On a class of discontinuous operators in Hilbert spaces

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Abstract. We construct a class of discontinuous operators in infinite-dimensional separable Hilbert spaces, answering a natural question which arises in comparing a fixed point theorem of Altman and Shinbrot ([1], [4]) with its improvement obtained by Ricceri ([2], [3]).

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In [4], M. Shinbrot gave a proof of the following fixed point theorem which was previously announced (without proof) by M. Altman in [1]:

Theorem A. *Let $(H, \langle\cdot,\cdot\rangle)$ be a separable real Hilbert space, and $\Psi : H \to H$ a sequentially weakly continuous operator. Assume that there is some $r > 0$ such that*

$$\langle \Psi(x), x\rangle \le r^2$$

for all $x \in H$ satisfying $\|x\| = r$.

Then, there exists $x^* \in H$ such that $x^* = \Psi(x^*)$ and $\|x\| \le r$.

In [3] (see also [2]), B. Ricceri obtained an extension of Theorem A to a class of discontinuous operators. His result was as follows:

Theorem B. *Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite-dimensional separable real Hilbert space; V the linear hull of an orthonormal base $\{e_n\}$ of H; $X \subseteq H$ a closed, bounded, convex set, with $0 \in \mathrm{int}(X)$. Further, let $\Psi : X \to H$ be an operator satisfying the following conditions:*

(i) *for each $y \in V$, the set*

$$\{x \in X \cap V : \langle x - \Psi(x), y\rangle \le 0\}$$

is finitely closed (that is, its intersection with any finite-dimensional linear subspace of H is closed);

(ii) *for each $n \in \mathbb{N}$, the set*

$$\{x \in X : \langle x - \Psi(x), e_n\rangle = 0\}$$

is weakly closed;

(iii) *for each $x \in V \cap \partial X$, one has*

$$\langle \Psi(x), x \rangle \leq \|x\|^2.$$

Then, there exists $x^* \in X$ such that $x^* = \Psi(x^*)$.

It is clear that the most natural (though less general) way to check (i) and (ii) is to assume that, for each $n \in \mathbb{N}$, the functional $x \to \langle \Psi(x), e_n \rangle$ be sequentially weakly continuous in X. To see this, take into account that, since H is separable and X is weakly compact, the relative weak topology on X can be deduced by a metric.

On the other hand, the most natural condition ensuring the sequential weak continuity of each functional $x \to \langle \Psi(x), e_n \rangle$ $(n \in \mathbb{N})$ is the sequential weak continuity of the operator Ψ, just as required in Theorem A.

Then, it is natural to ask whether there exist operators $\Psi : X \to H$ which, though not sequentially weakly continuous, satisfy condition (iii) and, at the same time, are such that, for each $n \in \mathbb{N}$, the functional $x \to \langle \Psi(x), e_n \rangle$ is sequentially weakly continuous.

The aim of this paper is to provide an affirmative answer to such a question.

Our main result is as follows:

Theorem 1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional separable real Hilbert space and $\{e_n\}$ an orthonormal base of H. Put*

$$Y = \{x \in H : \langle x, e_1 \rangle = 0\}.$$

Then, there exists an operator $\Phi : H \to H$ which has the following properties:

(a) $Y \subseteq \Phi^{-1}(0)$;

(b) *for each $n \in \mathbb{N}$, the functional $x \to \langle \Phi(x), e_n \rangle$ is weakly continuous;*

(c) $\langle \Phi(x), x \rangle = 0$ *for all $x \in H$;*

(d) $\limsup_{\|x\| \to 0} \|\Phi(x)\| = +\infty$.

Proof: For each $n \in \mathbb{N}$, define $\alpha_n : \mathbb{R} \to \mathbb{R}$ by

$$\alpha_n(t) = \begin{cases} t^{-4} & \text{if } |t| > n^{-\frac{1}{2}}, \\ n^2 & \text{if } (2n)^{-\frac{1}{2}} \leq |t| \leq n^{-\frac{1}{2}}, \\ 2^{\frac{1}{2}} n^{\frac{5}{2}} |t| & \text{if } |t| < (2n)^{-\frac{1}{2}}. \end{cases}$$

Note that each function α_n is continuous and non-negative. Moreover, for each $n \in \mathbb{N}$, $t \in \mathbb{R}$, one has

$$\alpha_n(t) \leq \alpha_{n+1}(t)$$

as well as

$$\sup_{n\in\mathbb{N}} \alpha_n(t) < +\infty.$$

Now, put

$$\varphi_n(t) = (\alpha_n(t) - \alpha_{n-1}(t))^{\frac{1}{2}}$$

with $\alpha_0(t) = 0$. Also, for each $x \in H$, $n \in \mathbb{N}$, set

$$\gamma_n(x) = \begin{cases} -\varphi_{\frac{n+1}{2}}(\langle x, e_1\rangle)\langle x, e_{n+1}\rangle & \text{if } n \text{ is odd,} \\ \varphi_{\frac{n}{2}}(\langle x, e_1\rangle)\langle x, e_{n-1}\rangle & \text{if } n \text{ is even.} \end{cases}$$

Fix $x \in H$. Clearly, the series

$$|\langle x, e_2\rangle|^2 + |\langle x, e_1\rangle|^2| + |\langle x, e_4\rangle|^2 + |\langle x, e_3\rangle|^2 + \dots$$

is convergent and the sequence

$$|\varphi_1(\langle x, e_1\rangle)|^2, |\varphi_1(\langle x, e_1\rangle)|^2, |\varphi_2(\langle x, e_1\rangle)|^2, |\varphi_2(\langle x, e_1\rangle)|^2 \dots$$

is bounded. So, by a classical result, the series

$$\begin{aligned} &|\gamma_1(x)|^2 + |\gamma_2(x)|^2 + |\gamma_3(x)|^2 + |\gamma_4(x)|^2 + \dots \\ &= |\varphi_1(\langle x, e_1\rangle)|^2|\langle x, e_2\rangle|^2 + |\varphi_1(\langle x, e_1\rangle)|^2|\langle x, e_1\rangle|^2 \\ &\quad + |\varphi_2(\langle x, e_1\rangle)|^2|\langle x, e_4\rangle|^2 + |\varphi_2(\langle x, e_1\rangle)|^2|\langle x, e_3\rangle|^2 + \dots \end{aligned}$$

is convergent. Then, by the Riesz-Fischer theorem, for each $x \in H$, the series

$$\gamma_1(x)e_1 + \gamma_2(x)e_2 + \gamma_3(x)e_3 + \gamma_4(x)e_4 + \dots$$

is convergent in H. For each $x \in H$, put

$$\Phi(x) = \sum_{n=1}^{\infty} \gamma_n(x)e_n.$$

So, for each $n \in \mathbb{N}$, one has

$$\gamma_n(x) = \langle \Phi(x), e_n\rangle.$$

Let us now prove that the operator $\Phi : H \to H$ just defined has properties (a)–(d). Property (a) follows at once observing that $\varphi_n(0) = \gamma_n(0) = 0$ for all $n \in \mathbb{N}$. Concerning (b), the weak continuity of each functional γ_n follows at once from the continuity of φ_n and the weak continuity of any continuous linear functional on H. For each $x \in H$, one has

$$\begin{aligned}\langle \Phi(x), x\rangle = & \sum_{n=1}^{\infty} \gamma_n(x)\langle x, e_n\rangle \\ = & -\varphi_1(\langle x, e_1\rangle)\langle x, e_2\rangle\langle x, e_1\rangle + \varphi_1(\langle x, e_1\rangle)\langle x, e_1\rangle\langle x, e_2\rangle \\ & -\varphi_2(\langle x, e_1\rangle)\langle x, e_4\rangle\langle x, e_3\rangle + \varphi_2(\langle x, e_1\rangle)\langle x, e_3\rangle\langle x, e_4\rangle + \dots\end{aligned}$$

Observe that $\sum_{n=1}^{2k} \gamma_n(x)\langle x, e_n\rangle = 0$ for each $k \in \mathbb{N}$, and so $\langle \Phi(x), x\rangle = 0$. That is, (c) is satisfied. Finally, let us check that (d) is satisfied too. To this end, fix $M > 0$ and $r \in]0, 1[$. We shall prove that there is $x \in H$, with $\|x\|^2 = r$, such that $\|\Phi(x)\|^2 > M$. Fix $p \in \mathbb{N}$, with $p > Mr^{-\frac{3}{2}}$. For each $n \in \mathbb{N}$, put

$$\eta_n = \begin{cases} \left(\frac{r}{2p}\right)^{\frac{1}{2}} & \text{if } n \leq 2p, \\ 0 & \text{if } n > 2p. \end{cases}$$

Finally, set

$$x = \sum_{n=1}^{\infty} \eta_n e_n.$$

Clearly, $\|x\|^2 = r$. Also, one has

$$\|\Phi(x)\|^2 = \frac{r}{p}\sum_{n=1}^{p} \varphi_n\left(\left(\frac{r}{2p}\right)^{\frac{1}{2}}\right) = \frac{r}{p}\alpha_p\left(\left(\frac{r}{2p}\right)^{\frac{1}{2}}\right) = r^{\frac{3}{2}}p > M.$$

This concludes the proof. □

Remark 1. Observe that, by (d), the operator Φ is even discontinuous with respect to the strong topology.

Applying Theorem B, via Theorem 1, we then get the following extension of Theorem A:

Theorem 2. *Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite-dimensional separable real Hilbert space, $X \subseteq H$ a closed, bounded, convex set, with $0 \in \mathrm{int}(X)$, and $\Psi : X \to H$ a sequentially weakly continuous operator such that*

$$\langle \Psi(x), x\rangle \leq \|x\|^2$$

for all $x \in \partial X$.

Then, for each operator $\Phi : H \to H$ as in Theorem 1, the operator $\Phi + \Psi$ is not sequentially weakly continuous and admits a fixed point in X.

From Theorem 2, in particular, we get the following surjectivity result:

Theorem 3. *Let $\Phi : H \to H$ be an operator as in Theorem 1. Then, the operator $x \to x - \Phi(x)$ is surjective.*

Proof: Fix $y \in H$ and choose $r > \|y\|$. Let $X = \{x \in H : \|x\| \leq r\}$, and put $\Psi(x) = y$ for all $x \in X$. Then, since, for each $x \in \partial X$, one has

$$\langle \Psi(x), x \rangle \leq \|y\| \|x\| \leq \|x\|^2,$$

one can apply Theorem 2, and so there exists $x^* \in X$ such that $x^* = y + \Phi(x^*)$, as claimed. □

We conclude observing that, when $\Psi : H \to H$ is an affine operator, Theorem B coincides substantially with Theorem A. In fact, we have the following result:

Theorem 4. *Let H, $\{e_n\}$, and X be as in Theorem B, and let $\Psi : H \to H$ be a linear operator such that, for each, the set*

$$\{x \in X : \langle x - \Psi(x), e_n \rangle = 0\}$$

is closed.

Then, Ψ is continuous.

Proof: First, observe that, if $A \subseteq H$ is a linear subspace such that $A \cap X$ is closed, then A is closed. Indeed, fix $r > 0$ so that $\{x \in H : \|x\| \leq r\} \subseteq X$. Let $x \in \overline{A} \setminus \{0\}$, and let $\{x_n\}$ be any sequence in $A \setminus \{0\}$ converging to x. Then, the sequence $\left\{ \frac{r x_n}{\|x_n\|} \right\}$ lies in $A \cap X$ and converges to $\frac{rx}{\|x\|}$. Since $A \cap X$ is closed, it follows that $\frac{rx}{\|x\|} \in A \cap X$, and so $x \in A$, as claimed. Consequently, by assumption, for each $n \in \mathbb{N}$, the hyperplane

$$\{x \in H : \langle x - \Psi(x), e_n \rangle = 0\}$$

is closed, and hence the functional $x \to \langle x - \Psi(x), e_n \rangle$ is continuous. Then, by Osgood's lemma, there is a non-empty open set $\Omega \subset H$ such that

$$\sup_{x \in \Omega} \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} |\langle x - \Psi(x), e_i \rangle|^2 < +\infty.$$

On the other hand, by Parseval's identity, we have

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^{n} |\langle x - \Psi(x), e_i \rangle|^2 = \|x - \Psi(x)\|^2$$

and so

$$\sup_{x \in \Omega} \|x - \Psi(x)\| < +\infty.$$

From this, of course, the continuity of Ψ follows. □

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