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Spaces X in which all prime z-ideals of $C(X)$ are minimal or maximal

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Abstract. Quasi P-spaces are defined to be those Tychonoff spaces X such that each prime z-ideal of $C(X)$ is either minimal or maximal. This article is devoted to a systematic study of these spaces, which are an obvious generalization of P-spaces. The compact quasi P-spaces are characterized as the compact spaces which are scattered and of Cantor-Bendixson index no greater than 2. A thorough account of locally compact quasi P-spaces is given. If X is a cozero-complemented space and every nowhere dense zeroset is a z-embedded P-space, then X is a quasi P-space. Conversely, if X is a quasi P-space and F is a nowhere dense z-embedded zeroset, then F is a P-space. On the other hand, there are examples of countable quasi P-spaces with no P-points at all. If a product $X \times Y$ is normal and quasi P, then one of the factors must be a P-space. Conversely, if one of the factors is a compact quasi P-space and the other a P-space then the product is quasi P. If X is normal and X and Y are cozero-complemented spaces and $f : X \longrightarrow Y$ is a closed continuous surjection which has the property that $f^{-1}(Z)$ is nowhere dense for each nowhere dense zeroset Z, then if X is quasi P, so is Y. The converse fails even with more stringent assumptions on the map f. The paper then closes with a number of open questions, amongst which the most glaring is whether the free union of quasi P-spaces is always quasi P.

Keywords: quasi P-space, P-space, scattered space, Cantor-Bendixson derivatives, nodec space, quasinormality

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1. Preliminaries

Let $C(X)$ denote the ring of continuous real-valued functions whose domain is the Tychonoff space X. This paper examines those spaces for which every prime z-ideal of $C(X)$ is either maximal or minimal, or, equivalently, for which each chain of prime z-ideals has no more than two elements. Consideration of lengths of such chains leads to a development of a notion of z-dimension; that will be explored elsewhere.

Much of what follows can be put in the context of f-rings or algebras, and will be phrased that way when convenient. There is a close connection between our work and the work of S. Larson in [La95], [La97a] and [La97b]; we shall feature some key theorems in those papers prominently. We will also make use of recent improvements in the latter due to C. Kimber [Ki01]. The notation and terminology is largely that of [GJ76] except that *absolutely convex ideals* of a $C(X)$ will be called *ℓ-ideals* as in [BKW77] and [D95].

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Throughout, every ring will be a commutative ring with identity element; unless otherwise indicated the reader may assume that the rings under discussion here are in fact *semiprime*, that is, that they have no nonzero nilpotent elements. All lattice-ordered rings that will arise in this paper will be f-rings; that is, subdirect products of totally ordered rings. Note that each $C(X)$ has the abovementioned properties.

All spaces are assumed to be Tychonoff. Recall that, for each $f \in C(X)$, $Z(f) = \{ x \in X : f(x) = 0 \}$, the *zeroset* of f, while $\mathrm{coz}(f) = X \setminus Z(f)$, the *cozeroset* of f. Likewise, we recall that $\mathrm{pos}(f) = \mathrm{coz}(f \vee 0) = \{ x \in X : f(x) > 0 \}$, while $\mathrm{neg}(f) = \mathrm{coz}(f \wedge 0) = \{ x \in X : f(x) < 0 \}$. We denote by βX the Stone-Čech compactification of X; αX stands for the one-point compactification of X (provided X is locally compact).

In this introductory section we review several of the pertinent elements which will form part of the subsequent presentation. We begin with annihilators in f-rings.

Definition & Remarks 1.1. (a) If S is a subset of the ring A, let

$$S^d = \{ a \in A : aS = \{0\} \}.$$

An ideal of the form S^d is called an *annihilator ideal*. It is well known that for a semiprime f-ring A the set $\mathcal{A}(A)$ of all annihilators of A forms a boolean algebra relative to inclusion, in which, for each $K \in \mathcal{A}(A)$, K^d is the complement. For convenience we write a^d for $\{a\}^d$, and $a^{dd} \equiv \{a^d\}^d$.

An annihilator of the form a^{dd} is called a *principal annihilator*. The subset of principal annihilators, $p\mathcal{A}(A)$, is a sublattice of $\mathcal{A}(A)$, a fact which follows from the identities below, which are easy to verify. For each $0 \leq a$, $b \in A$,

$$a^{dd} \wedge b^{dd} = (a \wedge b)^{dd} \;\text{ and }\; a^{dd} \vee b^{dd} = (a + b)^{dd}.$$

If the f-ring A is semiprime then, in terms of the additive lattice-ordered group structure of A, S^d may also be viewed as

$$S^d = \{ a \in A : |a| \wedge |b| = 0, \;\text{ for all }\; b \in S \}.$$

Indeed, if A is a lattice-ordered group, we may take the preceding description of S^d as its definition; in the language of lattice-ordered groups S^d is called the *polar* of S.

(b) Recall that a lattice-ordered group A is *projectable* if for each $a \in A$, $A = a^{dd} + a^d$.

Whenever $A = S^{dd} + S^d$ for an arbitrary subset S of A, then, since $S^{dd} \cap S^d = \{0\}$, we have, for each $f \in A$, a unique decomposition $f = f[S] + f[S^d]$, with $f[S] \in S^{dd}$ and $f[S^d] \in S^d$. We shall refer to $f[S]$ as *the projection of f on S*. If $S = \{a\}$, we use the notation $f[a]$ in place of $f[\{a\}]$. To recap the definition in the preceding paragraph then, A is projectable if and only if for each $f, g \in A$ there exist elements $f[g] \in g^{dd}$ and $f[g^d] \in g^d$ (which are necessarily uniquely determined by f) such that $f = f[g] + f[g^d]$.

For later use we record the following observation: if the lattice-ordered group A is projectable and C is any convex ℓ-subgroup — that is, a subgroup which is at once a sublattice and order-convex — then C too is projectable; (this follows from [D95, Theorem 18.4], which gives a necessary and sufficient condition for projectability which is preserved by passage to a convex ℓ-subgroup).

We also observe that $\mathbb{R}^D$, the lattice-ordered group of all real valued functions defined on a set D is projectable. This is well known, but also easy to see as soon as one observes that, for each $g \in \mathbb{R}^D$,

$$g^{dd} = \{\, f \in \mathbb{R}^D \;:\; \mathrm{coz}(f) \subseteq \mathrm{coz}(g) \,\} \quad \text{and} \quad g^d = \{\, f \in \mathbb{R}^D \;:\; \mathrm{coz}(f) \subseteq Z(g) \,\}.$$

(Note that $\mathbb{R}^D = C(D)$ when D is endowed with the discrete topology.) These descriptions make it clear that for each $f \in \mathbb{R}^D$, the functions defined by

$$f[g](x) = \begin{cases} f(x) & \text{if } x \in \mathrm{coz}(g), \\ 0 & \text{if } x \in Z(g), \end{cases} \qquad \text{and} \qquad f[g^d](x) = \begin{cases} 0 & \text{if } x \in \mathrm{coz}(g), \\ f(x) & \text{if } x \in Z(g), \end{cases}$$

are the projections of f which witness that $g^{dd} + g^d = \mathbb{R}^D$.

The above material will be used in a number of places further on: in the proof of the crucial Lemma 4.2, and in Section 6, in the proofs of Lemma 6.11 and 6.12. We will refer the reader to the preceding commentary prior to those results.

Next, we review some elementary notions in topology.

Definition & Remarks 1.2. (a) An ideal I of $C(X)$ is a *z-ideal* if $Z(g) \subseteq Z(h)$ and $g \in I$ imply that $h \in I$. By 2.7 and 14.7 of [GJ76] it follows, respectively, that maximal ideals and minimal prime ideals of $C(X)$ are z-ideals. As in [GJ76], if S is a subset of $C(X)$ then $\mathbf{Z}[S]$ denotes the set $\{\, Z(f) \;:\; f \in S \,\}$, and $\mathbf{Z}(X)$ the set of all zerosets of X. For the collection of all z-ideals of $C(X)$ we shall use $\mathcal{Z}(C(X))$.

(b) Recall from [GJ76, Chapter 1] or [W75, §10] that a subspace Y of a space X is said to be *C-embedded* (resp. *C^*-embedded*) *in* X if the map $f \mapsto f|_Y$ is a surjection of $C(X)$ onto $C(Y)$ (resp. $C^*(X)$ onto $C^*(Y)$). The subspace Y is said to be *z-embedded in* X if the map $Z \mapsto Z \cap X$ is a surjection of $\mathbf{Z}(X)$ onto $\mathbf{Z}(Y)$.

Clearly, C-embedding implies C^*-embedding, and C^*-embedding, in turn, implies z-embedding. It is well known that every closed subspace of a normal space is C-embedded. Moreover, by [W75, 10.7], every cozeroset and every Lindelöf subspace is z-embedded.

(c) A point p of a space X is called a *P-point* (resp. *almost P-point*) if whenever $f \in C(X)$ and $f(p) = 0$ it follows that $p \in \operatorname{int} Z(f)$ (resp. $\operatorname{int} Z(f) \neq \emptyset$). X is called a *P-space* (resp. an *almost P-space*) if every point is a P-point (resp. an almost P-point). Observe that X is almost P if and only if every zeroset of X is regular closed.

If the closure of each cozeroset (resp. open set) of X is open, then X is called *basically* (resp. *extremally*) *disconnected*. If disjoint cozerosets in X are completely separated, then X is said to be an *F-space*. It is shown in [GJ76] that every P-space and every extremally disconnected space is basically disconnected, and that each basically disconnected space is an F-space, and that none of the reverse implications hold.

(d) Our discussion will involve repeated references to the ideals O^p and M^p of $C(X)$, and in any case, it seems reasonable to include a brief review of the relationship between points of X and maximal ideals of $C(X)$. The notation is that of [GJ76]; see 7.2 and 7.12 of that reference.

For each $p \in \beta X$ then recall that

$$M^p = \{\, f \in C(X) \,:\, p \in \mathrm{cl}_{\beta X}\, Z_X(f) \,\},$$

is the maximal ideal of $C(X)$ associated with p. Also,

$$O^p = \{\, f \in C(X) \,:\, \mathrm{cl}_{\beta X}\, Z_X(f) \ \text{ is a neighborhood of } \ p \ \text{ in } \ \beta X \,\}$$

and the latter is the intersection of all the minimal prime ideals of $C(X)$ that are contained in M^p. The map $p \mapsto M^p$ defines a homeomorphism of βX onto $\mathrm{Max}(C(X))$, the space of all maximal ideals of $C(X)$, endowed with the hull-kernel topology.

Finally, when $p \in X$ one writes M_p (resp. O_p) in place of M^p (resp. O^p).

We come now to the central definitions of this article, prefaced by a review of some facts from Suzanne Larson's [La95] and [La97a].

Definition & Remarks 1.3. (a) An f-ring is defined to be *quasinormal* if the sum of any two minimal prime ideals is a maximal ℓ-ideal or the whole ring. This is equivalent, by [La97a, 1.3], to the following pair of conditions:

(i) the sum of any two semiprime ℓ-ideals is semiprime and
(ii) every nonmaximal prime ℓ-ideal contains a unique minimal prime ideal.

We will call a space X *quasinormal* if $C(X)$ is a quasinormal ring. (The reader is cautioned that this label does not imply a relaxation of normality in a topological space. "Normality" in this sense comes from the terminology in Riesz spaces.)

(b) Suppose X is a space, and M is a maximal ideal of $C(X)$. We say that $C(X)$ is *quasi P at M* if every prime z-ideal contained in M is minimal or else M. If $C(X)$ is quasi P at each of its maximal ideals, we call it a *quasi P-ring*, and say that X is a *quasi P-space*. We will also say that x is a *quasi P-point* of X if $C(X)$ is quasi P at M_x.

Let υX denote the Hewitt realcompactification of X; see [GJ76, Chapter 8]. Since $C(\upsilon X) \cong C(X)$ it follows that X is quasi P if and only if υX is quasi P.

Every P-space is a quasi P-space, since it is well known that X is a P-space precisely when every maximal ideal of $C(X)$ is a minimal prime ideal ([GJ76, 14.29]).

(c) Every quasi P-space is quasinormal. For, if X is not quasinormal, there are two distinct minimal prime ideals of $C(X)$ whose sum Q is contained properly in a maximal ideal M of $C(X)$. By [GJ76, 14.8], Q is a z-ideal, so M, Q, and one of the two minimal prime ideals form a chain of three z-ideals, showing that X is not a quasi P-space. The converse is false, as the reader will see in Proposition 2.7: for example, letting ω stand for the discrete space of natural numbers, $\beta\omega$ is quasinormal but not quasi P.

If X is compact then it is quasi P if and only if each $x \in X$ is quasi P, but the reader should note that any space Ψ belonging to the class of spaces described in [GJ76, 5I] is locally compact, pseudocompact and not normal. As we shall see in 4.4, such a Ψ is not quasi P, while every point of Ψ is quasi P.

A second example with this feature is given in Example 7.4.

(d) If X is an F-space then in $C(X)$ every maximal ideal contains a unique minimal prime ideal — by [GJ76, Theorem 14.25] — which makes it clear that any F-space is quasinormal. In [La95, Example 3.7] Larson shows that the one-point compactification of any infinite discrete space is quasinormal.

The image of a quasinonormal f-ring under a surjective homomorphism is quasinormal ([La95, Proposition 3.2]). Hence, every C-embedded subspace of a quasinormal space is quasinormal. In particular, every closed subspace of a normal quasinormal space is quasinormal.

The assertions made in the preceding paragraph clearly hold if “quasinormal” is replaced by “quasi P”.

It follows easily from Theorems 3.4 and 3.5 of [La97a] that if X is normal and realcompact, then βX is quasinormal if and only if X is quasinormal and O^p is prime for every $p \in \beta X \setminus X$. It is further asserted that the latter condition holds if X is locally compact and σ-compact. As will be seen below, this is not correct.

Theorem 1.4. *Consider the following assertions about a Tychonoff space X.*

(a) *O^p is a prime ideal for all $p \in \beta X \setminus X$.*
(b) *The intersection of the closures in X of any pair of disjoint cozerosets of X is compact.*

(c) *For every $f \in C(X)$, the intersection of the closures of* $\operatorname{pos}(f)$ *and* $\operatorname{neg}(f)$ *is compact.*

Then (b) *and* (c) *are equivalent,* (c) *implies* (a)*, and if X is normal or an F-space, then* (a) *implies* (b).

PROOF: The equivalence of (b) and (c) is immediate from the fact that for $g, h \in C(X)$, with $gh = 0$,

$$\operatorname{cl}(\operatorname{coz}(g)) \cap \operatorname{cl}(\operatorname{coz}(h)) = \operatorname{cl}(\operatorname{pos}(f)) \cap \operatorname{cl}(\operatorname{neg}(f)),$$

where $f = g^2 - h^2$.

Suppose O^p is not a prime ideal for some $p \in \beta X \setminus X$. By [GJ76, 2.9], there is an $f \in C^*(X)$ that changes sign on the trace on X of every neighborhood in βX of p. So the intersection of the closures in X of $\operatorname{pos}(f)$ and $\operatorname{neg}(f)$ cannot be compact. Hence (c) implies (a).

Assume that (a) holds and there is an $f \in C(X)$ and a point

$$p \in \operatorname{cl}_{\beta X}(\operatorname{cl}_X \operatorname{pos}(f) \cap \operatorname{cl}_X \operatorname{neg}(f)) \setminus X.$$

If X is normal,

$$(*) \qquad \operatorname{cl}_{\beta X}(\operatorname{cl}_X \operatorname{pos}(f) \cap \operatorname{cl}_X \operatorname{neg}(f)) = \operatorname{cl}_{\beta X} \operatorname{pos}(f) \cap \operatorname{cl}_{\beta X} \operatorname{neg}(f).$$

So $p \in \operatorname{cl}_{\beta X} \operatorname{pos}(f)$ and $p \in \operatorname{cl}_{\beta X} \operatorname{neg}(f)$. Note that we may assume that $f \in C^*(X)$. Also, since in an F-space disjoint cozerosets are completely separated, equation $(*)$ holds in any F-space.

It will be shown next that $p \notin \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\{\, x \in X : f(x) \le 0 \,\}$. For otherwise, there would be an open set $W \subseteq \beta X$ for which $p \in W \subseteq \operatorname{cl}_{\beta X}\{\, x \in X : f(x) \le 0 \,\}$. Now, $p \in \operatorname{cl}_{\beta X} \operatorname{pos}(\beta f)$, so that $W \cap \operatorname{pos}(\beta f) \neq \emptyset$. But

$$\operatorname{pos}(\beta f) \cap \operatorname{cl}_{\beta X}\{\, x \in X : f(x) \le 0 \,\} = \emptyset,$$

a contradiction. Similarly, $p \notin \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\{\, x \in X : f(x) \ge 0 \,\}$. Now, O^p is a prime z-ideal, making $\mathbf{Z}[O^p]$ a prime z-filter. Since, for each $x \in X$, either $f(x) \le 0$ or $f(x) \ge 0$, it follows that

$$p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\{\, x \in X : f(x) \le 0 \,\} \ \text{ or } \ p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\{\, x \in X : f(x) \ge 0 \,\},$$

and it has just been shown that this cannot happen. This shows that (a) implies (c), and hence (b), provided X is normal or an F-space. □

Remark 1.5. In a number of places in [La97a], it is asserted that if X is locally compact and σ-compact, then O^p is prime for every $p \in \beta X \setminus X$. This is not correct. For, if it were, then by this last theorem, the closures of the disjoint cozerosets

$$\{\,(x,y)\,:\,x<y\,\}\quad\text{and}\quad\{\,(x,y)\,:\,x>y\,\}$$

of $\mathbb{R}\times\mathbb{R}$ would have a compact intersection, which is not the case.

Coupling the previous theorem with [Ki01, Theorem 5.2] yields:

Theorem 1.6. *If X is normal and realcompact, then the following are equivalent.*

(a) *X is quasinormal and O^p is prime for every $p \in \beta X \setminus X$.*
(b) *X is quasinormal and the intersection of the closures of any pair of disjoint cozerosets of X is compact.*
(c) *βX is quasinormal.*

Last, in this introduction, we briefly review scattered spaces, for use in Section 4.

Definition & Remarks 1.7. (a) A topological space X is said to be *scattered* or *dispersed* (in French *clairsemé)* if each nonvoid subspace Y has an isolated point of Y. It is easy to see that if each nonempty closed subspace of X has an isolated point, then X is scattered. Many properties of scattered spaces are summarized in Z. Semadeni's memoir [Se59], his book [Se71], and in a paper by R. Levy and M. Rice [LR81].

A compact scattered space is necessarily zero-dimensional. The Stone dual is a *superatomic* boolean algebra: every homomorphic image has an atom. For a discussion of superatomic boolean algebras the reader is referred to [Ko89, §17].

As with quasi P-spaces, no space containing a copy of $\beta\omega$ can be scattered.

(b) Next, we review the Cantor-Bendixson sequence of a space. If Y is a space let $\mathrm{Is}(Y)$ denote its set of isolated points, and let: $Y^{(0)} = Y$, $Y^{(1)} = Y \setminus \mathrm{Is}(Y)$. For any ordinal η, let $Y^{(\eta+1)} = (Y^{(\eta)})^{(1)}$, and if η is a limit ordinal, let

$$Y^{(\eta)} = \bigcap_{\xi<\eta} Y^{(\xi)}.$$

The spaces $Y^{(\eta)}$ are called *Cantor-Bendixson derivatives of Y*. The reader will note that these derivatives form a decreasing transfinite sequence of closed subspaces of Y. From cardinality considerations there is an ordinal α such that $Y^{(\alpha)} = Y^{(\beta)}$, for each $\beta > \alpha$. Let $\mathrm{CB}(Y)$ denote the smallest ordinal for which $Y^{(\alpha)} = Y^{(\alpha+1)}$; this is the *CB-index* of a space Y.

Now, it is easily seen that Y is scattered if and only if $Y^{(\alpha)} = \emptyset$, for suitable α. If Y is compact and $\alpha = \mathrm{CB}(Y)$, then it is clear that $\bigcap_{\eta<\alpha} Y^{(\eta)}$ is nonempty.

It follows that α has a predecessor γ such that $Y^{(\gamma)}$ is finite and, hence, the last nonempty Cantor-Bendixson derivative. (To illustrate, if Y is compact then $\mathrm{CB}(Y) = 1$ means that Y itself is finite; $\mathrm{CB}(Y) = 2$ that $Y \setminus \mathrm{Is}(Y)$ is finite but nonvoid; and so on.)

Note that if Y is compact and scattered, then $\mathrm{CB}(Y) = 2$ if and only if Y is a finite topological sum of one-point compactifications of discrete spaces.

2. Basic properties of quasi P-spaces

As is customary in the theory of commutative rings, $\mathrm{Spec}(A)$ denotes the set of prime ideals of the ring A. In the lemma which follows we regard $\mathrm{Spec}(A)$ as a poset under inclusion. The lemma is easily proved from the definition of a prime ideal, keeping in mind that a prime ideal is necessarily a proper ideal. The result is well known; it appears as Exercise 15, p. 713, in [DF99].

Below $\mathrm{Spec}(A)$ is assumed to bear the hull-kernel topology; the reader is reminded that it is rarely Hausdorff.

Lemma 2.1. *Suppose that $A = A_1 \times A_2$, a direct product of commutative rings. Then* $\mathrm{Spec}(A)$ *is partitioned into clopen sets $\mathcal{S}_1$ and $\mathcal{S}_2$, where*

$$\mathcal{S}_i = \{ P \in \mathrm{Spec}(A) \,:\, O_i \subseteq P \}, \quad (i = 1, 2)$$

and $O_2 = A_1 \times \{0\}$ and $O_1 = \{0\} \times A_2$. Moreover, for each $i = 1, 2$, $\mathcal{S}_i$ is isomorphic as a poset to $\mathrm{Spec}(A_i)$ *via the map $P \mapsto P/O_i$.*

We interpret Lemma 2.1 in a ring of continuous functions. Observe that, for any set F, e_S stands for the characteristic function of the subset S of F.

Lemma 2.2. *Suppose that X is the topological sum of X_1 and X_2. Then*

(i) *$C(X) \cong C(X_1) \times C(X_2)$, via the ℓ-isomorphism $f \mapsto (f|_{X_1}, f|_{X_2})$.*

(ii) *The isomorphism in* (i) *induces a map Θ which is at once an isomorphism of posets and a homeomorphism from* $\mathrm{Spec}(C(X))$ *onto the free union* $\mathrm{Spec}(C(X_1)) \cup \mathrm{Spec}(C(X_2))$, *such that, for a prime ideal P of $C(X)$,*

$$\Theta(P) = \begin{cases} \{ f|_{X_1} \,:\, f \in P \} \in \mathrm{Spec}(C(X_1)), & \text{if } e_{X_2} \in P, \\ \{ f|_{X_2} \,:\, f \in P \} \in \mathrm{Spec}(C(X_2)), & \text{if } e_{X_1} \in P. \end{cases}$$

(iii) *Subject to the identification of βX with* $\mathrm{Max}(C(X))$ — *see* 1.2(d) — *Θ restricts to the natural homeomorphism from βX onto the topological sum of βX_1 and βX_2.*

Proof: That $f \mapsto (f|_{X_1}, f_{X_2})$ is an ℓ-isomorphism, as asserted, is easy to check; we leave it to the reader. Next, with the notation of Lemma 2.1, setting $A = C(X)$ and $A_i = C(X_i)$, $i = 1, 2$, we note that

$$O_i = \{ f \in C(X) \,:\, f \ \text{ vanishes on } \ X_i \} = e^d_{X_i},$$

and also that $e^d_{X_1} = e^{dd}_{X_2} = (e_{X_2})$, where (e_{X_i}) is the ideal generated by e_{X_i}. Then we have

$$\mathcal{S}_1 = \{\, P \in \operatorname{Spec}(C(X)) \,:\, e_{X_2} \in P \,\} \ \text{ and } \ \mathcal{S}_2 = \{\, P \in \operatorname{Spec}(C(X)) \,:\, e_{X_1} \in P \,\}.$$

Item (ii) should now be clear from Lemma 2.1. It is easy to see that (iii) follows from (ii), and we leave this to the reader as well. □

Proposition 2.3. *Let X be a space and K be a clopen subset with $p \in \operatorname{cl}_{\beta X} K$. Then*

(a) *M^p (in $C(X)$) is a minimal prime ideal if and only if M^p (in $C(K)$) is a minimal prime ideal;*

(b) *$C(X)$ is quasi P at M^p (relative to X) if and only if $C(K)$ is quasi P at M^p (relative to K).*

Proof: Note that X is the free union of K and $X \setminus K$ and apply Lemma 2.2. □

We now exhibit a class of quasi P-spaces that contains all one-point compactifications of discrete spaces. Exercise 14G in [GJ76] asserts that $\alpha\omega$ is quasi P, without using this terminology.

The reader should recall the following about commutative semiprime rings: suppose that P is a prime ideal of a such a ring A. Then it is minimal if and only if $a \in P$ implies that $a^d \nsubseteq P$. For rings of continuous functions this is shown in [HJ65]; it will be used presently.

If S is a subset of a partially ordered group G then we define $S^+ = \{\, x \in S : x \geq 0 \,\}$.

Lemma 2.4. *Let $X = D \cup \{p_\infty\}$, where D is an infinite P-space, and p_∞ is the unique non P-point of X. Then*

(a) *every prime z-ideal of $C(X)$ properly contained in M_{p_∞} is a minimal prime ideal;*

(b) *every free maximal ideal of $C(X)$ is a minimal prime ideal.*

Thus, X is a quasi P-space.

Proof: The final claim obviously follows from (a) and (b).

(a) On the contrary, suppose that Q is a prime z-ideal of $C(X)$ contained properly in $M = M_{p_\infty}$ that is not minimal. Then by the remark preceding this lemma, there is an $f \in Q^+$ such that $f^d \subseteq Q$. Choose also $g \in M^+ \setminus Q$ and note that $p_\infty \in \operatorname{cl}_X(\operatorname{coz}(g) \cap D)$. For otherwise, g vanishes on a neighborhood of p_∞ and lies in O_{p_∞}, and hence in Q.

Let $S(g,f) = \operatorname{coz}(g) \cap Z(f)$ and observe that it is a clopen subspace of the P-space D. We consider two cases:

I. $p_\infty \notin \operatorname{cl}_X(S(g,f))$. Then $S(g,f)$ is clopen in X and its characteristic function u is continuous. Because $u \equiv 0$ on $\operatorname{coz}(f)$, we have $u \in f^d \subseteq Q$, so that $f+u \in Q$. Since

$$Z(f+u) = Z(f) \cap Z(u) = Z(f) \cap (Z(g) \cup \operatorname{coz}(f)) \subseteq Z(g),$$

and Q is a z-ideal, it follows that $g \in Q$, contrary to assumption. So we must have:

II. $p_\infty \in \operatorname{cl}_X(S(g,f))$. Then define v on X by

$$v(x) = \begin{cases} g(x) & \text{if } x \in S(g,f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $v \in C(X)$ and $v(p_\infty) = 0$. Clearly, $vf = 0$, so it follows from our choice of f that $v \in Q$; consequently, $f + v \in Q$ as well. As in case I, $Z(f+v) \subseteq Z(g)$, whence $g \in Q$, again contrary to assumption.

These contradictions mean that (a) is satisfied.

(b) Suppose that $p \in \beta X \setminus X$. Clearly, βX is zero-dimensional, so p_∞ has a clopen X-neighborhood L whose complement L' contains p. It is also clear that $L \cap X$ and $L' \cap X$ form a clopen partition of X, and that $L' \cap X$ is a P-space. Furthermore,

$$C(X) \cong C(L \cap X) \times C(L' \cap X).$$

Invoking Proposition 2.3(a), we have that M^p is a minimal prime ideal of $C(X)$, as claimed. $\square$

We now put together Lemmas 2.2, 2.4 and induction.

Corollary 2.5. *Any finite topological sum of quasi P-spaces is a quasi P-space. In particular, any space in which all but a finite number of points are P-points is a quasi P-space.*

The reader will recall — see [GJ76, Chapter 14] — that X is an F-space if every cozeroset of X is C^*-embedded in X. We also recall the following generalization of F-spaces.

Definition & Remarks 2.6. (a) The space X is an *SV-space* if for each prime ideal P of $C(X)$, $C(X)/P$ is a valuation domain. (Note: a *valuation domain* is an integral domain in which, given any two ideals, one is contained in the other.) It is well known that every F-space is an SV-space ([GJ76, Theorem 14.27]). [HLMW94, 5.6] shows that every compact SV-space contains a copy of $\beta\omega$.

The *rank* of a point p in a space X, denoted $\operatorname{rk}_X(p)$, is defined to be the number of minimal prime ideals contained in M_p (see [HLMW94]); if this number is infinite, we write $\operatorname{rk}_X(p) = \infty$. (If the underlying space is understood we will drop the subscripting of the rank of a point.) A space X has *finite rank* if there is a finite number k such that $\operatorname{rk}(p) \leq k$, for each point $p \in X$. To say that $\mathrm{rk}_X(p) = n$ is to say that n is the maximum number of pairwise disjoint cozerosets of X having p in all their closures [HLMW94, Theorem 3.1]. Also — [HLMW94, 5.10] — if X is normal, then $\mathrm{rk}_X(p) = \mathrm{rk}_{\beta X}(p)$. Finally, it is shown in [HLMW94, Theorem 4.2] that an SV-space necessarily has finite rank.

(b) In [Ki01, Theorem 4.3], C. Kimber shows that if X is a normal space, then $C(X)$ is quasinormal if and only if for each pair of disjoint cozerosets U and V, $\mathrm{cl}_X U \cap \mathrm{cl}_X V$ is a P-space. Earlier, in [La97a, Theorem 3.5], S. Larson obtained this result in case X is compact. (In [Ki01], the terminology "2-boundary" is used to denote the interesection of the closures of a pair of disjoint cozerosets.)

With the following observation in hand, we are able to see that compact quasinormal spaces which are not quasi P abound.

Proposition 2.7. *No compact space containing a copy of $\beta\omega$ is a quasi P-space. In particular, no infinite compact SV-space, nor any infinite compact F-space is a quasi P-space.*

Proof: In [GJ60, Theorem 3.10], it is shown that the quasi P-points of $\beta\omega$ are the points of ω together with those points in $\beta\omega \setminus \omega$ that are P-points of the space $\beta\omega \setminus \omega$. Because every compact P-space is finite, this set cannot be all of $\beta\omega \setminus \omega$, so $\beta\omega$ is not a quasi P-space. The remaining assertions follow from 2.6, by appealing to 1.3(d). □

The following corollary is immediate from Proposition 2.7; it will be used in Section 6.

Corollary 2.8. *Any locally compact F-space that is quasi P is discrete.*

Proof: By Proposition 2.7 and 1.3(d), each point of such a space has a finite neighborhood and hence is discrete. □

Remark 2.9. By Proposition 2.7, any infinite P-space is an example of a quasi P-space whose Stone-Čech compactification is not quasi P.

After Theorem 5.9 an example will be given of a countable quasi P-space without any P-points, which is, in fact, extremally disconnected.

3. z-Embedded subspaces

This brief section contains an observation about the poset of prime z-ideals $\mathrm{Spec}_z(C(X))$ of $C(X)$, and its inverse image under the homomorphism induced by a continuous map to X. Ultimately, the goal is to describe the passage from $\mathrm{Spec}_z(C(X))$ to $\mathrm{Spec}_z(C(Y))$ when Y is a dense z-embedded subspace. We shall need this information later on.

We denote the set of minimal prime ideals of a ring A by $\mathrm{Min}(A)$. For a semiprime f-ring (and, indeed, more generally) $\mathrm{Min}(A)$ is a zero-dimensional Hausdorff space under the hull-kernel topology ([HJ65]).

Definition & Remarks 3.1. If $\tau : Y \longrightarrow X$ is a continuous mapping, and P is a z-ideal of $C(Y)$, then

$$\mathrm{Spec}_z(\tau)(P) \equiv \{\, g \in C(X) \,:\, g \cdot \tau \in P \,\}$$

is a prime z-ideal of $C(X)$, and the mapping $\mathrm{Spec}_z(\tau)$ is an order preserving map (under set inclusion) of $Spec_z(C(Y))$ into $Spec_z(C(X))$ that sends M_y to $M_{\tau(y)}$, for each $y \in Y$. See [Mo73, §5].

Let us examine $\mathrm{Spec}_z(\tau)$ more closely. Assume that τ is a z-embedding. Let $C(\tau) : C(X) \longrightarrow C(Y)$ be the induced ℓ-homomorphism, of restriction to Y, and K_τ denote its kernel. Note that, for each $P \in \mathrm{Spec}_z(C(Y))$,

$$\mathrm{Spec}_z(\tau)(P) = C(\tau)^{-1}(P) = \{\, f \in C(X) \,:\, f|_Y \in P \,\}.$$

Next, we wish to highlight a consequence of [Mo73, §8] and [Mo70, Theorem 6.2]:

> $\mathrm{Spec}_z(\tau)$ is an order preserving isomorphism of $\mathrm{Spec}_z(C(Y))$ into the subset $\mathcal{S}_\tau$ of $\mathrm{Spec}_z(C(X))$ consisting of all prime z-ideals containing K_τ.

Moreover, $\mathrm{Spec}_z(\tau)$ has the feature that if P is a prime z-ideal of $C(Y)$ contained in the maximal ideal M, then $\mathrm{Spec}_z(\tau)(P)$ is contained in the maximal ideal $\mathrm{Spec}_z(\tau)(M)$.

Now we have the following observations.

Proposition 3.2. *Suppose $\tau : Y \longrightarrow X$ is a z-embedding. Then*

(a) *if $C(X)$ is quasinormal (resp. quasi P), then so is $C(Y)$. In particular, every Lindelöf subspace and every cozeroset of a quasinormal (resp. quasi P) space is quasinormal (resp. quasi P);*

(b) *if Y is also dense in X, then the restriction of* $\mathrm{Spec}_z(\tau)$ *to* $\mathrm{Min}(C(Y))$ *is a homeomorphism onto* $\mathrm{Min}(C(X))$ *with respect to the hull-kernel topologies.*

Proof: The first sentence of (a) follows from 3.1, and the second sentence from the fact that Lindelöf subspaces and cozerosets of a space are z-embedded in that space, as is noted in of [W75, 10.7]. Part (b) is shown in [Mo70, 7.6]. □

Observe that Proposition 3.2(a) improves on Theorems 3.7 and 3.8 of [La97a].

4. Locally compact quasi P-spaces

We begin with a necessary condition for a compact space to be a quasi P-space. We remind the reader of the remarks in 2.6. Our goal is Theorem 4.3, describing locally compact quasi P-spaces and characterizing those that are normal.

Theorem 4.1. *Suppose that X is a compact space. Then,*

(I) *if X is quasi P we have that*
   (a) *the subset of X of all points p for which* $\mathrm{rk}(p) = 1$ *is cofinite;*
   (b) *each point p with* $\mathrm{rk}(p) > 1$ *has infinite rank;*
   (c) *each point with rank* 1 *is isolated; thus, X is scattered with* $\mathrm{CB}(X) \leq 2$.

(II) *X is a quasi P-space if and only if X is a finite topological union of a number of one-point compactifications of discrete spaces.*

Proof: Item I(a) follows immediately from [Ki01, Theorem 5.2]. To verify I(b), let $p \in X$, and assume that $\mathrm{rk}(p) > 1$. By I(a), there is a compact neighborhood K of p which excludes the other points of rank > 1. Note that any such K must be infinite. Observe that $\mathrm{rk}_X(p) = \mathrm{rk}_K(p)$. Note also that by [HLMW94, Corollary 1.8.2] the other points of K still have rank 1. Thus, if $\mathrm{rk}_K(p) < \infty$, then according to [La97b, Theorem 4.3], K is an SV-space and also quasi P. On the other hand, since p is not isolated in K, K is infinite, and we may then apply Proposition 2.7, to obtain a contradiction. This means that $\mathrm{rk}(p) = \infty$.

To prove I(c), observe that, by (a), the set S of points of rank 1 is open. For each $p \in S$, let C be a compact neighborhood of p contained in S. Then C is a quasi P-space consisting of points of rank 1, and hence an F-space ([GJ76, Theorem 14.25]). Using Proposition 2.7 once more, we conclude that C is finite, and so p is isolated in X. Then it is obvious that X is scattered with CB-index ≤ 2. The necessity in (II) is a consequence of (c) in (I). The converse follows from Lemma 2.4 and Corollary 2.5. □

For locally compact spaces we have a characterization, which improves a bit on Theorem 4.1. We shall need the following lemma. The reader is encouraged to review the comments in 1.1(b).

Lemma 4.2. *Suppose that X is a space with a dense set of isolated points, for which the first Cantor-Bendixson derivative, $X^{(1)}$, is C-embedded. Let $\rho : C(X) \longrightarrow C(X^{(1)})$ be the restriction homomorphism, and $K = \ker(\rho)$. Then, for each minimal prime ideal P of $C(X)$ that does not contain K, $P \cap K$ is maximal among the z-ideals of $C(X)$, which are properly in K.*

Proof: Note at the outset that ρ is surjective because $X^{(1)}$ is C-embedded. Also, we have

$$K = \{\, f \in C(X) \,:\, \mathrm{coz}(f) \subseteq \mathrm{Is}(X) \,\},$$

and we may then regard K as a convex ℓ-subgroup of $C(\mathrm{Is}(X))$. Now, as a lattice-ordered group, $C(\mathrm{Is}(X))$ is projectable by the remark in 1.1(b), as $C(\mathrm{Is}(X)) \cong \mathbb{R}^{\mathrm{Is}(X)}$. Since K is a convex ℓ-subgroup of $C(\mathrm{Is}(X))$, it follows — invoking 1.1(b) once more — that K is projectable as well.

Suppose now that P is a minimal prime ideal of $C(X)$ that does not contain K. Let $0 < f \in K \setminus P$. Pick $g > 0$ in K. Then, owing to the projectability, we have that $g[f] \in f^{dd}$ and $g[f^d]f = 0$, whence $g[f^d] \in P$. As for $g[f]$ and f, they are continuous functions on the discrete space $\mathrm{Is}(X)$ and so $Z(f) \subseteq Z(g[f])$, which implies that g lies in the z-ideal generated by f and $P \cap K$. This proves the lemma. □

Theorem 4.3. *Suppose that X is a locally compact space. If X is a quasi P-space then X is scattered of* CB-*index* ≤ 2. *The converse is true if X is assumed to be normal.*

If X is quasi P then each nonisolated point has infinite rank.

Proof: (Necessity) Suppose that $p \in X$ and that K is a compact regular closed neighborhood of p. Owing to the remarks in 1.3(d), K is quasi P, and hence, by Corollary 4.1, all but finitely many points of K are isolated in K. As $K = \mathrm{cl}_X \operatorname{int}_X K$, all such points are isolated in X. This proves that $\mathrm{Is}(X)$ is dense in X, and by shrinking K if necessary, that for each $p \in X^{(1)}$, there is a neighborhood of p which intersects $X^{(1)}$ in p only. Thus $X^{(1)}$ is discrete, proving that X is scattered of CB-index ≤ 2.

If $\mathrm{rk}_X(x) > 1$, then the above arguments are sufficient to show that x has infinite rank in some compact neighborhood, and therefore in X. This proves the final claim of the theorem.

(Sufficiency) Suppose that $\mathrm{CB}(X) \leq 2$, and assume that X is normal. We carry the notation of Lemma 4.2. If $\mathrm{CB}(X) = 1$, then X is discrete and there is nothing to show, so we may assume that $\mathrm{CB}(X) = 2$. Thus, we have $\mathrm{Is}(X)$ dense in X, with $X^{(1)}$ C-embedded in X, in view of the normality of X.

Now we recall (see [D95, Theorem 12.13]) that the trace map $Q \mapsto Q \cap K$ is an order-preserving bijection between the prime convex ℓ-subgroups of $C(X)$ that do not contain K and the proper prime convex ℓ-subgroups of K. It is easy to verify that Q is a z-ideal if and only if $Q \cap K$ is a z-ideal. We leave that to the reader.

Suppose, by way of contradiction, that there is a maximal ideal M in $C(X)$ and prime z-ideals $P_0 < P_1 < P_2$, contained in M. Without loss of generality, we may assume that P_0 is a minimal prime ideal. Since $C(X)/K \cong C(X^{(1)})$ is von Neumann regular, $P_1 \not\leq K$, and by tracing on K, we have $P_0 \cap K < P_1 \cap K$, both z-ideals which are properly contained in K, a contradiction to Lemma 4.2. □

Here are the details of the example mentioned in 1.3(c).

Example 4.4. Let Ψ be a member of the class of spaces discussed in [GJ76, 5I]. Each nonisolated point has a clopen neighborhood which is homeomorphic to $\alpha\omega$, which is quasi P. Thus, invoking Proposition 2.3, it follows that Ψ is quasi P at each of its nonisolated points, which means that it is quasi P at each of its points.

Next, we establish that any *noncompact* pseudocompact space X which is scattered with CB-index 2 is not quasi P. Since X is pseudocompact, $\upsilon X = \beta X$

([GJ76, 8A.4]). Thus,

$$C(X) \cong C(\upsilon X) \cong C(\beta X);$$

now if βX is scattered, its CB-index is at least 3, and so $C(X)$ is not quasi P, by Theorem 4.3. If βX is not scattered, then by the same theorem, $C(X)$ is not quasi P.

The upshot of this is that Ψ is not quasi P.

In Example 7.4, an example will be given of a locally compact, *realcompact*, nonnormal space X which is scattered with CB-index ≤ 2, which is not quasi P, but is quasi P at each of its points.

Remark 4.5. It is worth noting that there are locally compact quasi P-spaces for which $X^{(1)}$ is infinite. For instance, let X be the topological union of countably many copies of $\alpha\omega$, the one-point compactification of the discrete natural numbers. X is clearly scattered with CB-index 2. Here $X^{(1)} \cong \omega$ is C-embedded in X, and so by Theorem 4.3, X is quasi P.

The next result is an immediate consequence of Theorem 4.3.

Theorem 4.6. *If X is a locally compact normal quasi P-space, and D is a discrete space, then $D \times X$ is a quasi P-space.*

Note in the preceding theorem that the space $D \times X$ may also be described as the free union of $|D|$ copies of X.

This section is concluded with the following simple application of the last theorem.

Corollary 4.7. *A locally compact σ-compact space is a quasi P-space if and only if it is a countable free union of one-point compactifications of discrete spaces.*

Proof: A countable free union of one-point compactifications of discrete spaces is clearly locally compact and σ-compact, and is a closed subspace of the product of the one-point compactification of a discrete space of any cardinality exceeding that of each of the factor spaces and a countable discrete space. Because this latter product is normal, the resulting space is quasi P by Theorem 4.6.

Conversely, if X is a locally compact σ-compact quasi P-space, then X is scattered and $\mathrm{CB}(X) \leq 2$, and $X^{(1)}$ is countable and discrete. Because each $x \in X^{(1)}$ has a compact neighborhood meeting no other point of $X^{(1)}$, it has one homeomorphic to the one-point compactification of a discrete space with x as its only nonisolated point, so the desired conclusion holds. $\square$

5. Quasi P-spaces X for which $\mathrm{Min}(C(X))$ is compact

We remind the reader that if $S \subseteq C(X)$, then $S^d = \{\, g \in C(X) : gS = \{0\}\,\}$; see Section 1. The space $\mathrm{Min}(C(X))$ will denote the set of minimal prime ideals of $C(X)$ endowed with the hull-kernel topology. The following facts about

$\mathrm{Min}(C(X))$ appear in [HJ65]; in fact, most of the equivalences listed in Proposition 5.1 are shown in [HJ65] for semiprime f-rings. Note that f is a zero-divisor if and only if $\mathrm{int}_X Z(f) \neq \emptyset$.

Proposition 5.1.

(a) *If $P \in \mathrm{Min}(C(X))$ and $f \in P$, then f^d is not contained in P. In particular, each member of a minimal prime ideal is a zero divisor. Thus no element of $\mathbf{Z}[P]$ is nowhere dense.*

(b) $\mathrm{Min}(C(X))$ *is a zero-dimensional Hausdorff space which is countably compact, but need not be compact.*

(c) *The following are equivalent in $C(X)$:*

  (i) $\mathrm{Min}(C(X))$ *is compact;*

  (ii) *for each $f \in C(X)$, there is a $g \in C(X)$ such that $f^{dd} = g^d$;*

  (iii) *if every element of a (proper) prime ideal is a zero-divisor, then it is minimal;*

  (iv) *for each $f \in C(X)$, there is a $g \in C(X)$ such that* $\mathrm{cl}(\mathrm{int}(Z(f))) = \mathrm{cl}(\mathrm{coz}(g))$*;*

  (v) *for each $f \in C(X)$,* $\mathrm{cl}(\mathrm{coz}(f))$ *is the closure of the interior of a zeroset;*

  (vi) *for each cozeroset U of X there exists a cozeroset V of X such that $U \cup V$ is dense in X and $U \cap V = \emptyset$.*

Remark 5.2. (a) Perhaps because of 5.1(c)(vi), such spaces are called *cozero-complemented*. We observe that the equivalence of (i), (ii) and (iii) of Proposition 5.1(c) is known in any semiprime f-ring. Indeed, in [CM90] it is shown for arbitrary lattice-ordered group that the space of minimal prime convex ℓ-subgroups is compact precisely when 5.1(c)(ii) is satisfied.

The claim of Proposition 5.1(a) was already noted prior to Lemma 2.4, for arbitrary commutative semiprime rings.

(b) If the closure of every cozeroset of a space X is a zeroset, in particular, if X is perfectly normal or basically disconnected, then $\mathrm{Min}(C(X))$ is compact. A space in which every regular closed set is a zeroset is called an *Oz-space*; see [Bl76]. We note that a space is Oz if and only if each of its open subspaces is z-embedded. It follows from Proposition 5.1(c) that if X is an Oz-space, then $\mathrm{Min}(C(X))$ is compact.

(c) Recall that a space satisfies the *countable chain condition*, or is a *ccc-space*, or *has countable cellularity* if any collection of pairwise disjoint open subsets is countable. Any separable space is a *ccc*-space, but the cube $[0,1]^\alpha$ and the generalized Cantor space $\{0,1\}^\alpha$ are *ccc* for any cardinal α, and they both fail to be separable if the exponent α exceeds 2^ω. In fact, continuous images of dense subspaces of *ccc*-spaces are *ccc*. See 2N(6) and 3RST of [PW88].

Combining Theorem 1.1 and Proposition 4.3 of [HM93] yields immediately that every *ccc*-space has a compact space of minimal prime ideals that is extremally disconnected. The arguments of [HM93] also show that every Oz-space has this property.

We now proceed to examine consequences of the hypothesis that $\mathrm{Min}(C(X))$ be compact.

Proposition 5.3.

(a) *If X is a quasi P-space, then any prime z-filter that contains a nowhere dense zeroset is a z-ultrafilter.*

(b) *The converse of* (a) *holds if* $\mathrm{Min}(C(X))$ *is compact.*

PROOF: (a) The prime z-filter determines a prime z-ideal Q that is contained in a unique maximal ideal M and contains a minimal prime ideal P. By Proposition 5.1(a), $Q \neq P$, so that $Q = M$, since X is a quasi P-space.

(b) Suppose P is a minimal prime ideal contained properly in a prime z-ideal Q. By Proposition 5.1(c)(iii) the z-filter $\mathbf{Z}[Q]$ contains an element with empty interior, and so by assumption must be a z-ultrafilter. Thus, Q is a maximal ideal, proving (b). □

Definition & Remarks 5.4. (a) A space in which every nowhere dense subspace is closed is called a *nodec space*; (equivalently, a space is nodec if each of its closed nowhere dense sets is discrete.) This definition was introduced by van Douwen in [vD93].

(b) It is shown in [HM93, 2.6(d)] that if $\mathrm{Min}(C(X))$ is compact and extremally disconnected, and $|X|$ is nonmeasurable, then each almost P-point of X is isolated. (For a definition of measurable cardinal and the reason why it is consistent with ZFC to assume that all cardinals are nonmeasurable, see [GJ76, Chapter 12].)

Observe that Theorem 5.5(a) below appears implicitly in [HM93, §2].

Theorem 5.5. *Suppose that X is cozero-complemented.*

(a) *Every almost P-point of X is a P-point. Thus, an almost P-space with a compact space of minimal prime ideals is a P-space.*

(b) *If Y is a dense z-embedded subspace of X, then* $\mathrm{Min}(C(Y))$ *is compact.*

(c) *If every nowhere dense zeroset of X is a z-embedded P-space, then X is a quasi P-space.*

PROOF: (a) If p is an almost P-point of X, then $\operatorname{int} Z(f) \neq \emptyset$ for every $f \in M_p$. So by Theorem 5.1(c)(iii), M_p is a minimal prime ideal, and it follows that p is a P-point (see [GJ76, 4L]).

(b) follows immediately from Proposition 3.2(b).

(c) Suppose I is a prime z-ideal of $C(X)$ that is not minimal and is contained in the maximal ideal M^p for some $p \in \beta X$. Then $\mathbf{Z}[I]$ contains a nowhere dense zeroset F, which by assumption is a P-space. Suppose T is a zeroset of X such that $p \in \mathrm{cl}_{\beta X} T$. It suffices to show that $T \in \mathbf{Z}[I]$. To see this, note first that $T \cap F$ is a zeroset of F and hence is clopen in the P-space F. Hence $F \setminus T \in \mathbf{Z}(F)$ and by assumption there is an $S \in \mathbf{Z}(X)$ such that $S \cap F = F \setminus T$. To continue,

$$p \in \mathrm{cl}_{\beta X} T \cap \mathrm{cl}_{\beta X} F = \mathrm{cl}_{\beta X}(T \cap F).$$

So $p \notin \mathrm{cl}_{\beta X}(F \setminus T)$, and it follows that $F \setminus T \notin \mathbf{Z}[I]$. But

$$(F \setminus T) \cup (F \cap T) = F \in \mathbf{Z}[I],$$

and I is prime, so that $F \cap T \in \mathbf{Z}[I]$, and hence $T \in \mathbf{Z}[I]$. Thus, by [GJ76, Theorem 6.4], $I = M^p$, and the result follows. □

Theorem 5.6. *If F is a nowhere dense z-embedded zeroset of a quasi P-space X, then F is a P-space.*

Proof: It follows easily from Proposition 3.2(a) that the map τ that sends the prime z-ideal I of $C(F)$ to $\{ f \in C(X) \,:\, f|_F \in I \}$ is an order preserving injection of the set of prime z-ideals of $C(F)$ into the set of prime z-ideals of $C(X)$. Suppose there were distinct prime z-ideals $I \subset J$ of $C(F)$. Then $\tau(I)$ is contained properly in $\tau(J)$, and since both $\mathbf{Z}[\tau(I)]$ and $\mathbf{Z}[\tau(J)]$ contain the nowhere dense zeroset F, neither is a minimal prime ideal. Because X is a quasi P-space, both ideals are maximal and hence equal, contrary to the fact that τ is an injection. Hence F must be a P-space. □

The following is immediate from Theorems 5.5(c) and 5.6.

Corollary 5.7. *If every nowhere dense zeroset of X is z-embedded (in particular, if X is normal) and X is cozero-complemented, then X is a quasi P-space if and only if each of its nowhere dense zerosets is a P-space.*

Next, we look at quasi P-spaces with certain countability conditions.

Proposition 5.8. *Suppose that every point of the quasi P-space X is a G_δ-point. Then every nowhere dense zeroset of X is discrete. If X is also ccc then it is nodec. If, in addition, X is first countable, then it is scattered of* CB*-index* ≤ 2.

Proof: Suppose that X is a quasi P-space, yet has a nowhere dense set S with a point $p \in \mathrm{cl}\, S \setminus S$. By replacing S by $\mathrm{cl}\ S \setminus \{p\}$, we may assume that $\mathrm{cl}\ S = S \cup \{p\}$. Let Q denote a minimal prime ideal of $C(\mathrm{cl}\ S)$ contained in $\{ f \in C(\mathrm{cl}\ S) \,:\, f(p) = 0 \}$, and let

$$Q' = \{ f \in C(X) \,:\, f[Z] = \{0\} \text{ for some } Z \in \mathbf{Z}[Q] \}.$$

By Proposition 5.1(a), each element of $\mathbf{Z}[Q]$ and hence of the prime z-filter $\mathbf{Z}[Q']$ is infinite, while $\mathbf{Z}(M_p)$ contains the one element zeroset $\{p\}$ since X has countable pseudocharacter. Thus M_p contains the prime z-ideal Q' properly. If $S = Z(g)$ for some $g \in C(X)$, then clearly $g \in Q'$. Using Proposition 5.1(a) again, we conclude that Q' contains a minimal prime ideal of $C(X)$ properly. This contradicts that X is a quasi P-space, and we must then conclude that S is discrete.

If X is a *ccc*-space, any open set contains a cozeroset densely. Thus, if T is a nowhere dense closed set, it is contained in a nowhere dense zeroset $Z(h)$. By the arguments above, $Z(h)$ is discrete, proving that T is discrete and that X is nodec.

Now assume that X is also first countable, but fails to be scattered. Then it has a closed subset A without isolated points that contains a convergent sequence and its limit. By the remarks in 5.4(a) this cannot happen in a nodec space. If $\mathrm{CB}(X) > 2$, then $X \setminus \mathrm{Is}(X)$ contains a convergent sequence and its limit, again contradicting that the space is nodec. □

We are now able to characterize some kinds of quasi P-spaces that need not be locally compact.

Theorem 5.9.

(a) *If X is normal and either ccc or basically disconnected, then X is a quasi P-space if and only if each of its nowhere dense zerosets is a P-space.*
(b) *If X is perfectly normal, then X is a quasi P-space if and only if it is nodec.*
(c) *If X is metrizable, then X is a quasi P-space if and only if it is scattered with* $\mathrm{CB}(X) \leq 2$.

Proof: (a) From the remarks in 5.2 X is cozero-complemented. Therefore, (a) follows from Corollary 5.7.

(b) is a consequence of Corollary 5.7, when one realizes that in a perfectly normal space, every closed set is a zeroset and every P-point is isolated.

(c) The necessity follows immediately from 5.2 and Proposition 5.8, and the sufficiency follows from (b) and the fact that scattered spaces of CB-index ≤ 2 are nodec. □

Next, we refer to the example promised at the conclusion of Section 2. It is due to van Douwen.

Remark 5.10. Example 3.3 of [vD93] is a countable (hence perfectly normal) nodec, extremally disconnected space. By Theorem 5.9(b) this space is quasi P. Since the example has no isolated points and is countable, it has no P-points. Note as well that it is *nowhere locally compact*; that is, there are no points with compact neighborhoods.

To verify all this the reader should examine, in [vD93], Definition 1.1, Theorem 1.2(a) and its proof, Fact 1.14, and Theorem 2.2.

Remark 5.11. An example will be given below (Example 7.5) of a space satisfying the hypotheses of Theorem 5.9(a) that fails to be nodec.

6. Product spaces

In this section, conditions under which a product of two spaces is quasinormal or quasi P are given. First we show that requiring even the former imposes severe restrictions on the factor spaces. Note that if the product of two spaces is normal, quasinormal, or quasi P, then so is each of the factor spaces. This is a consequence of the following two facts: first, if X and Y are any two spaces, and $y \in Y$, then $X \times \{y\}$ is C-embedded in $X \times Y$; second, all three properties under consideration are invariant under passing to a C-embedded subspace.

Theorem 6.1. *Suppose $X \times Y$ is normal.*

(a) *If $X \times Y$ is quasinormal and X is not an F-space, then Y is a P-space. Thus, if this product is quasinormal and neither X nor Y is a P-space, then both are F-spaces. In particular, if $X \times X$ is normal and quasinormal, then it is an F-space.*

(b) *If $X \times Y$ is a locally compact quasi P-space, then X or Y is discrete. In particular, if $X \times X$ is a locally compact quasi P-space, then X is discrete.*

(c) *If $X \times Y$ is a compact quasi P-space, then X or Y is finite, and the other is a finite free union of one-point compactifications of discrete spaces. In particular, if $X \times X$ is a compact quasi P-space, then X is finite.*

Proof: (a) Because X is not an F-space, there are disjoint cozerosets U, V of X with closures having nonempty intersection. So, $U \times Y$ and $V \times Y$ are disjoint cozerosets of the quasinormal space $X \times Y$ whose closures in $X \times Y$ have a nonempty intersection S. By Theorem 4.3 of [Ki01], S is a P-space. If $(x, y) \in S$, then $\{x\} \times Y$ is a P-space contained in S that is a homeomorphic copy of Y. The second assertion is now obvious.

(b) Because quasi P-spaces are quasinormal, it follows from (a) that if X is not an F-space, then Y is a locally compact P-space, and hence, by Corollary 2.8, is discrete. Otherwise, X is a locally compact F-space that is quasi P, and hence is discrete (see Theorem 4.3).

(c) This follows immediately from (a), using facts from Section 2 and Theorem 4.6.

□

Remark 6.2. Let $p \in \beta\omega\backslash\omega$ and X denote the subspace $\omega \cup \{p\}$ of $\beta\omega$. Because X is an F-space with exactly one non P-point, and because $X \times X$ is countable and hence normal, we may apply [La97a, Theorem 3.10], to conclude that $C(X \times X)$ is quasinormal. This shows that we cannot conclude that quasinormality of $X \times X$ implies that X is a P-space. Also, $X \times X$ is a not a quasi P-space. For, if

$$T = \{\, f \in C(X \times X) \,:\, Z(f) \text{ contains a neighborhood in } \{p\} \times X \text{ of } (p,p) \,\}$$

then T is a prime z-ideal of $C(X \times X)$ that is not minimal, which is contained properly in $M_{(p,p)}$. So, any minimal prime ideal of $C(X \times X)$ contained properly in T, together with T and $M_{(p,p)}$, forms a chain of three distinct prime z-ideals in $C(X \times X)$. Thus, the product of two scattered F-spaces of CB-index ≤ 2 need not be a quasi P-space.

We will next show that if a product of two spaces is a normal quasi P-space, then at least one of the factors is a P-space. (We do not know whether the hypothesis of normality is necessary.) We begin with some lemmas.

Lemma 6.3. *If $X \times Y$ is a quasi P-space and Y is an F-space that is not a P-space, then X is an almost P-space.*

Proof: By way of contradiction, suppose that there is a proper dense cozeroset $\mathrm{coz}(f)$ of X, and let $s \in Z(f)$. Define $F \in C(X \times Y)$ by letting $F(x,y) = f(x)$ for all $(x,y) \in X \times Y$. Then $\{s\} \times Y \subseteq Z(F)$ and $\mathrm{int}_{X \times Y} Z(F) = \emptyset$. Because Y is an F-space that not a P-space, there a $t \in C(Y)$ such that the prime z-ideal O_t is contained properly in the maximal ideal M_t of $C(Y)$. Now let

$$O_t(X \times Y) = \{\, G \in C(X \times Y) \,:\, G(s,\cdot) \in O_t \,\}.$$

(Note that here we identify $C(Y)$ with $C(\{s\} \times Y)$.) Then the maximal ideal $M_{(s,t)}$ of $C(X \times Y)$ contains the prime z-ideal $O_t(X \times Y)$ properly. The latter cannot be a minimal prime ideal, as $F \in O_t(X \times Y)$, yet F is not a zero-divisor in $C(X \times Y)$. Let Q be any minimal prime ideal of $C(X \times Y)$ contained in $O_t(X \times Y)$; then

$$\{\, Q,\, O_t(X \times Y),\, M_{(s,t)} \,\}$$

is a chain of three distinct prime z-ideals, contradicting that $X \times Y$ is quasi P. $\square$

Lemma 6.4. *If the product of two spaces is a normal quasi P-space, then either one is a P-space or else both X and Y are F-spaces and almost P-spaces. In particular, if $X \times X$ is a normal quasi P-space, then X is both an F-space and an almost P-space.*

Proof: By Theorem 6.1(a), if neither factor is a P-space, then both are F-spaces. Combining this with the previous lemma yields that both are also almost P-spaces. $\square$

The next lemma tells us that the P-spaces are the *hereditarily* almost P-spaces. This result is stated as Theorem 9 in [Ve73], with our condition (c) replaced by "every regular closed subspace of X is almost P". Very little is proved in [Ve73], and we have not found the characterization anywhere in English. The short proof we have provided seems, therefore, appropriate.

Lemma 6.5. *For a Tychonoff space X, the following are equivalent.*

(a) *X is a P-space.*
(b) *Every subspace of X is almost P.*
(c) *Every closed subspace of X is almost P.*

Proof: That (a) implies (b) is well known, and it is obvious that (b) implies (c).

Now assume (c) and suppose, by way of contradiction, that $p \in X$ is not a P-point. Then there is an $f \in C(X)$ such that $p \in Z(f) \setminus \operatorname{int}_X Z(f)$. Now let $Y \equiv \operatorname{cl}_X \operatorname{coz}(f)$; observe that $Z_Y(f|_Y)$ is nowhere dense in Y, and so Y is a closed subspace of X that fails to be an almost P-space. □

We are ready for the anticipated result.

Theorem 6.6. *If $X \times Y$ is a normal quasi P-space, then X or Y is a P-space.*

Proof: If one factor is not an F-space, then the other is a P-space by Theorem 6.1. Otherwise, by Lemma 6.4, both are F-spaces that are also almost P-spaces. On the other hand, according to Lemma 6.5, if neither is a P-space, then X has a closed subspace X' and Y has a closed subspace Y' that are not almost P-spaces. Using Lemma 6.4 again, $X' \times Y'$ is not a quasi P-space, despite being a closed subspace of the normal quasi P-space $X \times Y$. This contradiction completes the proof. □

The next corollary is straightforward, and the proof is left to the reader. One should observe, however, that as far as we know, (c) in the corollary might be without content; see Question 9.1.

Corollary 6.7.

(a) *If $X \times X$ is a normal quasi P-space then X is a P-space.*
(b) *If $X \times Y$ is a perfectly normal quasi P-space then one of the factors is discrete and the other is nodec.*
(c) *If $X \times X$ is an infinite connected quasi P-space, then it fails to be normal.*

Remark 6.8. Let X denote the product of the discrete space ω and its one-point compactification $\alpha\omega$. Then X is a locally compact scattered normal space of CB-index 2, and hence X is quasi P, contrary to the assertion made in [La97a, Example 3.6(1)]. This example shows also that [La97a, Corollary 3.11] is incorrect.

These errors stem from the incorrect assertion that for every locally compact σ-compact space X, O^p is prime, for every $p \in \beta X \setminus X$. The reader may refer to the comments in 1.5 above. No other errors seem to exist in this otherwise excellent paper.

We conclude the section with a sufficient condition for a product of two spaces to be quasi P. As we now know, for normal products, a standing assumption should be — and will be — that one of the two factors is a P-space.

First we present a pair of lemmas (Lemmas 6.9 and 6.12) on how to piece together P-spaces to form a quasi P-space. The lemmas themselves are intriguing. The reader is reminded of the notation of 3.1 for the z-embedding τ of a subspace Y in X. In particular, for such a τ, $\mathcal{S}_\tau$ stands for the set of all prime z-ideals of $C(X)$ containing the kernel of $C(\tau)$, K_τ.

Lemma 6.9. *Suppose that X is a Tychonoff space having a dense, open P-subspace S, such that the embedding $\tau : X \setminus S \longrightarrow X$ is a C-embedding. Then if $P \in \mathcal{S}_\tau$ $C(\tau)(P)$ is a prime z-ideal of $C(X \setminus S)$.*

Proof: Let $Y = X \setminus S$. As Y is assumed to be C-embedded in X the homomorphism $C(\tau)$ is surjective. Thus, from elementary algebra, if $P \in \mathcal{S}_\tau$ then $Q = C(\tau)(P)$ is a prime ideal of $C(Y)$. The question is whether it is a z-ideal. We show this now. We remind the reader that $C(\tau)$ is the restriction homomorphism to Y.

Suppose that $f, g \in C(X)$ with $f \in P$ and $Z(f) \cap Y \subseteq Z(g) \cap Y$. To show that Q is a z-ideal we must prove that $g \in P$. Without loss of generality, f and g may be taken to be positive. Now, let h be defined on X as follows:

$$h(x) = \begin{cases} g(x) & \text{if } x \in S \cap Z(f), \\ 0 & \text{otherwise.} \end{cases}$$

We prove that h is a continuous function. Note first that, since S is a P-space, $S' = S \cap Z(f)$ is clopen in S, so that both S' and $S \setminus S'$ are open in X. This means that h is continuous at any point of S.

Continuity at the points of Y remains to be settled. By way of contradiction, suppose that there is a $y \in Y$ and positive number δ such that, for each neighborhood U of y, there is a $p_U \in S \cap U$ with $h(p_U) = g(p_U) \geq \delta$. Then $g(y) \geq \delta$ as well, and therefore not zero, which means that $f(y) > 0$ as well. Now select a neighborhood W of y such that, for each $x \in W$,

$$(*) \qquad |f(y) - f(x)| < f(y).$$

Letting $x = p_W$, we have $f(x) = 0$, for otherwise, $h(x) = h(p_W) = 0$, by definition of h, which is contrary to the choice of p_W. However, $f(x) = 0$ contradicts $(*)$. This proves that $h \in C(X)$, and it is clear that it lies in K_τ.

Next consider $g - h$. Note that $Z(f) \subseteq Z(g - h)$, while $h \in K_\tau$ and therefore in P. This implies that $g - h \in P$, whence g is also in P. This proves that Q is a prime z-ideal of $C(Y)$, and, clearly, the claim of the lemma. □

Remark 6.10. Returning to 3.1 for a moment, that discussion shows that, under the conditions of Lemma 6.9, $\mathrm{Spec}_z(\tau)$ is an order isomorphism of the poset $\mathrm{Spec}_z(X \setminus S)$ onto $\mathcal{S}_\tau$.

Recall that an f-ring A is projectable if every principal annihilator of A is a summand. The reader should also review the notation of Remark 1.1(b).

The following lemma is well known. We prove it here for completeness.

Lemma 6.11. *For each space X, $C(X)$ is projectable if and only if X is basically disconnected.*

Proof: First, observe that, for each $f \in C(X)$,

$$f^{dd} = \{g \in C(X) : \operatorname{coz}(g) \subseteq \operatorname{cl}_X \operatorname{coz}(f)\} \ \text{ and } \ f^d = \{g \in C(X) : \operatorname{coz}(g) \subseteq Z(g)\}.$$

Now if $C(X)$ is projectable then $1 = 1[f] + 1[f^d]$, for each $f \in C(X)$. It is easy to check that $1[f]$ and $1[f^d]$ are mutually annihilating idempotents of the ring $C(X)$. Also we have the partition $X = \operatorname{coz}(1[f]) \cup \operatorname{coz}(1[f^d])$, into clopen sets. It follows that $\operatorname{coz}(1[f]) = \operatorname{cl}_X \operatorname{coz}(f)$, proving that $\operatorname{cl}_X \operatorname{coz}(f)$ is clopen and that X is basically disconnected.

Conversely, if X is basically disconnected, $\operatorname{cl}_X \operatorname{coz}(f)$ and $\operatorname{int}_X Z(f)$ partition X into clopen sets. The characteristic functions of these sets, e_1 and e_2, satisfy $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. We leave it to the reader to check that $e_1 = 1[f]$ and $e_2 = 1[f^d]$. Finally, for each $g \in C(X)$, we have that $g = g\,1[f] + g\,1[f^d]$, and $g\,1[f] \in f^{dd}$ and $g\,1[f^d] \in f^d$, because each annihilator is an ideal. Thus, $C(X) = f^{dd} + f^d$, proving that $C(X)$ is projectable. □

Lemma 6.12. *Suppose that X is a Tychonoff space having a dense, open P-subspace S, such that $X \setminus S$ is C-embedded and also a P-space. Then X is quasi P.*

Proof: We shall keep to the notation of Lemma 6.9 and its proof. We begin by proving the following.

(†) Suppose that P is a prime z-ideal of $C(X)$ that does not contain K_τ. Then $P \cap K_\tau$ is maximal among the z-ideals of $C(X)$ which are properly contained in K_τ.

First observe that the dense embedding of S in X induces an ℓ-embedding ρ of $C(X)$ in $C(S)$ by restriction of functions. We shall strain the notation a bit, and identify K_τ with its image under ρ. Note as well that, for the balance of this argument, we shall treat $C(S)$ as an ℓ-group, ignoring the ring structure.

Since S is a P-space it is also basically disconnected. Now K_τ is a convex ℓ-subgroup of $C(S)$: for if $|g| \le |h|$, and $h \in K_\tau$, with $g \in C(S)$, then extending g to X by setting $g(y) = 0$, for each $y \in Y$, extends it continuously to X. Evidently, $g \in K_\tau$. The importance of this is that K_τ too is projectable.

Now to the proof of (†). It is clear that $P \cap K_\tau$ is a z-ideal of $C(X)$, and properly contained in K_τ. To establish (†) it suffices to show that for each $f \in K_\tau \setminus P$, the z-ideal of $C(X)$ generated by f and $P \cap K_\tau$ is K_τ.

The reader is encouraged at this point to review the projection notation defined in 1.1(b). To begin, $K_\tau \subseteq f^{dd} + f^d$, so that, if $b \in K_\tau$, we have $b = b[f] + b[f^d]$, with $b[f] \in f^{dd}$ and $b[f^d] \in f^d$. Since $b[f^d]f = 0$, we have that $b[f^d] \in P \cap K_\tau$. Finally, we assert that $Z_X(f) \subseteq Z_X(b[f])$: for $b[f] \in f^{dd}$ means that

$$\operatorname{coz}_S(b[f]) \subseteq \operatorname{cl}_S \operatorname{coz}_S(f|_S) = \operatorname{coz}_S(f|_S),$$

the latter identity coming from the assumption that S is a P-space. Since $f \in K_\tau$, we get, in fact, that $\mathrm{coz}_X(b[f]) \subseteq \mathrm{coz}_X(f)$, whence $Z_X(f) \subseteq Z_X(b[f])$. This fact shows that $b[f]$ lies in the z-ideal of $C(X)$ generated by f, which proves (†).

Now suppose that $P_0 \subset P_1 \subset P_2$ are distinct prime z-ideals of $C(X)$. Since Y is assumed to be a P-space, we have that every prime ideal is maximal, as pointed out in 1.3(b). Thus Lemma 6.9 shows that P_0 and P_1 cannot contain K_τ, whereas (†) implies that P_1 and P_2 must contain K_τ. This is a contradiction, proving that every prime z-ideal of $C(X)$ is either maximal or minimal. The proof of this lemma is now complete. □

Theorem 6.13. *Suppose that X is either a compact or a metrizable quasi P-space, and Y is any P-space. Then $X \times Y$ is quasi P.*

PROOF: Since X is compact (resp. metrizable) it is scattered of CB-index ≤ 2 (in the metrizable instance by Theorem 5.9(c)). To say that $\mathrm{CB}(X) = 1$ is to say that X is finite (resp. discrete), in either case making $X \times Y$ a free union of copies of Y. Since a free union of P-spaces is a P-space, the claim is established. We therefore suppose that $\mathrm{CB}(X) = 2$.

Then $X \times Y$ is composed of $\mathrm{Is}(X) \times Y$, which is a free union of P-spaces (and therefore a P-space) and $(X \setminus \mathrm{Is}(X)) \times Y$, which is a finite union of P-spaces and also a P-space. Next, observe that $\mathrm{Is}(X) \times Y$ is open and dense in $X \times Y$; the complement, $(X \setminus \mathrm{Is}(X)) \times Y$ is C-embedded in $X \times Y$. By Lemma 6.12, $X \times Y$ is quasi P. □

Remark 6.14. We do not know if the product of a locally compact space quasi P-space X and a P-space Y must be quasi P, even if X is normal. The best we seem to be able to do follows. The reader is referred to [En89, Chapter 5] for the definition and basic properties of *paracompact* spaces. For our purposes, the reader need only know from 5.1.27 of this reference that:

> *Every locally compact paracompact space is a free union of locally compact σ-compact spaces.*

Coupled with Corollary 4.7 and Theorem 6.13, the above comment yields:

Theorem 6.15. *If X is a locally compact paracompact quasi P-space and Y is a P-space, then $X \times Y$ is quasi P.*

The following is worth noting.

Remark 6.16. We use $\alpha\omega$ to denote the one-point compactification of the discrete natural numbers. Let Y be any P-space. Then according to Theorem 6.13, $\alpha\omega \times Y$ is a quasi P-space. Note also that $\alpha\omega$ is neither almost P nor an F-space.

In closing this section, we look ahead to Question 9.5 about free unions of quasi P-spaces. Technically, what follows may be viewed as a result about products.

Let us call a space *essentially a P-space* if all of its points except possibly one are P-points. By Lemma 2.4 every space which is essentially a P-space is a quasi P-space.

The proposition which follows is easily proved using Lemma 6.12.

Proposition 6.17. *A free union of essentially P-spaces is quasi P.*

Proof: Suppose that $\{ X_i : i \in I \}$ is a set of essentially P-spaces, and let X be their free union. If, for $i \in I$, X_i is not a P-space, denote the lone non-P-point of X_i by p_i, and let $Y_i = X_i \setminus \{p_i\}$. If X_i is a P-space, let $Y_i = X_i$. Finally, let $S = \bigcup_i Y_i$.

It is easily checked that S, being a free union of P-spaces, is a P-space. Moreover, S is open and dense in X, with $X \setminus S$ C-embedded and discrete. Thus, by Lemma 6.12, X is a quasi P-space. □

7. Some nonnormal examples

Most of the theorems we have used to create quasi P-spaces assume normality conditions, and it is natural to ask about the extent to which such assumptions are needed. The examples that follow in this section answer some of these questions while raising others.

The first result is of independent interest and will be used below; it should be compared to Theorem 4.3.

Theorem 7.1. *Suppose X is a scattered space of* CB*-index* 2 *for which* $\mathrm{Is}(X)$ *is countable. Then the following are equivalent.*

(a) *X is a quasi P-space.*
(b) *If $X^{(1)}$ is partitioned into two infinite subsets A and B, and one of them is a zeroset of X, then so is the other.*

Proof: (a) ⇒ (b): Assume that (b) fails, and $X^{(1)}$ is partitioned into two infinite subsets A and B, with $A \in \mathbf{Z}(X)$ but $B \notin \mathbf{Z}(X)$. Suppose A and B were completely separated in X. Then there would be disjoint zerosets Z_1 and Z_2 of X such that $Z_1 \supseteq A$ and $Z_2 \supseteq B$. Because $\mathrm{Is}(X)$ is countable, it is a cozeroset of X, which implies that $X^{(1)}$, and hence $X^{(1)} \cap Z_2 = B$, are in $\mathbf{Z}(X)$, contrary to the assumption that $B \notin \mathbf{Z}(X)$. Hence these two sets are not completely separated, so there is a $p \in \mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X} B$.

Let
$$\mathcal{F} = \{ Z \cap X^{(1)} : Z \in \mathbf{Z}[O^p] \} \cup \{B\},$$
and observe that since $\mathbf{Z}[O^p]$ is a z-filter and $p \in \mathrm{cl}_{\beta X} B$, $\mathcal{F}$ is closed under finite intersection. Let $\mathcal{U}$ denote an ultrafilter on the set $X^{(1)}$ containing $\mathcal{F}$, and let
$$\mathcal{P} = \{ Z \in \mathbf{Z}(X) : \text{ there is a } U \in \mathcal{U} \text{ such that } U \subseteq Z \cap X^{(1)} \}.$$

It is not difficult to verify that $\mathcal{P}$ is a prime z-filter contained in $\mathbf{Z}[M^p]$. Because it contains the nowhere dense zero set $X^{(1)}$, $\mathcal{P}$ cannot be a minimal prime z-filter. Moreover, $\mathcal{P}$ cannot be a maximal z-filter. For, A is a zeroset of X such that $p \in \mathrm{cl}_{\beta X} A$, which implies that $A \in \mathbf{Z}[M^p]$. But, if $A \in \mathcal{P}$, there is a $U \in \mathcal{U}$ contained in A, and because $B \in \mathcal{F} \subseteq \mathcal{U}$ and $A \cap B = \emptyset$, $\mathcal{U}$ would fail to be a filter. We conclude that $\mathbf{Z}[M^p]$ and any minimal prime z-filter contained in $\mathcal{P}$ is a chain of three prime z-filters, contradicting that X is a quasi P-space.

(b) $\Rightarrow$ (a): X is *ccc*, and so, by the remarks in 5.2(c), $\mathrm{Min}(C(X))$ is compact. Note also that because $\mathrm{Is}(X)$ is countable, $X^{(1)} \in \mathbf{Z}(X)$. Suppose $\mathcal{P}$ is a prime z-filter that is not minimal, and contained in the maximal z-filter $\mathbf{Z}[M^q]$ for some $q \in \beta X$. Because $\mathcal{P}$ is not minimal and mX is compact, it contains a nowhere dense zeroset, which must be contained in $X^{(1)}$. Therefore $X^{(1)} \in \mathcal{P}$. It suffices to show that $\mathcal{P} = \mathbf{Z}[M^q]$, that is, that if $q \in \mathrm{cl}_{\beta X} Z$ for some zeroset Z, then $Z \in \mathcal{P}$.

Because q is in the closure in βX of both Z and $X^{(1)}$, it is in $\mathrm{cl}_{\beta X}(Z \cap X^{(1)})$. We now consider three cases.

(i) *$Z \cap X^{(1)}$ is finite.* Then $q \in Z \cap X^{(1)} \subseteq Z \in \mathcal{P}$, and so $X^{(1)} \setminus Z \in \mathbf{Z}(X) \setminus \mathbf{Z}[M^q]$. As $X^{(1)} \in \mathcal{P}$ and $\mathcal{P}$ is prime, $Z \cap X^{(1)} \in \mathcal{P}$, proving that $Z \in \mathcal{P}$.

(ii) *$Z \cap X^{(1)}$ is cofinite in $X^{(1)}$.* Then $X^{(1)} = (Z \cap X^{(1)}) \cup F$ for some finite subset F of $X^{(1)}$ disjoint from $Z \cap X^{(1)}$. Note that $F \in \mathbf{Z}(X)$; for each point of F is in a clopen set disjoint from $Z \cap X^{(1)}$, so that their finite union G is a zeroset disjoint from $Z \cap X^{(1)}$. Therefore, $G \cap X^{(1)} = F \in \mathbf{Z}(X)$. But $q \notin F$, which implies that $F \notin \mathcal{P}$.
Because $\mathcal{P}$ is prime, we conclude that $Z \cap X^{(1)}$ and hence Z is in $\mathcal{P}$.

(iii) *Both $Z \cap X^{(1)}$ and $X^{(1)} \setminus Z$ are infinite.* By assumption, because $Z \cap X^{(1)} \in \mathbf{Z}(X)$, one has as well that $X^{(1)} \setminus Z \in \mathbf{Z}(X)$. Hence $Z \cap X^{(1)}$ and $X^{(1)} \setminus Z$ have disjoint closures in βX. Because $q \in \mathrm{cl}_{\beta X}(Z \cap X^{(1)})$, it is not in $\mathrm{cl}_{\beta X}(X^{(1)} \setminus Z)$. It follows that $X^{(1)} \setminus Z$ is not in $\mathbf{Z}[M^p]$ and hence is not in $\mathcal{P}$. Because $\mathcal{P}$ is prime, this implies that $Z \cap X^{(1)} \in \mathcal{P}$ whence Z too is in $\mathcal{P}$.

The proof that (b) implies (a) is now complete. □

The next two lemmas will be used below. The first one appears as [O80, Theorem 7.3]. Lemma 7.3 appears implicitly, but not explicitly, in [vR82, §11]. A proof is given for the sake of completeness.

We remind the reader that a function $f : X \longrightarrow \mathbb{R}$ is a *Baire*-1 *function* (or that $f \in B_1(X)$) if it is the pointwise limit of a sequence of continuous functions.

Lemma 7.2. *If $f \in B_1(\mathbb{R})$, then the set of real numbers at which f fails to be continuous is of the first category;* (*i.e., is a countable union of nowhere dense closed sets*).

Lemma 7.3. *If $f \in B_1(\mathbb{R})$, then $Z(f)$ is not a countable dense subspace of $\mathbb{R}$.*

Proof: Let S denote a countable dense subspace of $\mathbb{R}$. It will be shown that $Z(f) \neq S$. For, otherwise, if $r \in \mathbb{R} \setminus S$, $f(r) \neq 0$, and we may assume $f(r) > 0$. If f is continuous at r, there is a $\delta > 0$ such that if $|x - r| < \delta$, then

$$|f(x) - f(r)| < \frac{f(r)}{4}.$$

So $f(x) \neq 0$ if $x \in (r - \delta, r + \delta)$. Because S is dense in $\mathbb{R}$ this cannot happen, so that f is discontinuous at each point of $\mathbb{R} \setminus S$. By Lemma 7.2 there is a set E of first category containing $\mathbb{R} \setminus S$. Hence $S \cup E = \mathbb{R}$. Since both S and E are of the first category, $\mathbb{R}$ is a countable union of nowhere dense closed subsets, contrary to the Baire Category Theorem. This contradiction completes the proof. □

Example 7.4. A first-countable, nonnormal, locally compact, realcompact, separable, scattered space with CB-index 2, that is not a quasi P-space, but which is quasi P at each of its points.

In [Mr70, 1.2], S. Mrowka presents a first countable locally compact space T constructed on a subset of $\{ (x, y) \in \mathbb{R} \times \mathbb{R} : y \geq 0 \}$. Further,

(a) T has a countable dense subspace N of isolated points;
(b) $T \setminus N$ is a copy of $\mathbb{R}$ with the discrete topology;
(c) if $f \in C(T)$, then $f|_{T \setminus N}$ is a Baire-1 function on $\mathbb{R}$ with respect to its usual topology.

In [Mr70, 1.1], it is shown that T is realcompact and (a), (b), and (c) hold. Moreover, T is nonnormal because it is separable but has a closed discrete subspace of cardinality $\mathfrak{c}$ of the continuum. Finally, T is not a quasi P-space by Theorem 7.1 and Lemma 7.3. For, if S is a countable dense subspace of $\mathbb{R}$ with its usual topology, then $S \notin \mathbf{Z}[T]$, but $\mathbb{R} \setminus S \in \mathbf{Z}[T]$. However, as with Example 4.4, T is quasi P at each of its points.

Example 7.5. A separable, scattered, quasi P-space X of CB-index 2 of cardinality ω_1 that fails to be normal or realcompact, such that its realcompactification υX is Lindelöf, $\mathrm{CB}(\upsilon X) = 3$, and υX fails to be nodec.

In [BSV, 3.5], the authors give an example of a subspace $D \cup \{p\}$ of $\beta\omega \setminus \omega$, such that D is a discrete space of cardinality ω_1 and the neighborhoods of p are cocountable in D. Let $X = \omega \cup D$, and let $Y = X \cup \{p\}$, considered as subspaces of $\beta\omega$. The fact that every zeroset of $D \cup \{p\}$ is countable or cocountable has three consequences:

(i) no two disjoint uncountable subspaces of X that are contained in D can be completely separated in X, and so X is not normal;
(ii) X is a quasi P-space. To see this we make a number of observations. First, since X is C^*-embedded in Y, it is also z-embedded in Y. Now we show that Theorem 7.1 applies.

If complementary uncountable subsets of D are zerosets of X, there exist zerosets of Y, and hence of $D \cup \{p\}$, whose traces on D form complementary uncountable sets. This contradicts the fact that in $D \cup \{p\}$ neighborhoods of p are cocountable.

Next, note that every cocountable subset of D is a zeroset of X. Hence (to apply Theorem 7.1) we must show that every countable subset C of D is a zeroset of X. As to that, note that the characteristic function e of $(D \setminus C) \cup \{p\}$ in $D \cup \{p\}$ is continuous. Now $D \cup \{p\}$ is Lindelöf and hence C-embedded in the extremally disconnected space Y, so that e extends to $f \in C^*(Y)$. Then $C = Z(f|_X)$ and $f|_X \in C(X)$.

(iii) The restriction of every $f \in C(X)$ is constant on a cocountable subset of D, whence $Y = \upsilon X$ and hence the Lindelöf (realcompact) space Y is also a quasi P-space.

Clearly Y is scattered of CB-index 3. Since D is a nowhere dense subspace of Y that fails to be closed, Y is not nodec.

8. Mapping theorems

We turn next to the question of when continuous images of quasi P-spaces are again quasi P.

Definition & Remarks 8.1. Suppose that $f : X \longrightarrow Y$ is a continuous surjection. Recall that f is called *irreducible* if it is continuous, closed, and no proper closed subset of X is mapped by f onto Y. It is well known that f is irreducible precisely when the induced embedding (as f-rings) $C(f) : C(Y) \longrightarrow C(X)$ is order dense; that is to say, when for each $h > 0$ in $C(X)$ there is a $k > 0$ in $C(Y)$ such that $C(f)(k) = k \cdot f \leq h$.

We remind the reader — see Remark 1.1(a) — that the set $\mathcal{A}(C(X))$ of all annihilators of $C(X)$ is a boolean algebra under inclusion. Observe then that f is irreducible if and only if the "trace" map $P \mapsto C(f)^{\leftarrow}(P)$ is an isomorphism from $\mathcal{A}(C(X))$ onto $\mathcal{A}(C(Y))$. (The reader may combine [BKW77, 11.1.15] and [BH87, Theorem 2.4] to get the measure of these assertions.)

Definition & Remarks 8.2. Suppose $f : X \longrightarrow Y$ is a continuous mapping.

(a) If the inverse image $f^{\leftarrow}(Z)$ of each nowhere dense closed subspace (resp. nowhere dense zeroset) Z of Y is nowhere dense in X, then f is called a *skeletal map* (resp. *z-skeletal map*).

(b) The notion of a skeletal map is introduced and discussed in I, Section 6, and III, Section 1 of [MR69]. It is noted in [MR69] that a continuous surjection $f : X \longrightarrow Y$ is skeletal if the image of each open subset of X set has a nonempty interior. A proof that irreducible maps are skeletal appears in [PW88, §6.5].

Theorem 8.3. *Suppose $f : X \longrightarrow Y$ is a z-skeletal, continuous, closed surjection. Assume that X is normal, and both* $\mathrm{Min}(C(X))$ *and* $\mathrm{Min}(C(Y))$ *are compact. Then if X is a quasi P-space, Y is quasi P as well.*

Proof: As is noted in [Bu80, Table III], a closed continuous image of a normal space is normal. So, by Corollary 5.7, it suffices to show that every nowhere dense zeroset of Y is a P-space. Suppose $L \in \mathbf{Z}(Y)$ is nowhere dense. Because f is z-skeletal, so is the zeroset $f^{\leftarrow}(L)$ of X. Since $\mathrm{Min}(C(X))$ is compact, by Corollary 5.7, this zeroset is a P-space. Thus, if $Z \in \mathbf{Z}(L)$, then the zeroset $f^{\leftarrow}(Z)$ is open in $f^{\leftarrow}(L)$. Thus $f^{\leftarrow}(L) \setminus f^{\leftarrow}(Z)$ is closed in $f^{\leftarrow}(L)$, and hence in X. But

$$f^{\leftarrow}(L) \setminus f^{\leftarrow}(Z) = f^{\leftarrow}(L \setminus Z).$$

Because f is a closed map, it follows that $L \setminus Z$ is closed and Z is open, proving that L is a P-space. We conclude that X is a quasi P-space. □

As noted earlier, perfectly normal spaces have compact spaces of minimal prime ideals, and by [Bu80, Table III], a closed continuous image of a (perfectly) normal space is (perfectly) normal. Also, because a continuous image of a *ccc*-space is *ccc* and *ccc*-spaces have compact spaces of minimal prime ideals, we deduce from Theorems 5.8 and 5.9, and the preceding result:

Corollary 8.4. *Suppose $f : X \longrightarrow Y$ is a continuous closed z-skeletal surjection. If X is a perfectly normal quasi P-space or a normal ccc-space that is quasi P, then so is Y.*

To conclude this section let us see what happens when we strengthen the irreducibility of a map.

Definition & Remarks 8.5. The reader is reminded that a continuous surjection $f : X \longrightarrow Y$ is *perfect* if it is a closed map and $f^{\leftarrow}(\{p\})$ is compact, for each $p \in Y$.

Assume throughout that the surjection $f : X \longrightarrow Y$ is perfect and irreducible. Once more we consider the induced embedding $C(f) : C(Y) \longrightarrow C(X)$, but for convenience in this situation we suppress the label $C(f)$, and regard $C(Y)$ embedded in $C(X)$ as a subring and sublattice. Note that the so-called trace map of 8.1 is now to be viewed as $P \mapsto C(Y) \cap P$.

(a) Call f *ω_1-irreducible* if for each cozeroset U of X there is a cozeroset V of Y such that $f^{\leftarrow}(V)$ is dense in U. This concept has been called *sequentially irreducible* (in [BH87]) and $Z^{\#}$-irreducible (in [HVW87]). Note that f is ω_1-irreducible if and only if under the trace map in 8.1, $P \in \mathcal{A}(C(X))$ is countably generated if and only if $C(Y) \cap P$ is countably generated; in fact, one has [HM93, Proposition 2.2], which is summarized here as follows:

> *Suppose that $f : X \longrightarrow Y$ is irreducible. Then it is ω_1-irreducible if and only if for each $h \in C(X)$ there is a $g \in C(Y)$ such that $h^{dd} = g^{dd}$.*

In [CM90] such an embedding (such as $C(Y) \subseteq C(X)$) is said to be *rigid*. Observe that $C(f)$ induces a rigid embedding precisely when tracing $P \mapsto P \cap C(Y)$ induces a lattice-isomorphism between the lattices of principal annihilators.

(b) It is a consequence of [CM90, Proposition 2.3] that if $C(Y) \subseteq C(X)$ is rigid then tracing $P \mapsto P \cap C(Y)$ induces a homeomorphism from $\mathrm{Min}(C(X))$ onto $\mathrm{Min}(C(Y))$.

Contrast this with the following example: let D be an uncountable discrete space and f denote the perfect irreducible map that sends βD onto αD by identifying all the points of the remainder of βD to the point at infinity of αD. As βD is basically disconnected, $\mathrm{Min}(C(\beta D)) \cong \mathrm{Max}(C(\beta D)) = \beta D$ is compact, while it is known that $\mathrm{Min}(C(\alpha D))$ is not.

The following is an immediate consequence of Theorem 8.3.

Corollary 8.6. *Suppose that $f : X \longrightarrow Y$ is perfect and ω_1-irreducible. Assume that X is normal and cozero-complemented. If X is quasi P, then so is Y.*

The converse to Corollary 8.6 is false; here is an example. It also shows that the converse of Corollary 8.4, involving *ccc*-spaces, is also false.

Example 8.7. The map $\beta\omega \longrightarrow \alpha\omega$ which collapses the remainder of $\beta\omega$ to the point at infinity in $\alpha\omega$ is perfect and ω_1-irreducible. However, $\alpha\omega$ is quasi P, whereas $\beta\omega$ is not. Note that the rings of continuous functions of these examples have a (common) extremally disconnected minimal prime ideal space, $\beta\omega$. These examples are *ccc*-spaces.

A particular instance of Corollary 8.6 gives us another interesting example.

Corollary 8.8. *Suppose X is normal and $\mathrm{Min}(C(X))$ is compact. Let F be a compact subset of a nowhere dense zero set of X. Suppose that Y is obtained from X by collapsing F to a point, and $f : X \longrightarrow Y$ is the quotient map. If X is quasi P, then so is Y.*

Proof: Simply observe that the quotient map under consideration is perfect and ω_1-irreducible. □

Here is a simple application of Corollary 8.8.

Example 8.9. Take two copies E_1 and E_2 of a countable, nowhere locally compact, extremally disconnected quasi P-space E and attach them together at a point. (The reader is reminded of the example of van Douwen; see Remark 5.10.) One gets a countable quasi P-space X with no P-points that fails to be an F-space. It should be noted that X is an SV-space.

9. Open questions

There are many interesting open questions on quasi P-spaces. Here is a sample.

Question 9.1. *Are there any infinite connected quasi P-spaces, or must all of them be zero-dimensional?*

The results given above imply that no infinite locally compact or metrizable quasi P-space could be connected. Indeed, in an infinite connected quasi P-space, no point can have a compact neighborhood. Also, if there were a normal quasi P-space that is not totally disconnected, it would have an infinite component that is a connected quasi P-space. This question appears to be quite difficult, and even a consistent answer would be welcome.

Question 9.2. *Most of the results in Section* 4 *depend on the assumption of local compactness. What happens if we weaken the local compactness to: the set of points with no compact neighborhoods is compact?*

The latter condition is equivalent to assuming that the space in question has a locally compact remainder in any compactification.

Here is an example with some relevance to this question. Taking $X = \alpha\omega$ and $Y = \lambda D \equiv D \cup \{\lambda\}$ (with D discrete and uncountable); the neighborhoods of λ are the subsets of Y which have countable complement in D. It is well known that Y is a P-space. Thus, $X \times Y$ is a quasi P-space by Theorem 6.13. However, the CB-index is 3; this is easy to check. The point is that in $X \times Y$ the set of points with no compact neighborhoods is not compact, as it is a copy of Y.

Question 9.3. *We have assumed often that various spaces are normal in order to carry out proofs of theorems; especially in the product theorems of Section* 6*. To what extent is this assumption necessary?*

Theorems 6.13 and 6.15 inspire the following questions.

Question 9.4. *Must the product of a locally compact* (*normal*) *quasi P-space and a P-space be quasi P?*

Question 9.5. *Must the free union of an infinite family of quasi P-spaces be a quasi P-space?*

With some assumptions one is able to answer this question in the affirmative. In what follows let X be the free union of the spaces X_i $(i \in I)$.

(i) Note that X is cozero-complemented if and only if each X_i is cozero-complemented; apply Proposition 5.1(c)(iv). Also, X is normal if and only if each X_i is normal. Now apply Corollary 5.7: assume each X_i is normal and cozero-complemented. Then if each X_i is quasi P it follows that X is quasi P.

(ii) In particular, the free union of any number of perfectly normal quasi P-spaces is quasi P.

(iii) See Proposition 6.17: suppose, in addition, that each X_i is quasi P and both locally compact and paracompact. By Corollary 4.7 and the remarks in 6.14, each X_i is a free union of one-point compactifications of discrete spaces. Relabelling, it follows from Proposition 6.17 that X is quasi P.

In general, though, we are unable to answer this question even for a countably infinite free union.

References

[BSV81] Balcar B., Simon P., Vojtáš P., *Refinement properties and extensions of filters in boolean algebras*, Trans. Amer. Math. Soc. **267** (1981), 265–283.

[BH87] Ball R.N., Hager A.W., *Archimedean kernel-distinguishing extensions of archimedean ℓ-groups with weak unit*, Indian J. Math. **29** (3) (1987), 351–368.

[BKW77] Bigard A., Keimel K., Wolfenstein S., *Groupes et Anneaux Réticulés*, Lecture Notes in Math. **608**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[Bl76] Blair R.L., *Spaces in which special sets are z-embedded*, Canad. J. Math. **28** (1976), 673–690.

[Bu80] Burke D., *Closed Mappings*, Surveys in Gen. Topology, Academic Press, New York, 1980, pp. 1–32.

[CH70] Comfort W., Hager A., *Estimates for the number of continuous functions*, Trans. Amer. Math. Soc. **150** (1970), 619–631.

[CM90] Conrad P., Martinez J., *Complemented lattice-ordered groups*, Indag. Math. (N.S.) **1** (1990), 281–297.

[D95] Darnel M., *Theory of Lattice-Ordered Groups*, Pure & Appl. Math. **187**, Marcel Dekker, New York, 1995.

[DF99] Dummit D.S., Foote R.M., *Abstract Algebra*, 2nd edition, Prentice Hall, 1999.

[vD93] van Douwen E., *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–139.

[vDP79] van Douwen E., Pryzmusiński T., *First countable and countable spaces all compactifications of which contain $\beta\mathbb{N}$, Fund. Math.* **52** (1979), 229–234.

[En89] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[GJ60] Gillman L., Jerison M., *Quotient fields of residue class rings of function rings*, Illinois J. Math. **4** (1960), 425–436.

[GJ76] Gillman L., Jerison M., *Rings of Continuous Functions*, Grad. Texts Math. **43**, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[HM93] Hager A., Martinez J., *Fraction-dense algebras and spaces*, Canad. J. Math. **45** (1993), 977–996.

[HJ65] Henriksen M., Jerison M., *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (1965), 110–130.

[HLMW94] Henriksen M., Larson S., Martinez J., Woods R.G., *Lattice-ordered algebras that are subdirect products of valuation domains*, Trans. Amer. Math. Soc. **345** (September 1994), no. 1, 195–221.

[HVW87] Henriksen M., Vermeer J., Woods R.G., *Quasi-F covers of Tychonoff spaces*, Trans. Amer. Math. Soc. **303** (2) (1987), 779–803.

[Ki01] Kimber C., *m-Quasinormal f-rings*, J. Pure Appl. Algebra **158** (2001), 197–223.

[Ko89] Koppelberg S., *Handbook of Boolean Algebras, I.*, J.D. Monk, Ed., with R. Bonnet; Elsevier, Amsterdam-New York-Oxford-Tokyo, 1989.

[La95] Larson S., *A characterization of f-rings in which the sum of semiprime ℓ-ideals is semiprime and its consequences*, Comm. Algebra **23** (1995), no. 14, 5461–5481.

[La97a] Larson S., *Quasi-normal f-rings*, in Proc. Ord. Alg. Structures (Curaçao, 1995), W.C. Holland & J. Martinez, Eds., Kluwer Acad. Publ., Dordrecht, 1997, pp. 261–275.

[La97b] Larson S., *f-Rings in which every maximal ideal contains finitely many minimal prime ideals*, Comm. Algebra **25** (1997), 3859–3888.
[Le77] Levy R., *Almost P-spaces*, Canad. J. Math. **29** (1977), 284–288.
[LR81] Levy R., Rice M., *Normal spaces and the G_δ-topology*, Colloq. Math. **44** (1981), 227–240.
[Mn69] Mandelker M., *F'-spaces and z-embedded subspaces*, Pacific J. Math. **28** (1969), 615–621.
[MR69] Mioduszewski J., Rudolph L., *H-closed and extremally disconnected Hausdorff spaces*, Dissertationes Math. **LXVI** (1969, Warsaw).
[Mo70] Montgomery R., *Structures determined by prime ideals of rings of functions*, Trans. Amer. Math. Soc. **147** (1970), 367–380.
[Mo73] Montgomery R., *The mapping of prime z-ideals*, Symp. Math. **17** (1973), 113–124.
[Mr70] Mrowka S., *Some comments on the author's example of a non-R-compact space*, Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys. **18** (1970), 443–448.
[O80] Oxtoby J., *Measure and Category*, 2nd edition, Springer-Verlag, Berlin-Heidelberg-New York, 1980.
[PW88] Porter J., Woods R.g., *Extensions and Absolutes of Hausdorff Spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1988.
[vRS82] van Rooij A., Shikof W., *A Second Course in Real Analysis*, Cambridge Univ. Press, Cambridge, England, 1982.
[Se59] Semadeni Z., *Sur les ensembles clairsemés*, Rozprawy Mat. **19** (1959), Warsaw.
[Se71] Semadeni Z., *Banach Spaces of Continuous Functions*, Polish Scientific Publishers, Warsaw, 1971.
[T68] Telgársky R., *Total paracompactness and paracompact dispersed spaces*, Bull. Acad. Polon. Sci. **16** (1968), 567–572.
[Ve73] Veksler A.G., *P'-points, P'-sets and P'-spaces. A new class of order-continuous measures and functionals*, Soviet Math. Dokl. **14** (1973), 1440–1445.
[W75] Weir M., *Hewitt-Nachbin Spaces*, North Holland Publ. Co., Amsterdam, 1975.

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