On the number of intersections of two polygons

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Abstract. We study the maximum possible number $f(k,l)$ of intersections of the boundaries of a simple k-gon with a simple l-gon in the plane for $k,l \geq 3$. To determine the number $f(k,l)$ is quite easy and known when k or l is even but still remains open for k and l both odd. We improve (for $k \leq l$) the easy upper bound $kl - l$ to $kl - \lceil k/6 \rceil - l$ and obtain exact bounds for $k = 5$ ($f(5,l) = 4l - 2$) in this case.

Keywords: geometry, polygon, intersection, combinatorial complexity

Classification: 52C45, 52C10

1. Introduction

To determine the maximum complexity of union of two or more geometric objects in the plane is among basic extremal geometric problems, see e.g. [1], [2], [3], [5], [6] and [7]. Let $k,l \geq 3$ be given integer numbers. We are interested in the problem of determining the maximum possible number $f(k,l)$ of intersections of a simple k-gon and a simple l-gon. This problem was studied in [2] — the cases when k or l is even are solved there, but an unrecoverable error appears in the case of k and l being both odd. Similar problem was also studied in [4].

We introduce basic definitions and notation in Section 2. The difficulty of determining the number $f(k,l)$ depends on the parity of k and l. If one of these numbers is even, the problem is quite easy and solved; we survey the previous results in Section 3.

We deal with the case of k and l being both odd in Section 4. The bounds $kl - k - l + 3 \leq f(k,l) \leq kl - l$ (for $k \leq l$) are proved in [2]. If $k,l \geq 7$ we improve the upper bound and prove $f(k,l) \leq kl - \lceil \frac{k}{6} \rceil - l$ for $k \leq l$ (Theorem 4). The conjecture is that the lower bound $kl - k - l + 3$ is tight.

We focus on the number of intersections of an l-gon and a pentagon for l odd in Section 5. We prove that $f(5,l) = 4l - 2$ (Theorem 5).

The general problem of determining $f(k,l)$ for k and l both odd, $k,l \geq 7$ remains open.

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2. Definitions and notation

We mean by a *segment* a closed line segment. If A and B are points of the plane, then we write AB for the segment connecting them. We assume throughout the whole paper that all the end-points of all the segments are in general position. Two segments *intersect* if they share an interior point and we call that point their *intersection*. The set of segments is *intersection-free* if no two of segments of the set intersect. Two segments *miss* each other if they do not intersect.

The sequence of non-intersecting segments $A_1A_2, A_2A_3, \dots, A_kA_{k+1}$ where $A_i \neq A_j$ for $i \neq j$ is called a *path*; we write shortly $A_1A_2 \dots A_kA_{k+1}$ instead of $A_1A_2, A_2A_3, \dots, A_kA_{k+1}$ in the paper. The *length* of the path is the number of segments which it contains. If $A_{k+1} = A_1$, then the sequence forms a k*-gon*; if we do not want to emphasize the length of the sequence, we say a *polygon* instead of a k*-gon*. *Intersections of two polygons* are the intersections of segments which form their boundaries. For given $k, l \geq 3$ we denote $f(k,l)$ the maximum number of intersections of some k-gon and l-gon.

Let $\mathcal{A}$ and $\mathcal{B}$ be two intersection-free sets of segments. Then their *non-intersection graph* is a bipartite graph with the vertex set $\mathcal{A} \cup \mathcal{B}$ such that a segment of $\mathcal{A}$ is joined by an edge to a segment of $\mathcal{B}$ iff these two segments do not intersect; the segments of $\mathcal{A}$ ($\mathcal{B}$) are not mutually joined by edges. We state two easy observations to get familiar with these definitions:

Observation 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two intersection-free sets of segments; let $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$. Then the non-intersection graph of $\mathcal{A}'$ and $\mathcal{B}'$ is the subgraph of the non-intersection graph of $\mathcal{A}$ and $\mathcal{B}$ induced by the vertex set $\mathcal{A}' \cup \mathcal{B}'$.*

Observation 2. *Let $ABCD$ be a path which forms together with the segment DA a convex quadrilateral and let p be any segment in the plane. Then the vertex p has degree at least 1 in the non-intersection graph of $\mathcal{A} = \{AB, BC, CD\}$ and $\mathcal{B} = \{p\}$.*

PROOF: If the degree of p is zero, then p intersects all the three segments AB, BC and CD which is impossible since A, B, C and D form a convex quadrilateral.

□

3. Intersections with N-gons for even N's

The following theorems and lemmas are proved (using a different notation) in [2]. Note that the upper bounds in Theorem 1 and Theorem 2 easily follow from Lemma 1.

Lemma 1. *Let K be a k-gon and let p be a line. The number of intersections of p with the segments of K is even and at most k. Thus no line intersects all the segments of a polygon with odd number of vertices.*

Theorem 1. *Let $k, l \geq 3$ be two even integers. Then $f(k,l) = kl$ (the lower bound is shown by Figure 1).*

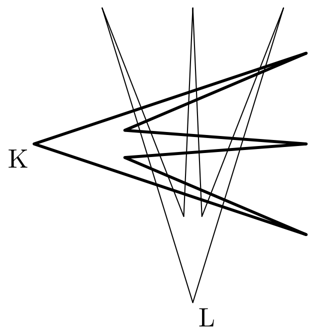

Figure 1: A k-gon K and an l-gon L with kl intersections for k and l both even.

Theorem 2. *Let $k \geq 3$ be even and let $l \geq 3$ be odd. Then $f(k,l) = k(l-1)$ (the lower bound is shown by Figure 2).*

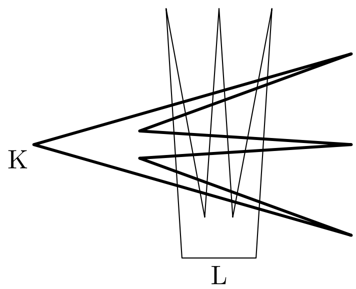

Figure 2: A k-gon K and an l-gon L with $k(l-1)$ intersections for even k and odd l.

Note: Figure 1 and Figure 2 can be generalized for arbitrary k, l by an easy trick. We can substitute one segment of the polygon K (or L) which looks like I by three segments which look like a narrow N. This can obviously be done in such a way that we obtain sufficiently many new intersections.

4. Intersections with N-gons for odd N's

We assume $k \leq l$ in the whole section.

The following lower bound on $f(k,l)$ is also shown in [2].

Theorem 3. *Let $k, l \geq 3$ be two odd integers. Then $f(k,l) \geq (k-1)(l-1)+2 = kl - k - l + 3$ (see Figure 3).*

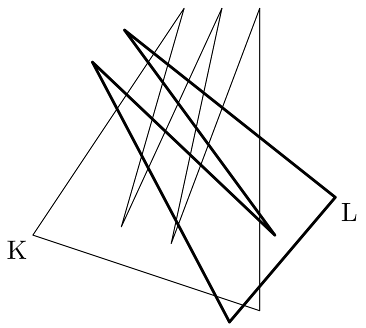

Figure 3: A k-gon K and an l-gon L with $kl - k - l + 3$ intersections for both k and l odd.

We focus our attention on proving an upper bound on the number of intersections of two polygons with odd number of vertices in the rest of this section. There is an easy upper bound $kl - l$ (Theorem 2) which gives the exact value for $k = 3$ (in this case, it is equal to the lower bound). For $k \geq 7$ we improve this bound to $kl - \lceil k/6 \rceil - l$ in Theorem 4. We first prove several lemmas on the number of intersections of a path and a non-intersecting set of segments:

Lemma 2. *Let $ABCDE$ be a path such that the point D is inside the triangle ABC. Then there is no intersection-free set $\mathcal{B}$ of four segments such that the non-crossing graph of $\mathcal{A} = \{AB, BC, CD, DE\}$ and $\mathcal{B}$ forms a perfect matching.*

Proof: Let us suppose there exists such a $\mathcal{B}$. Let p be the segment which misses only CD and let q be the segment which misses only DE. We can suppose that p and q do not share an end-point, since we could shorten them otherwise. Note that p and q do not intersect each other ($\mathcal{B}$ is intersection-free) and they both intersect AB and BC. Hence the segments p, q, AB and BC split the plane into three regions (see Figure 4); let T be the triangle and Q be the quadrilateral. The point E has to lie inside the triangle T, since DE intersects p but not q. Thus the segments CD and DE split the quadrilateral Q into two pentagons; let P be the convex one of them. It is either $P \cap AB = \emptyset$ or $P \cap BC = \emptyset$.

Let us suppose first that $P \cap AB = \emptyset$. Let the vertices of P be B', C', D', D and E' where $B', C' \in BC$, $D' \in CD$ and $E' \in DE$. Let r be the segment of $\mathcal{B}$ which misses only AB; r does not intersect p or q, but it intersects BC, CD and DE; thus it has to intersect CD between D' and D and DE between D and E'. Then r intersects the segments of the convex pentagon P more than twice, but this is impossible. The case that $P \cap BC = \emptyset$ is symmetric. We have just proved that no such set $\mathcal{B}$ can exist. $\square$

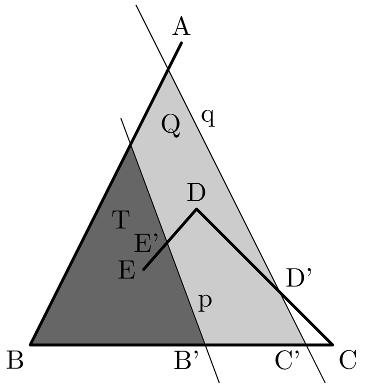

Figure 4: The triangle ABC, the point D inside it, the triangle T and quadrilateral Q, and the forced position of the point E.

Lemma 3. *Let $ABCDEFG$ be a path. Then there is no intersection-free set $\mathcal{B}$ of six segments such that the non-crossing graph of $\mathcal{A} = \{AB, BC, CD, DE, EF, FG\}$ and $\mathcal{B}$ forms a perfect matching.*

Proof: Let us suppose the existence of such a $\mathcal{B}$. The lines BC, CD and BD split the plane into seven regions (see Figure 5): α, β, γ, δ, ϵ, ζ and η.

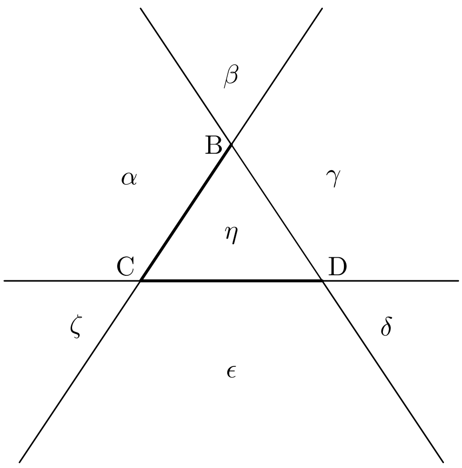

Figure 5: The seven regions α, β, γ, δ, ϵ, ζ and η in which the lines BC, CD and BD split the plane.

The point E cannot lie in α, since BC and DE do not intersect, it cannot lie in β due to Observation 1 and Lemma 2 used for the path $EDCBA$, it cannot lie in γ due to Observation 2 used for the path $BCDE$, it cannot lie in ϵ, since any segment intersecting both BC and DE would not miss CD, and it cannot lie in η due to Observation 1 and Lemma 2 used for the path $BCDEF$. Thus the point E is either in δ or in ζ.

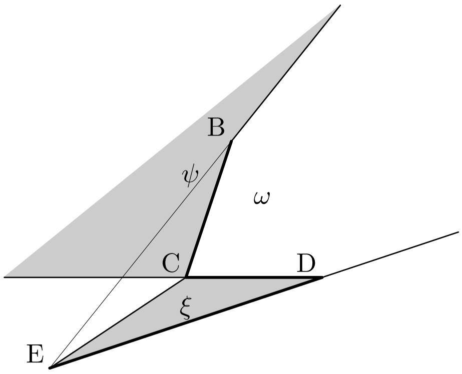

Figure 6: The regions ξ, ψ and ω; ξ and ψ are marked by gray.

We first deal with the case that the point E lies in ζ. Consider the regions ξ, ψ and ω as shown in Figure 6. The point F cannot lie in ξ due to Observation 1 and Lemma 2 used for the path $CDEFG$, it cannot lie in ψ due to Observation 1 and Lemma 2 used for the path $FEDCB$ and it cannot lie in ω, since EF crosses neither BC nor CD. If F were outside ξ, ψ and ω, it would be impossible to intersect all the segment except for DE by a segment. We conclude that E cannot lie in ζ.

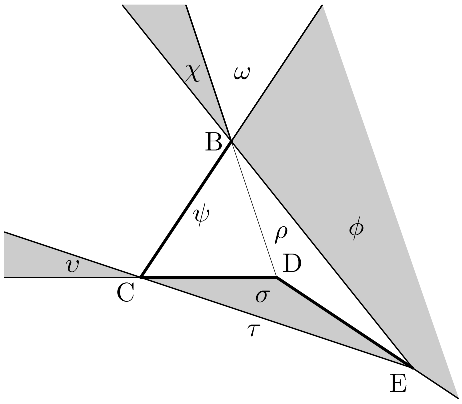

Figure 7: The regions σ, τ, υ, ϕ, χ, ψ, ρ and ω; the regions σ, υ, ϕ and χ are marked by gray.

The point E has to lie in δ according to the discussion above. Consider the regions σ, τ, υ, ϕ, χ, ψ, ρ and ω as shown in Figure 7. We discuss the position of F.

The point F cannot lie in σ due to Observation 1 and Lemma 2 used for the path $CDEFG$, it cannot lie in τ due to Observation 2 used for the path $CDEF$ and it cannot lie in υ due to Observation 1 and Lemma 2 used for the path $FEDCB$. The point F cannot lie in ϕ or ω, since there would be no segment intersecting BC, DE and EF which misses CD and the point F cannot lie in ρ, since there would be no segment intersecting BC, CD and EF which misses DE. Thus the point F has to lie either in ψ or χ; these two cases are "symmetric" with respect to changing the orientation of the given path (we take the path $GFEDCBA$ instead of $ABCDEFG$) — see Figure 8. We may thus assume w.l.o.g. that F is in ψ.

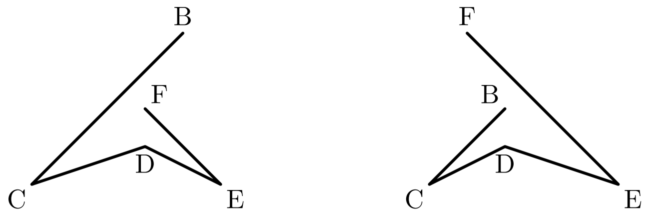

Figure 8: The cases that the point F is in ψ or χ are symmetric.

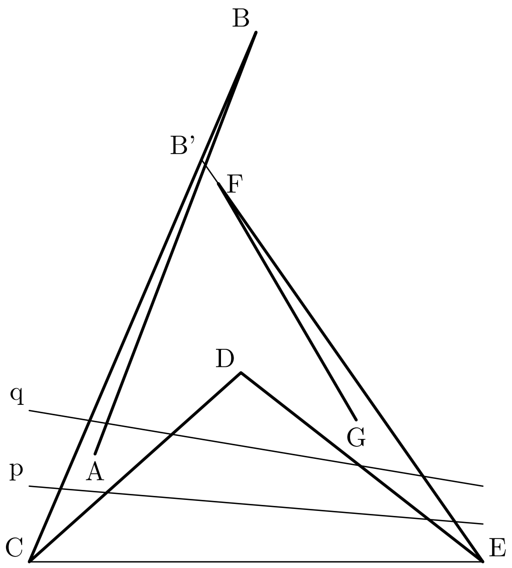

Figure 9: The segments p and q and the triangle T'.

The situation is shown in Figure 9. Let B' be the intersection of the lines BC and EF and let T be the triangle $B'CE$. Every line intersecting both CD and DE has to intersect also $B'C$ and EF. If A were outside T, then there could not be a segment which misses only BC and a segment which misses only EF

at once. A similar argument holds also for the point G. Thus both the points A and G are inside T. Let p be the segment of $\mathcal{B}$ which misses only AB and let q be the segment which misses only FG. Since p (q) intersects CD and DE, the intersection with BC is between B' and C. The triangle T is splitted into three regions by p and q, see Figure 9; let T' be the triangular one. We first deal with the case that one side of T' is formed by q. Since FG misses q, the point G has to lie inside T'. But then, the segments FG and p do not intersect. If one side of T' is formed by p, then A has to lie inside T' and AB and q do not intersect. Both cases are impossible. But this was the very last possible configuration and thus the set $\mathcal{B}$ does not exist. $\square$

Lemma 3 is the best possible in the sense that there is a path of length five and a set of five non-intersecting segments such that their non-intersection graph is a perfect matching (see Figure 10).

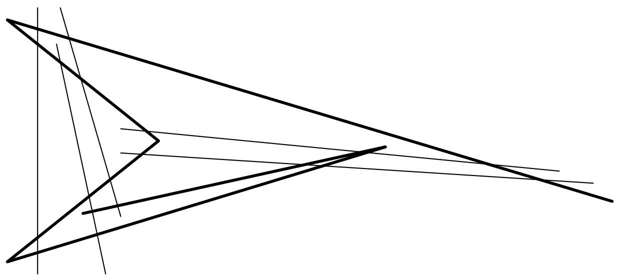

Figure 10: A path of length five and a set of five non-intersecting segments such that their non-intersection graph is a perfect matching.

Lemma 4. *Let $ABCDEFG$ be a path and let $\mathcal{B}$ be a set of non-intersecting segments. The following holds for at least one segment r among AB, BC, CD, DE, EF and FG: If a segment q of $\mathcal{B}$ intersects at least five segments of the path, then it intersects r.*

Proof: If the lemma does not hold, then for each segment p of AB, BC, CD, DE, EF and FG, there exists a segment of $\mathcal{B}$ which misses only p. But then these six segments of $\mathcal{B}$ and the path $ABCDEFG$ contradict Lemma 3. $\square$

Theorem 4. *Let $k, l \geq 7$ be odd. Then $f(k,l) \leq kl - \lceil k/6 \rceil - l$.*

Proof: Let K be any k-gon and let L be any l-gon. Let $\mathcal{A}$ be the set of segments forming the polygon K and let $\mathcal{B}$ be the set of segments forming the polygon L; let G be a non-intersection graph of $\mathcal{A}$ and $\mathcal{B}$. Each vertex of G has degree at least one, since no segment can intersect all the segments of a polygon with odd number of vertices due to Lemma 1.

Let $p_1, \dots, p_k$ be the segments of K. Let r_1 be the segment for the path $p_1p_2p_3p_4p_5p_6$ with the properties described in Lemma 4, $r_1 = p_{i_1}$ for some $1 \leq i_1 \leq 6$. Let r_2 be the segment for the path $p_{i_1+1}p_{i_1+2}p_{i_1+3}p_{i_1+4}p_{i_1+5}p_{i_1+6}$ with the properties described in Lemma 4, $r_2 = p_{i_2}$ for some $i_1 + 1 \leq i_2 \leq i_1 + 6$. By repeating this procedure we can find segments r_j, $j = 1, 2, \dots, \lceil k/6 \rceil$, it is easy to see that the path for $r_{\lceil k/6 \rceil}$ contains none of the segments $r_1, r_2, \dots, r_{\lceil k/6 \rceil - 1}$.

Each segment of $\mathcal{B}$ misses at least one segment of $\mathcal{A}$ different from the segments $r_1, r_2, \dots, r_{\lceil k/6 \rceil}$; otherwise due to Lemma 4 it would have to intersect also $r_{\lceil k/6 \rceil}$, $r_{\lceil k/6 \rceil - 1}, \dots, r_2$ and r_1 hence it would have to intersect all the segments of $\mathcal{A}$ which is impossible due to Lemma 1 since the segments of $\mathcal{A}$ form a polygon with odd number of vertices.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be the set $\{r_1, \dots, r_{\lceil k/6 \rceil}\}$. The degree of each vertex of $\mathcal{B}$ in the subgraph of G induced by $(\mathcal{A} \setminus \mathcal{A}') \cup \mathcal{B}$ is at least one, since each segment of $\mathcal{B}$ misses at least one segment of $\mathcal{A} \setminus \mathcal{A}'$; thus this subgraph contains at least l edges. The degree of each vertex of $\mathcal{A}'$ in the subgraph of G induced by $\mathcal{A}' \cup \mathcal{B}$ is at least one, since no segment can intersect all the segments of $\mathcal{B}$; this subgraph contains at least $\lceil k/6 \rceil$ edges. The whole graph G thus contains at least $\lceil k/6 \rceil + l$ edges and thus the number of intersections of K and L is at most $kl - \lceil k/6 \rceil - l$. $\square$

5. Intersections with pentagons

Observation 3. *Let K be a polygon. Then there exist three consecutive vertices A, B, C of K such that the interior angle $\angle ABC$ is convex and no vertex of K lies in the interior of the triangle $\triangle ABC$.*

PROOF: As a well known fact every (not necessarily convex) polygon has a triangulation. Let us consider any triangulation T of K. Let G be a graph with the vertex set consisting of all triangles of T; two distinct triangles are connected by an edge iff they are sharing a common segment. Then G is obviously a tree and we can take A, B, C to be vertices of any leaf of G. $\square$

Lemma 5. *Let K be a k-gon for $k \geq 5$ odd. Then there are four consecutive vertices A, B, C, D of K such that one of the four following conditions holds:*

(1) *No line p intersects all the segments AB, BC and CD.*
(2) *Every line p which intersects the segments AB and BC intersects also the segment CD.*
(3) *Every line p which intersects the segments AB and CD intersects also the segment BC.*
(4) *Every line p which intersects the segments BC and CD intersects also the segment AB.*

Proof: Take the points A, B, C constructed in the Observation 3 and let D be the vertex neighboring with C (different from B). The lines AB, BC and CA split the plane into the six regions $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ where the vertex D can lie (due to the choice of the points A, B, C no vertex of K lies in the interior of the triangle $\triangle ABC$) (see Figure 11). We can w.l.o.g. suppose that D does not lie in the region ζ (if it does we can take $A' = C$, $B' = B$, $C' = A$ and D' the vertex neighboring with C' different from B', then D' cannot lie in $\zeta' = \delta$ unless the segments CD and $C'D'$ intersect which is impossible). The vertex D cannot lie in the region α because AB and CD cannot intersect. If D lies in β then condition (2) holds. If D lies in γ then condition (3) holds. If D lies in δ then condition (4) holds. Finally if D lies in ϵ then condition (1) holds. □

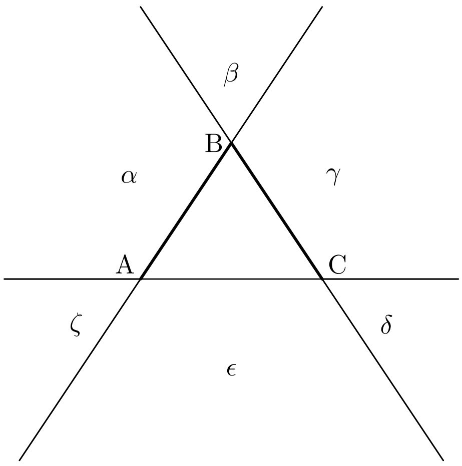

Figure 11: The six regions $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ where the vertex D can lie.

Corollary 1. *Let $K = ABCDE$ be a pentagon. There are four segments among AB, BC, CD, DE and EA such that no line intersects all of these four segments.*

Proof: Suppose that A, B, C, D are the vertices constructed in Lemma 5. If condition (1) of Lemma 5 holds then no line intersects all the segments AB, BC, CD hence we can take e.g. the segments AB, BC, CD and DE. If condition (2) holds then no line intersects all the segments AB, BC, DE, EA (if such line exists then it also intersects the segment CD by condition (2) which is impossible due to Lemma 1). The cases when conditions (3) or (4) hold are similar. □

Theorem 5. *Let $l \geq 5$ be odd. Then $f(5, l) = 4l - 2$.*

Proof: The inequality $f(5, l) \geq 4l - 2$ is assured by Theorem 3. It remains to prove the upper bound for the number of intersections. Let L be any l-gon and K

be any pentagon. Each segment of L has at most 4 intersections with segments of K due to Lemma 1. Thus K and L cannot have more than $4l$ intersections. The number of intersections of K and L is even: Imagine that we pass along the boundary of K; each time we intersect a segment of L, we either enter or leave the interior region of L and we cannot enter or leave it in any other way. Thus if the number of intersections is more than $4l-2$, it has to be $4l$ and each segment of L intersects exactly four segments of K. But there are four segments of K which cannot be intersected by a line (more likely by a segment) due to Corollary 1; let p be the remaining segment of K. Each of the segments of L has to intersect p (otherwise it would intersect the remaining four segments of K which is impossible). But p cannot intersect all the segments of L due to Lemma 1 — thus K and L have at most $4l-2$ intersections. □

Note: It is quite obvious that Corollary 1 and thus the upper bound in Theorem 5 can be easily generalized to all k-gons for $k \geq 5$ odd. We obtain $f(k,l) \leq kl-l-2$ for $5 \leq k \leq l$ and k, l odd. However for $k > 5$ Theorem 4 gives us the same or even better bound.

6. Conclusion

We have focused on the case of k and l being both odd. We proved an exact bounds on $f(k,l)$ when k is 3 or 5 and l is arbitrary odd. The case that k and l are both odd and at least seven remains open also for $k=l=7$. We improved the simple upper bound of $kl-l$ to $kl-\lceil k/6\rceil-l$. The original conjecture is that there are no k-gon and l-gon with more than $kl-k-l+3$ intersections; the construction of a k-gon and an l-gon with $kl-k-l+3$ intersections is described in Theorem 3. The gap between the lower and the upper bound is still linear in the number of vertices thus the natural task might be: Improve the upper (lower) bound on the maximum possible number of intersections of two polygons.

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