Contractive projections and Seever's identity in complex f-algebras

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Abstract. In this paper we give necessary and sufficient conditions in order that a contractive projection on a complex f-algebra satisfies Seever's identity.

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1. Introduction

Let T be a contractive projection on $C_0(X)$, where X is a locally compact Hausdorff space. Seever [11] showed that if T is positive, then T satisfies the Seever's identity, that is,

$$(S) \qquad T(fTg) = T(TfTg) \quad \text{for all} \quad f, g \in C_0(X).$$

Some years later, Wulbert [16] proved that if X is compact and if the range of T has a weakly separating quotient, then (S) holds. After that, Huijsmans and de Pagter [6] extended the aforementioned Seever's result to the more general case of semiprime f-algebras satisfying the Stone condition. Recently, Triki [12] improved this result by showing that the same result holds true for arbitrary Archimedean f-algebras without any extra conditions. In [4] Friedman and Russo found a necessary and sufficient condition in order that a contractive projection on $C_0(X)$ verifies the Seever's identity. They proved, in the same paper, that Wulbert's condition (the range of T has a weakly separating quotient) and therefore the condition that T is positive, is not necessary. They gave the following example: Let $A = C([-2,-1] \cup [1,2])$ and let $\chi = \chi_{[1,2]}$. Then T defined by $Tf(x) = \frac{\chi(x)\ f(x) - \chi(-x)\ f(-x)}{2}$ is a contractive projection which verifies (S), but the range $R(T)$ does not have a weakly separating quotient.

In the present work we shall consider complex f-algebras, with additional hypotheses, and we shall give necessary and sufficient condition in order that (S) holds.

2. Preliminaries

For terminology and elementary properties of vector lattices and f-algebras we refer to the standard works [9], [10] and [15]. All vector lattices and lattice-ordered algebras under consideration are supposed to be Archimedean. The topologies considered on these spaces are the order topology (cf. [9, Sections 16 and 65]) and the (relatively) uniform topology (cf. [9, Sections 16 and 63]).

The (real) algebra A is called a *lattice-ordered algebra* (briefly, *ℓ-algebra*) if A is simultaneously a vector lattice such that the ordering and the multiplication are compatible (i.e., $f, g \in A^+$ implies $fg \in A^+$). The ℓ-algebra A is called an *f-algebra* whenever $f \wedge g = 0$ and $0 \leq h \in A$ imply $fh \wedge g = hf \wedge g = 0$. Since every Archimedean f-algebra is commutative, we deal only with commutative f-algebras in this paper. The ℓ-algebra A is said to be *semiprime* if 0 is the only nilpotent element of A. Every unital f-algebra is semiprime.

Let A and B be vector lattices. The operator $\pi : A \to B$ is called *order bounded* if the image under π of an order bounded set is again an order bounded set (notation $\pi \in \mathcal{L}_b(A, B)$). The operator $\pi : A \to B$ is said to be *positive* if $\pi(A^+) \subset B^+$.

Let A be a uniformly complete semiprime f-algebra. According to [2], for all elements $f = a + ib$ in the complexification $A_{\mathbb{C}} = A + iA$ of A, the familiar modulus

$$|f| = \sup \{a \cos \theta + b \sin \theta : 0 \leq \theta \leq 2\pi\}$$

is equal to $\sqrt{a^2 + b^2}$. This implies several properties of the complexification. We list some of them:

(i) $|fg| = |f||g|$ for all $f, g \in A_{\mathbb{C}}$. In particular $\left|f^k\right| = |f|^k$ for all $f \in A_{\mathbb{C}}$ $(k = 1, 2, \dots)$.

(ii) $A_{\mathbb{C}}$ is a "complex f-algebra", that is, $|f| \wedge |g| = 0$ implies $|hf| \wedge |g| = |fh| \wedge |g| = 0$ for all $h \in A_{\mathbb{C}}$.

(iii) $|f| \wedge |g| = 0$ if and only if $fg = 0$.

It is said that the sequence $(f_n,\ n = 1, 2, \dots)$ in $A_{\mathbb{C}}$ is order convergent to the element $f \in A_{\mathbb{C}}$ whenever there exists a sequence $p_n \downarrow 0$ in A such that $|f_n - f| \leq p_n$ holds for all n. This will be denoted by $f_n \to f$.

If T is a linear operator on A, then T has a unique extension as a linear operator on $A_{\mathbb{C}}$ by defining

$$T(f + ig) = Tf + iTg$$

for all f and g in A. It follows that all the results proved in this paper hold true for real operators. The linear operator $T = T_1 + iT_2$ on $A_{\mathbb{C}}$ is called *order bounded* (notation $T \in \mathcal{L}_b(A_{\mathbb{C}})$) if $T_1, T_2 \in \mathcal{L}_b(A)$. An operator $T \in \mathcal{L}_b(A_{\mathbb{C}})$ is said to be *order continuous* (or *σ-order continuous*) whenever T_1, T_2 are so. For further details on complex f-algebras we refer the reader to [2] and [15].

Finally, we recall that a linear map T on a semiprime complex f-algebra $A_{\mathbb{C}}$ is said to be *contractive* if $|f|^2 \leq |f|$ implies $|T(f)|^2 \leq |T(f)|$. Obviously, if A has a unit element e, then T is contractive whenever $|f| \leq e$ implies $|T(f)| \leq e$.

3. Seever's identity in unital complex f-algebras

We begin this section by recalling the Seever's identity. We say that a linear operator T on a complex f-algebra $A_{\mathbb{C}}$ satisfies the *Seever's identity* (S), if

$$(S) \qquad T(fTg) = T(TfTg), \quad \text{for all} \quad f, g \in A_{\mathbb{C}}.$$

The main topic of this paper is the connection between contractive projections and operators satisfying (S). Our first proposition in this context is the following.

Proposition 1. *Let A be a unital f-algebra and T be an operator on $A_{\mathbb{C}}$ satisfying (S). Then T^2 is a projection that verifies (S).*

PROOF: Let e denote the unit element of A and $f, g \in A_{\mathbb{C}}$. It follows from

$$T^3 f = T(eT^2 f) = T(TeTf) = T(eTf) = T^2 f$$

that T^2 is projection.

On the other hand

$$T^2(T^2 f T^2 g) = T^2(TfT^2 g) = T^2(fT^2 g)$$

and thus T^2 satisfies the Seever's identity. □

Remark 1. An operator T, which satisfies (S), needs not be a projection (although T^2 is projection). Take, for instance, $A = \mathbb{R}^3$ with the pointwise operations and $T : A \to A \quad (x, y, z) \longmapsto (0, x, z)$.

Let A be a σ-Dedekind complete f-algebra with unit element e and T be a σ-order continuous contractive projection on $A_{\mathbb{C}}$. So, $T(e) = a + ib$ for some $a, b \in A$. We put

$$u = \inf\{\left(a^+\right)^n : n = 1, 2, \dots\}$$

which exists in A because A is σ-Dedekind complete. This infimum satisfies the following.

Lemma 1. (i) *u is idempotent (i.e., $u^2 = u$).*
(ii) *$uTf \in A^+$ for all $f \in A^+$.*

PROOF: (i) Since $a^+ \leq |Te| \leq e$, it follows that the sequence $((a^+)^n, n = 1, 2, \dots)$ is decreasing and then

$$u^2 = \inf\{\left(a^+\right)^{2n} : n = 1, 2, \dots\} = \inf\{\left(a^+\right)^n : n = 1, 2, \dots\} = u$$

where the first equality is obtained by using the fact that the multiplication by a positive element in A is order continuous.

(ii) We prove first that $u = uTe$. To this end, denote by B_u the band generated by u in A. Since $A = B_u + B_u^d$ (recall that A is σ-Dedekind complete and therefore it verifies the principal projection property [9, Theorem 25.1]), there exist $a_1, b_1 \in B_u$ and $a_2, b_2 \in B_u^d$ such that $a = a_1 + a_2$ and $b = b_1 + b_2$. It follows from $|a_1 + ib_1| \wedge |a_2 + ib_2| = 0$ that

$$|Te| = |(a_1 + a_2) + i\,(b_1 + b_2)| = |a_1 + ib_1| + |a_2 + ib_2|\,.$$

Hence

$$a_1^2 + b_1^2 = |a_1 + ib_1|^2 \leq |Te|^2 \leq e.$$

So

$$(ua_1)^2 + (ub_1)^2 \leq u.$$

Moreover

$$ua = u\,(a_1 + a_2) = ua_1 \quad \text{and} \quad ub = u\,(b_1 + b_2) = ub_1,$$

then

$$(ua)^2 + (ub)^2 \leq u.$$

Observe that $au = (a^+ - a^-)u = u$. This together with the last inequality yields $(ub)^2 = 0$ and so $ub = 0$, as A is semiprime. Thus $u = uTe$, as required. Now, we define $\widetilde{T} : A_{\mathbb{C}} \to (B_u)_{\mathbb{C}}$ by putting $\widetilde{T}f = uTf$ for all $f \in A_{\mathbb{C}}$. Obviously, $\widetilde{T}$ is an order bounded contractive operator mapping e, the unit element of A, onto u the unit element of B_u. Hence, according to [13, Proposition 4.1], $\widetilde{T}$ is positive. This implies the desired result. $\square$

Lemma 2. *Let A be a σ-Dedekind complete f-algebra with unit element e and let T be a σ-order continuous contractive projection on $A_{\mathbb{C}}$. If T satisfies Seever's identity and if we put $Te = a + ib$ $(a, b \in A)$, then*

(i) $T(b^n) = 0$ *for all* $n \in \{1, 2, \dots\}$;
(ii) $T(a^m b^n) = 0$ *for all* $n, m \in \mathbb{N}$ *such that* $n \geq 1$;
(iii) $Te = T(a^n)$ *for all* $n \in \mathbb{N}$.

Proof: First, we note that since T is a projection, by setting $g = e$ in the Seever's identity, we get

$$Tf = T(eTf) = T(fTe)$$

for all $f \in A_{\mathbb{C}}$. It follows that

$$Tf = T(f(Te)^n) \tag{1}$$

for all $f \in A_{\mathbb{C}}$ and all $n \in \mathbb{N}$.

(i) The proof is by induction on n. Since A is an f-algebra, we have

$$\left|(Te)^n(e-|Te|^2)\right| = \left||Te|^n - |Te|^{n+2}\right|. \tag{2}$$

The sequence $(|Te|^n)_{n\in\mathbb{N}}$ is order convergent in A, because A is σ-Dedekind complete and $|Te|\le e$. Then it follows from (2) that

$$(Te)^n\,(e-|Te|^2) \longrightarrow 0.$$

Hence the σ-order continuity of T implies that

$$T((Te)^n\,(e-|Te|^2)) \to 0.$$

Therefore (1) implies that $T(e-|Te|^2)=0$. Consequently, again by using (1) (for $n=1$) we get

$$Te = T(\overline{Te})$$

and so $Tb = 0$. This shows that (i) holds for $n=1$. Assume now that $T(b^n)=0$ (for every operator satisfying the hypothesis of this lemma) and prove that $T(b^{n+1})=0$. To this end, define the mapping $\widetilde{T}: A_{\mathbb{C}} \to A_{\mathbb{C}}$ by $\widetilde{T}(f) = T(\overline{Te}f)$ for all $f\in A_{\mathbb{C}}$. It is straightforward to show that $\widetilde{T}$ is a σ-order continuous contractive projection satisfying Seever's identity and $\widetilde{T}(e) = Te = a+ib$. It follows that $\widetilde{T}(b^n)=0$ and therefore

$$T(ab^n) - iT(b^{n+1}) = 0. \tag{3}$$

On the other hand, it follows from $T(b^n)=0$ and

$$T(b^n) = T(b^nTe) = T(b^n(a+ib))$$

that

$$T(ab^n) + iT(b^{n+1}) = 0. \tag{4}$$

Combining (3) and (4), we infer that $T(b^{n+1})=0$, which finishes the induction step.

(ii) We proceed again by induction. Since $T(b^n) = T(ab^n)+iT(b^{n+1})$ for all $n\in\mathbb{N}$, by (i) we get $T(ab^n)=0$ for all $n\in\{1,2,\dots\}$. Now let $m\in\mathbb{N}$ and assume that $T(a^mb^n)=0$ for all $n\in\{1,2,\dots\}$. We shall prove that $T(a^{m+1}b^n)=0$ for all $n\in\{1,2,\dots\}$. We have

$$\begin{aligned} T\left(a^mb^n\right) &= T\left(a^mb^nTe\right) = T\left(a^mb^n(a+ib)\right) \\ &= T\big(a^{m+1}b^n\big) + iT\big(a^mb^{n+1}\big). \end{aligned}$$

This implies the desired result.

(iii) It follows from (i) that $T(e) = T(a)$. Using (ii), we deduce that

$$T(a) = T(a(Te)^n) = T\Big(a\sum_{k=0}^{n} a^k (ib)^{n-k}\Big) = T\left(a^{n+1}\right)$$

for all $n \in \mathbb{N}$. This completes the proof. $\square$

For the sake of simplicity, we introduce the following definition that will be used for later purposes.

Definition 1. *Let A be an f-algebra and T a projection on $A_{\mathbb{C}}$. We say that T is almost positive if there exists an order projection π_T on $A_{\mathbb{C}}$ such that*

$$T(\pi_T Tf) = Tf \quad \textit{for all} \quad f \in A_{\mathbb{C}}$$

and

$$\pi_T Tf \in A^+ \quad \textit{for all} \quad f \in A^+.$$

In this case, we say that T is an almost positive projection with an order projection π_T.

Of course, a positive projection is almost positive.

The proposition below gives another characterization of almost positive projections. Its proof is easy and consequently omitted.

Proposition 2. *Let A be an f-algebra and let T be a projection on $A_{\mathbb{C}}$. Then T is almost positive if and only if T can be written in the form $T = T_1 + T_2$, where*

(i) *T_1 is a positive projection given by $T_1 = \pi T$, where π is an order projection;*

(ii) *T_2 is a linear operator on $A_{\mathbb{C}}$ such that $T_1 T_2 = 0$ and $T_2^2 = 0$.*

The following result shows that an almost positive projection T is uniquely determined by its range $R(T)$ and the positive projection $T_1 = \pi_T T$.

Proposition 3. *Let A be an f-algebra and let T, T' be almost positive projections on $A_{\mathbb{C}}$ with order projections π_T and $\pi_{T'}$ respectively. If $R(T) = R\left(T'\right)$ and if π_T, $\pi_{T'}$ satisfy $\pi_T T = \pi_{T'} T'$, then $T = T'$.*

Proof: We have

$$T = T\pi_T T = T\pi_{T'} T' = T(T' - (\mathrm{I} - \pi_{T'})T'),$$

where I is the identity mapping. From $R(T) = R\left(T'\right)$, it follows that $TT' = T'$. Hence

$$T = T' - T(\mathrm{I} - \pi_{T'})T' \tag{1}$$

multiplying by $(\mathrm{I}-\pi_{T'})T'$, we get

$$\begin{aligned} T(\mathrm{I}-\pi_{T'})T' &= T'(\mathrm{I}-\pi_{T'})T' - T(\mathrm{I}-\pi_{T'})T'(\mathrm{I}-\pi_{T'})T' \\ &= 0. \end{aligned}$$

Thus (1) implies that $T = T'$. $\square$

Remark 2. Recall that if A is a unital f-algebra, then order projections on $A_{\mathbb{C}}$ are multiplications by idempotent elements in A. Thus a projection T on $A_{\mathbb{C}}$ is almost positive if and only if there exists $u \in A$ such that $u^2 = u$, $T(uTf) = Tf$ for all $f \in A_{\mathbb{C}}$ and $uTf \in A^+$ for all $f \in A^+$.

We are now in a position to prove the main result of this section.

Theorem 1. *Let A be a σ-Dedekind complete f-algebra, with unit element and let T be a σ-order continuous contractive projection on $A_{\mathbb{C}}$. Then T satisfies the Seever's identity if and only if T is almost positive.*

Proof: Suppose that T verifies the Seever's identity (S) and let e denote the unit element of A. Put $Te = a + ib$ $(a, b \in A)$. By Lemma 1, $u = \inf\{(a^+)^n : n = 1, 2, \dots\}$ verifies $u^2 = u$ and $uTf \in A^+$ for all $f \in A^+$. We claim that $Tu = Te$. Indeed, since A is σ-Dedekind complete and $a^- \leq e$, there exists $v \in A^+$ such that $\left(a^-\right)^n \downarrow v$. Consequently

$$a^{2n} = \left(a^+\right)^{2n} + \left(a^-\right)^{2n} \to u + v$$

and

$$a^{2n+1} = \left(a^+\right)^{2n+1} - \left(a^-\right)^{2n+1} \to u - v.$$

Thus the σ-order continuity of T together with (iii) of Lemma 2 imply $Tv = 0$ and so $Tu = Te$. Hence $Tf = T(T(u)f) = T(uTf)$. So by Remark 2, T is almost positive.

Conversely, we consider the operator $\widetilde{T} : A_{\mathbb{C}} \to A_{\mathbb{C}}$ defined by $\widetilde{T}f = uTf$ for all $f \in A_{\mathbb{C}}$. Then the hypothesis imply that $\widetilde{T}$ is a positive contractive projection. Thus by [12, Theorem 3.4] Seever's identity holds for $\widetilde{T}$. Therefore

$$\begin{aligned} \widetilde{T}(f\widetilde{T}g) &= \widetilde{T}(\widetilde{T}ef\widetilde{T}g) = \widetilde{T}(u^2TefTg) \\ &= \widetilde{T}(uTefTg) = \widetilde{T}(\widetilde{T}efTg) \\ &= \widetilde{T}(fTg) = uT(fTg) \end{aligned}$$

for all $f, g \in A_{\mathbb{C}}$.
Similarly,

$$\widetilde{T}(\widetilde{T}f\widetilde{T}g) = uT(TfTg).$$

Then the identity (S) for $\widetilde{T}$ yields

$$uT(fTg) = uT(TfTg).$$

Applying T and using the hypothesis, we get

$$T(fTg) = T(TfTg)$$

and we are done. $\square$

The assumption that T is σ-order continuous in the above theorem cannot be dropped as it is shown in the next example.

Example 1. Let A be the Dedekind completion of $C([-1,1])$. Then there exists a one-one lattice homomorphism $\varphi : A \to \mathbb{R}^{[-1,1]}$ such that φ preserves the unit element of A (see [15, Theorem 83.18] and its proof). In view of [7, Corollary 5.5], φ is an algebra homomorphism and consequently A can be embedded as a sub-f-algebra in $\mathbb{R}^{[-1,1]}$. Let $T : A \to A$ be the operator defined by $Tf = f(1)g_1 - f(-1)g_2$, where g_1 and g_2 are given respectively by:

$$g_1(x) = \begin{cases} 0 & \text{for } \; -1 \leq x \leq 1/3 \\ \dfrac{3}{2}x - \dfrac{1}{2} & \text{for } \; 1/3 \leq x \leq 1 \end{cases}$$

and

$$g_2(x) = \begin{cases} \dfrac{4}{3}x + \dfrac{1}{3} & \text{for } \; -1 \leq x \leq 0 \\ \dfrac{-1}{3}x + \dfrac{1}{3} & \text{for } \; 0 \leq x \leq 1. \end{cases}$$

It is an easy task to verify that T is a contractive projection satisfying the Seever's identity. Assume that there exists $u \in A$ such that $u^2 = u$, $uTf \in A^+$, for all $f \in A^+$, and $T(uTf) = Tf$ for all $f \in A$. Put $u = \chi_S$, $S \subset [-1,1]$. Let $n \in \mathbb{N}$ and consider f_n the positive function in A defined by:

$$f_n(x) = \begin{cases} n^2x^2 & \text{for } \; -1 \leq x \leq 0 \\ nx & \text{for } \; 0 \leq x \leq 1. \end{cases}$$

Then the inequality $\chi_S Tf_n \geq 0$ implies $\chi_S(x)g_1(x) \geq n\chi_S(x)g_2(x)$, for all $x \in [-1,1]$. This being true for all $n \in \mathbb{N}$, the Archimedean property yields $S \subset [-1,0[\cup \{1\}$, as $g_2(x) > 0$ for all $x \in [0,1[$. Hence we deduce from the equality

$$\chi_S = \sup\{f \in C[-1,1] \quad 0 \leq f \leq \chi_S\}$$

that $\chi_S(1) = 0$. Therefore

$$f(1) = T(f)(1) = T(\chi_S Tf)(1) = 0$$

for all $f \in A$, which is wrong.

4. Seever's identity for nonunital complex f-algebras

In this section we give a nonunital version of Theorem 1. To this end, we make some preparations. First we recall some definitions and properties of orthomorphisms.

Let A be a vector lattice. The order bounded operator $\pi : A \longrightarrow A$ is called *orthomorphism* if $|f| \wedge |g| = 0$ implies $|\pi(f)| \wedge |g| = 0$. The collection $\mathrm{Orth}(A)$ of all orthomorphisms on A is an Archimedean f-algebra with respect to the usual vector spaces operations and composition as multiplication. Besides, the identity mapping I on A is the unit element of $\mathrm{Orth}(A)$. The principal order ideal in $\mathrm{Orth}(A)$ generated by the identity mapping I is called the *center* of A and is denoted by $\mathrm{Z}(A)$. If A is Dedekind complete then $\mathrm{Orth}(A)$ and $\mathrm{Z}(A)$ are likewise Dedekind complete. If A is a semiprime f-algebra, then the mapping ρ defined from A into $\mathrm{Orth}(A)$ by $\rho(f) = \pi_f$, where $\pi_f(g) = fg$ for all $g \in A$, is an injective f-algebra homomorphism. Throughout this section a semiprime f-algebra A will be identified with $\rho(A)$. If T is a linear operator on a semiprime complex f-algebra, then T is contractive if and only if $|T(f)| \leq \mathrm{I}$ whenever $|f| \leq \mathrm{I}$. We shall denote by A_b the subalgebra of all bounded elements in A, i.e., $A_b = \{f \in A, |f| \leq \alpha\,\mathrm{I}; \alpha \in \mathbb{R}^+\}$. More details about orthomorphisms can be found in [15].

Now recall that a vector sublattice F of a vector lattice E is order dense if $x = \sup\{y : 0 \leq y \leq x, y \in F\}$ for every $x \in E^+$ (cf. [3, Section 14]). It follows from [8, Proposition 2.1] that A_b is order dense in $\mathrm{Z}(A)$.

To prove the first proposition in this section, we need the following result for the proof of which we refer to [3, Theorem 17.B].

Theorem 2. *Let E and G be vector lattices and F an order dense vector sublattice of E. Suppose that $T : F \to G$ is a positive order continuous linear map such that*

$$\widetilde{T}x = \sup\{Ty : y \in F,\ 0 \leq y \leq x\}$$

exists in G for every $x \in E^+$. Then T has a unique extension to a positive order continuous linear map from E to G.

Proposition 4. *Let A be a semiprime Dedekind complete f-algebra and let $T : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ be an order continuous contractive operator. Then there exists an order continuous contractive operator $\widetilde{T} : (\mathrm{Z}(A))_{\mathbb{C}} \longrightarrow (\mathrm{Z}(A))_{\mathbb{C}}$ such that $\widetilde{T}f = Tf$ for all $f \in (A_b)_{\mathbb{C}}$.*

Proof: First, consider the case that T is positive. Let T_b denote the restriction of T to A_b, $T_b : A_b \to \mathrm{Z}(A) f \longmapsto Tf$. Since $\mathrm{Z}(A)$ is Dedekind complete and T is contractive,

$$\sup\{Tf : f \in A,\ 0 \leq f \leq \pi\}$$

exists in $\mathrm{Z}(A)$ for every $\pi \in \mathrm{Z}(A)^+$. Moreover A_b is order dense in $\mathrm{Z}(A)$. Then, by Theorem 2, T_b extends to an order continuous operator from $\mathrm{Z}(A)$ into $\mathrm{Z}(A)$.

There exists, therefore, an order continuous operator $\widetilde{T}$ from $(\mathrm{Z}(A))_{\mathbb{C}}$ into $(\mathrm{Z}(A))_{\mathbb{C}}$ such that $\widetilde{T}f = Tf$ for all $f \in (A_b)_{\mathbb{C}}$. Now consider the case where T is arbitrary. Then

$$T = T_1 + iT_2 = (T_1^+ - T_1^-) + i(T_2^+ - T_2^-)$$

where T_i^+, T_i^- $(i \in \{1,2\})$ are positive contractive and order continuous operators. Therefore applying the above result for each of T_i^+ and T_i^-, we prove the existence of an order continuous operator $\widetilde{T} : (\mathrm{Z}(A))_{\mathbb{C}} \to (\mathrm{Z}(A))_{\mathbb{C}}$ such that $\widetilde{T}f = Tf$ for all $f \in (A_b)_{\mathbb{C}}$. It remains to show that $\widetilde{T}$ is contractive. Let $\pi \in (\mathrm{Z}(A))_{\mathbb{C}}$ such that $|\pi| \leq \mathrm{I}$. It follows from the order density of A_b in $\mathrm{Z}(A)$ that there exists a directed system $\{a_j\ j \in J\}$ in A^+ for which $0 \leq a_j \uparrow \mathrm{I}$. Then $\left|\pi a_j - \pi\right| \downarrow 0$ and $\left|\pi a_j\right| \leq \mathrm{I}$. Hence the order continuity of $\widetilde{T}$ together with the contractivity of T imply that $\left|\widetilde{T}\pi\right| \leq \mathrm{I}$ and so $\widetilde{T}$ is contractive. □

We are now in a position to prove our main result.

Theorem 3. *Let A be a semiprime Dedekind complete f-algebra and let T be an order continuous contractive projection on $A_{\mathbb{C}}$. Then T satisfies the Seever's identity if and only if T is almost positive.*

Proof: First, suppose that T satisfies Seever's identity. By Proposition 4 there exists an order continuous contractive operator $\widetilde{T} : (\mathrm{Z}(A))_{\mathbb{C}} \to (\mathrm{Z}(A))_{\mathbb{C}}$ such that $\widetilde{T}f = Tf$ for all $f \in (A_b)_{\mathbb{C}}$. It follows from the order density of A_b in $\mathrm{Z}(A)$ and the order continuity of $\widetilde{T}$ that $\widetilde{T}$ is a projection satisfying Seever's identity. Therefore by Theorem 1 (applied to the pair $(\mathrm{Z}(A), \widetilde{T})$), there exists $\pi \in \mathrm{Z}(A)$ such that $\pi^2 = \pi$, $\pi\widetilde{T}f \in \mathrm{Z}(A)^+$ for all $f \in \mathrm{Z}(A)^+$ and $\widetilde{T}(\pi\widetilde{T}f) = \widetilde{T}f$ for all $f \in (\mathrm{Z}(A))_{\mathbb{C}}$. Then using the order density of A_b in A and the order continuity of T, we get the desired result.

The proof of the "only if" part follows the same lines as the proof of the "if" part and therefore is omitted. □

5. The case where A has a point separating order dual

Throughout this section A' will denote the order dual of A. Recall that if A is an f-algebra with point separating order dual, then a multiplication can be introduced in the complexfication of the order bidual A'' of A (the so called Arens multiplication). Which is accomplished in three steps: Given $x, y \in A$, $f \in A'$ and $F, G \in A''$, we define $f \cdot x \in A'$, $G \cdot f \in A'$ and $F \cdot G \in A''$ by the equations

(1) $$(f \cdot x)(y) = f(xy)$$

(2) $$(G \cdot f)(x) = G(f \cdot x)$$

(3) $$(F \cdot G)(f) = F(G \cdot f).$$

Then A'' is a Dedekind complete f-algebra with respect to this multiplication (see [5, Theorem 2.8]). The band of all order continuous linear functionals on A' (denoted by $(A')'_n$) is also a Dedekind complete f-algebra ([8, Theorem 4.4]). From now on, A denotes an f-algebra with point separating order dual A'.

Let the map $\sigma : A \to A''$ denote the canonical evaluation map, that is, $\sigma(g)(\mu) = \mu(g)$ where $\mu \in A'$. Then σ is an injective algebra homomorphism. The extension of σ as a linear operator from $A_{\mathbb{C}}$ into $A''_{\mathbb{C}}$ is also denoted by σ. We have $\sigma(A_{\mathbb{C}}) \subset ((A')'_n)_{\mathbb{C}}$. If $T = T_1 + iT_2$ is an order bounded operator on $A_{\mathbb{C}}$, then the biadjoint T'' ($T'' = T''_1 + iT''_2$) verifies $T'' \circ \sigma = \sigma \circ T$. Moreover T'' maps $((A')'_n)_{\mathbb{C}}$ into itself. The restriction of T'' to $((A')'_n)_{\mathbb{C}}$ is also denoted by T''.

Proposition 5. *The operator T satisfies the Seever's identity if and only if T'' does.*

Proof: Suppose that T verifies (S) and let $F, G \in A''_{\mathbb{C}}$ (or $((A')'_n)_{\mathbb{C}}$). Then

$$T''(F \cdot T''G)(f) = F(T''G \cdot T'f) \quad \text{for all} \quad f \in A'_{\mathbb{C}}. \tag{1}$$

$$(T''G \cdot T'f)(a) = G(T'(T'f \cdot a)) \quad \text{for all} \quad a \in A_{\mathbb{C}} \tag{2}$$

$$\begin{aligned} T'(T'f \cdot a)(b) &= (T'f \cdot a)(Tb) = f(T(aTb)) \\ &= f(T(TaTb)) = T'(T'f \cdot Ta)(b) \end{aligned}$$

for all $b \in A_{\mathbb{C}}$. This implies that $T'(T'f \cdot a) = T'(T'f \cdot Ta)$.
Then (2) implies that

$$\begin{aligned} (T''G \cdot T'f)(a) &= G(T'(T'f \cdot Ta)) = (T''G \cdot T'f)(Ta) \\ &= T'(T''G \cdot T'f)(a). \end{aligned}$$

for all $a \in A_{\mathbb{C}}$. Consequently

$$T''G \cdot T'f = T'(T''G \cdot T'f).$$

Combining this with (1) we get

$$\begin{aligned} T''(F \cdot T''G)(f) &= F(T'(T''G \cdot T'f)) = T''F(T''G \cdot T'f) \\ &= T''(T''F \cdot T''G)(f) \end{aligned}$$

for all $f \in A'_{\mathbb{C}}$. Thus

$$T''(F \cdot T''G) = T''(T''F \cdot T''G)$$

and we are done.
Conversely, suppose that T'' satisfies the identity (S). Then

$$\begin{aligned}\sigma(T(fTg)) &= T''(\sigma(f)\cdot T''(\sigma(g)))\\ &= T''(T''(\sigma(f))\cdot T''(\sigma(g)))\\ &= T''(\sigma(TfTg)) = \sigma(T(TfTg))\end{aligned}$$

for all $f,g\in A_{\mathbb{C}}$, where we use that $\sigma\circ T = T''\circ\sigma$. Since σ is injective, we infer that $T(fTg) = T(TfTg)$, which completes the proof. □

Recall that an upward directed net $\{a_j\ j\in J\}$ is said to be a weak approximate unit if

$$f(b) = \sup\{f(a_j b) : j\in J\}$$

for all $b\in A^+$ and all $f\in (A')^+$ (cf. [8, Definition 7.1]). If A is a semiprime f-algebra, such that A has a weak approximate unit, then $(A')'_n$ is semiprime ([14, Theorem 4.3]). We recall also that if $T : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ is order bounded, then $T'' : ((A')'_n)_{\mathbb{C}} \longrightarrow ((A')'_n)_{\mathbb{C}}$ is order continuous (see [15, Section 97]).

Now combining Proposition 5 and Theorem 3, we get the following theorem.

Theorem 4. *Let A be a semiprime f-algebra with point separating order dual, and such that A has a weak approximate unit. Suppose that T is an order bounded contractive projection on $A_{\mathbb{C}}$. Then T satisfies the Seever's identity if and only if the restriction of T'' to $((A')'_n)_{\mathbb{C}}$ is almost positive.*

Let $A = C_0(X)$ be the collection of all continuous functions on a locally compact Hausdorff space X with values in a field F which is either the real or complex numbers and let T be a contractive projection on A. In the proof of [4, Theorem 1] Friedman and Russo have defined an order projection M on A'', which verifies $T''MT'' = T''$. In the same paper they proved that T satisfies the Seever's identity (S) if and only if MT'' is positive (see [4, Theorem 3]). Thus the authors proved that if T verifies (S), then T'' is almost positive. Since $A = C_0(X)$ satisfies the hypothesis of Theorem 4 and $A'' = (A')'_n$, we can see clearly that the aforementioned Friedman and Russo's result is a consequence of Theorem 4.

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