spectral family* of a real-valued function f on a set X, defined as

$$f^{-1}[(-,\lambda)] \quad (\lambda \in \mathbb{R}),$$

and a characterization when an $\mathbb{R}$-indexed family of subsets of X is of this form, in particular with continuous f if X is a topological space. The manner in which this notion is employed to obtain the required suprema for given sets of functions has a certain formal similarity with the technique used here, involving the trail introduced in the proof of Proposition 1. On the other hand, though, the absence of points and the replacement of subsets by abstract frame elements does make the present situation rather different, and the proofs presented here are certainly not just automatic translations of the classical proofs. One specific feature which is different from the latter, and which plays a key rôle in the present arguments, is the use of the cozero map; this confirms a general perception that the latter is a particularly suitable tool in the pointfree setting.

3. The integer-valued case

We first recall some of the basic facts concerning the ring $\mathcal{Z}L$ of integer-valued continuous functions on a frame L. Its elements are the maps $\alpha : \mathbb{Z} \to L$ such that

$$\alpha(k) \wedge \alpha(l) = 0 \ \text{ for } \ k \neq l \ \text{ and } \ \bigvee\{\alpha(k) \mid k \in \mathbb{Z}\} = e,$$

and its operations are derived from the ℓ-ring operations of $\mathbb{Z}$ as follows:

for $\diamond = +, \cdot, \wedge, \vee$ and $\alpha, \beta : \mathbb{Z} \to L$ as above,

$$\alpha \diamond \beta(k) = \bigvee\{\alpha(l) \wedge \beta(m) \mid l \diamond m = k\},$$

$(-\alpha)(k) = \alpha(-k)$, and

for each $m \in \mathbb{Z}$, $\mathbf{m}(k) = e$ if $k = m$ and $\mathbf{m}(k) = 0$ otherwise.

These operations satisfy all identities which hold for their counterparts in $\mathbb{Z}$ so that $\mathcal{Z}L$ is a commutative ℓ-ring with unit, just like $\mathcal{R}L$. To compare the two, note that there is the obvious frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathfrak{P}(\mathbb{Z})$ taking (p, q) to $\{k \in \mathbb{Z} \mid p < k < q\}$, and the elements of $\mathcal{Z}L$ are then easily identified with those elements of $\mathcal{R}L$ which factor through this, providing an ℓ-ring embedding $\mathcal{Z}L \to \mathcal{R}L$. We note in passing that, for compact L, the image of $\mathcal{Z}L$ in $\mathcal{R}L$ is just the subring of $\mathcal{R}L$ generated by its idempotents.

Regarding the partial order in $\mathcal{Z}L$ we have, entirely analogous to the case of $\mathcal{R}L$, that $\alpha \leq \beta$ iff $\alpha(k, -) \leq \beta(k, -)$ iff $\beta(-, k) \leq \alpha(-, k)$ where

$$\alpha(k, -) = \bigvee\{\alpha(l) \mid l > k\} \text{ and } \alpha(-, k) = \bigvee\{\alpha(l) \mid l < k\}.$$

Similarly, the cozero map from $\mathcal{Z}L$ to L is given by

$$\operatorname{coz}(\alpha) = \bigvee\{\alpha(k) \mid k \in \mathbb{Z} - \{0\}\}$$

and it obviously satisfies the same rules listed earlier for $\operatorname{coz} : \mathcal{R}L \to L$.

Next, we need a result which corresponds to defining some $\varphi \in \mathcal{R}L$ by means of a trail.

Lemma 3. *For any map $t : \mathbb{Z} \to BL$ such that $t(k) \leq t(l)$ whenever $k \leq l$ and $\bigvee\{t(k) \mid k \in \mathbb{Z}\} = e = \bigvee\{t(k)^* \mid k \in \mathbb{Z}\}$, $\varphi : \mathbb{Z} \to L$ given by $\varphi(k) = t(k)^* \wedge t(k+1)$ belongs to $\mathcal{Z}L$.*

Proof: If $k \neq l$ then $k < l$, say, and $\varphi(k) \wedge \varphi(l) \leq t(k+1) \wedge t(l)^* \leq t(l) \wedge t(l)^* = 0$. Further, an easy induction shows that $\bigvee\{\varphi(k) \mid l \leq k \leq m\} = t(l)^* \wedge t(m)$ whenever $l \leq m$, hence $\bigvee\{\varphi(k) \mid l \leq k\} = t(l)^*$ and then $\bigvee\{\varphi(k) \mid k \in \mathbb{Z}\} = e$, by the second and third of the defining properties of t. ☐

Now we have the following analogues of Proposition 1 and 2.

Proposition 3. *For any zero-dimensional frame L,*

(1) *$\mathcal{Z}L$ is order complete iff L is extremally disconnected, and*
(2) *$\mathcal{Z}L$ is σ-complete iff $a^* \vee a^{**} = e$ for all $a \in L$ which are countable joins of complemented elements.*

Proof: (1)($\Rightarrow$) For any $a \in L$, let S be the set of all $\varphi \in \mathcal{Z}L$ such that $\varphi|c = \mathbf{1}$ and $\varphi|c^* = \mathbf{0}$ for some complemented $c \leq a$. Then S is trivially bounded above, and non-void since $\varphi(1) = c$ and $\varphi(0) = c^*$ defines $\varphi \in S$ for each complemented $c \leq a$. Hence we have $\gamma = \bigvee S$ in $\mathcal{Z}L$. It follows that $\gamma|a = \mathbf{1}$ by zero-dimensionality while $\gamma|a^* = \mathbf{0}$ by Lemma 2, and the proof of Proposition 1 then shows that $a^* \vee a^{**} = e$.

($\Leftarrow$) Given $S \subseteq \mathcal{Z}L$ non-void and bounded above, put

$$t(k) = (\bigvee\{\alpha(k,-) \mid \alpha \in S\})^*$$

for each $k \in \mathbb{Z}$. Then $t(k) \in BL$ since L is extremally disconnected, $t(k) \leq t(l)$ whenever $k \leq l$, and

$$\bigvee\{t(k) \mid k \in \mathbb{Z}\} = e = \bigvee\{t(k)^* \mid k \in \mathbb{Z}\}$$

by the same argument as in the proof of Proposition 1. Consequently, Lemma 3 provides $\gamma \in \mathcal{Z}L$ such that $\gamma(k) = t(k)^* \wedge t(k+1)$. Now, for each $\alpha \in S$,

$$\alpha(k,-) \leq t(k)^* = \gamma(k,-)$$

by the proof of Lemma 3, showing that γ is an upper bound of S. On the other hand, for any upper bound $\beta \in \mathcal{Z}L$ of S and any $k \in \mathbb{Z}$, $\beta(k,-)^* \leq t(k)$, hence $t(k)^* \leq \beta(k,-)$ since $\beta(k,-)$ is complemented, and therefore $\gamma(k,-) \leq \beta(k,-)$, showing that $\gamma \leq \beta$. In all, then, $\gamma = \bigvee S$ in $\mathcal{Z}L$.

(2)($\Rightarrow$) If $a = \bigvee K$ for some *countable* $K \subseteq BL$ then the S in the above proof of (1)($\Rightarrow$) may be taken countable, hence we still have $\gamma \in \mathcal{Z}L$ for which $\gamma|a = \mathbf{1}$ and $\gamma|a^* = \mathbf{0}$, and again conclude that $a^* \vee a^{**} = e$.

($\Leftarrow$) If S in the above proof of (1)($\Leftarrow$) is countable the $t(k)$ defined there are still complemented because the join involves only countably many complemented elements, and all the other arguments then apply, showing that S has a join in $\mathcal{Z}L$. □

Corollary 4. *For any zero-dimensional frame L, $\mathcal{Z}L$ is σ-complete iff BL is σ-complete.*

Proof: ($\Rightarrow$) For any countable $K \subseteq BL$, $(\bigvee K)^{**}$ is complemented by (2) of the proposition and hence the join of K in BL.

($\Leftarrow$) For any $a = \bigvee K$ in L where $K \subseteq BL$ is countable, let $b \in BL$ be the join of K in BL. Then $a \leq b$ and hence $a^{**} \leq b$. On the other hand, for any complemented $c \leq a^*$, $b \wedge c = 0$ since a Boolean σ-algebra is a σ-frame, and therefore $b \wedge a^* = 0$ by zero-dimensionality, showing that $b \leq a^{**}$. Thus $b = a^{**}$ and consequently $a^* \vee a^{**} = e$, and (2) of the proposition then implies that $\mathcal{Z}L$ is σ-complete. □

Remark 4. Just as in the case of $\mathcal{R}L$, the arguments used here concerning σ-completeness immediately extend to show that the following are equivalent for any zero-dimensional frame L where κ is any cardinal:

(1) $\mathcal{Z}L$ is κ-complete,
(2) $a^* \vee a^{**} = e$ for all $a \in L$ which are joins of κ-sets of complemented elements, and
(3) BL is κ-complete.

One might add here that, contrary to the case of Remark 2, this result does not require the use of any choice principle in view of the fact that, for any complemented $c \leq a$ we have a canonical function φ such that $\mathbf{0} \leq \varphi \leq \mathbf{1}$, $\varphi|c = \mathbf{1}$, and $\varphi|a^* = \mathbf{0}$.

We close this section with a result concerning the relation between the σ-completeness of $\mathcal{R}L$ and $\mathcal{Z}L$.

Proposition 4. *For any zero-dimensional frame L, $\mathcal{R}L$ is σ-complete iff $\mathcal{Z}L$ is σ-complete and L is strongly zero-dimensional.*

Proof: ($\Rightarrow$) L is basically disconnected by Proposition 2, and since countable joins of complemented elements are cozero elements Proposition 3 shows $\mathcal{Z}L$ is σ-complete. Further, L is strongly zero-dimensional because βL is basically disconnected by Corollary 3, and any basically disconnected completely regular frame is zero-dimensional since each of its elements a is the join of the cozero elements $c \prec\prec a$.

($\Leftarrow$) Since any cozero element in a compact frame is Lindelöf any such element in a compact zero-dimensional frame is a countable join of complemented elements. Hence under the present hypothesis this holds for $\operatorname{Coz} \beta L$ and consequently for $\operatorname{Coz} L$. Thus L is basically disconnected by Proposition 3, and $\mathcal{R}L$ is σ-complete by Proposition 2. $\square$

We note we have not been able to settle whether the σ-completeness of $\mathcal{Z}L$ implies that of $\mathcal{R}L$.

As far as classical topology is concerned, the results obtained in this section appear to be new even in the spatial case: it seems that the order completeness properties of the ring of integer-valued continuous functions on a space have not been studied before even though they are, clearly, a significant aspect of this ring. Incidentally, comparing Remark 4 with some material of Stone [10] concerning the κ-completeness of $C(X)$ shows that the integer-valued case provides the more conclusive result.

4. An application

To begin with, recall that the compact zero-dimensional frames, like the compact completely regular frames, are coreflective in **Frm**, the coreflection obviously given by the subframe ζL of βL generated by the complemented elements of βL, but also realized autonomously as the ideal lattice $\mathfrak{J}(BL)$ of BL with the coreflection map $\mathfrak{J}(BL) \to L$ given by taking joins in L (Banaschewski [2]).

In the following, a frame homomorphism $h : L \to M$ is called an $\mathcal{R}^*$-*isomorphism* or a $\mathcal{Z}^*$-*isomorphism* whenever $\mathcal{R}^*h : \mathcal{R}^*L \to \mathcal{R}^*M$ or $\mathcal{Z}^*h : \mathcal{Z}^*L \to \mathcal{Z}^*M$ is an isomorphism.

Note that the coreflection maps $\beta L \to L$ and $\zeta L \to L$ for completely regular L and zero-dimensional L, respectively, are examples of such: for βL this was discussed earlier and for ζL it follows from the simple observation that each $\alpha \in \mathcal{Z}^* L$ has finite image in L and hence factors through some finite zero-dimensional subframe of L.

Regarding $\mathcal{R}^*$-isomorphisms we have

Lemma 4. *A frame homomorphism $h : L \to M$ is an $\mathcal{R}^*$-isomorphism iff $\beta h : \beta L \to \beta M$ is an isomorphism.*

Proof: ($\Rightarrow$) For any frame N, βN is realized by a frame constructed functorially from the ℓ-ring $\mathcal{R}^* N$ in one way or the other (Banaschewski–Mulvey [5], Banaschewski [4]) and hence if $\mathcal{R}^* h : \mathcal{R}^* L \to \mathcal{R}^* M$ is an isomorphism the same holds for $\beta h : \beta L \to \beta M$.

($\Leftarrow$) Obvious since the coreflection map $\beta M \to M$ is an $\mathcal{R}^*$-isomorphism. $\square$

Proposition 5. *For any completely regular frame L, the following are equivalent:*

(1) *L is extremally disconnected;*
(2) *every dense onto $h : L \to M$ is an $\mathcal{R}^*$-isomorphism;*
(3) *every dense onto $h : L \to M$ is a $\mathcal{Z}^*$-isomorphism.*

Proof: To begin with, note that any dense onto $f : K \to N$ with extremally disconnected K maps BK onto BN: if $c \in BN$ and $f(b) = c$ then $f(b^{**}) = f(b)^{**} = c$, by the proof of Corollary 2 and since c is complemented, and here $b^{**} \in BK$ by extremal disconnectedness.

(1)$\Rightarrow$(2) For $h : L \to M$ as stated, consider the commuting square

$$\begin{tikzcd} \beta L \arrow[r, "\beta h"] \arrow[d, "\beta_L"'] & \beta_M \arrow[d, "\beta_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

Here βL is extremally disconnected by Corollary 2 and $h\beta_L$ is dense onto so that the above observation applies. Consequently, for any complemented $c \in \beta M$ there exist complemented $a \in \beta L$ such that $h\beta_L(a) = \beta_M(c)$, hence $\beta_M \beta h(a) = \beta_M(c)$ and finally $\beta h(a) = c$ since β_M is dense. Now M is extremally disconnected by the proof of Corollary 2 and then βM is extremally disconnected by the same corollary; this makes βM zero-dimensional and therefore βh onto since its image contains all the complemented elements of βM.

(2)$\Rightarrow$(3) Again by Lemma 4 $\beta h : \beta L \to \beta M$ is an isomorphism, hence $\zeta h : \zeta L \to \zeta M$ is also an isomorphism, and since $\zeta M \to M$ is a $\mathcal{Z}^*$-isomorphism, as noted, the same holds for $h : L \to M$.

(3)⇒(1) We verify the Stone identity. For any $a \in L$, if $s = a^* \vee a^{**}$ then the homomorphism $\nu = (\cdot) \wedge s : L \to \downarrow s$ is dense onto and hence a $\mathcal{Z}^*$-isomorphism by hypothesis. Now, a^* is trivially complemented in $\downarrow s$ and we therefore have $\gamma \in \mathcal{Z}L$ such that $\gamma(1) = a^*$ and $\gamma(0) = a^{**}$. Let then $\gamma = \nu\varphi$ for some $\varphi \in \mathcal{Z}^*L$. It follows that φ is an idempotent:

$$\nu\varphi^2 = (\nu\varphi)^2 = \gamma^2 = \gamma = \nu\varphi$$

and ν can be cancelled from this since it is dense. Next, the obvious calculation shows $b = \mathrm{coz}(\varphi)$ is complemented, and then

$$b \wedge s = \nu(\mathrm{coz}(\varphi)) = \mathrm{coz}(\nu\varphi) = \mathrm{coz}(\gamma) = a^*$$

so that $a \wedge b \wedge s = 0$. It follows that $a \wedge b = 0$, hence $b \leq a^*$, and in all then $b = a^*$, showing that $a^* \vee a^{**} = e$, as desired. □

Remark 5. The above equivalence (1)≡(2) is also proved by Ball and Walters-Wayland [1] who derive it from a general characterization of the onto homomorphisms $L \to M$ which are $\mathcal{R}^*$-surjections, that is, for which the corresponding $\mathcal{R}^*L \to \mathcal{R}^*M$ is onto.

It extends a familiar result which is usually expressed in the form: a Tychonoff space is extremally disconnected iff every dense subspace is C^*-embedded (Gillman–Jerison [6]). On the other hand, (1)≡(3) seems to be new even for topological spaces. Regarding the proof of the spatial form of (1)≡(2), the present (1)⇒(2) is substantially different in that it is based on Lemma 4 and hence has a more categorical character. On the other hand, the present (3)⇒(1) is the natural pointfree adaptation of the classical argument for (2)⇒(1).

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Department of Mathematics and Statistics, McMaster University,
Hamilton L8S 4K1, Canada

Department of Mathematics, Sogang University, Seoul 121-742, Korea

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