# On the local moduli space of locally homogeneous affine connections in plane domains

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*Abstract.* Classification of locally homogeneous affine connections in two dimensions is a nontrivial problem. (See [5] and [7] for two different versions of the solution.) Using a basic formula by B. Opozda, [7], we prove that all locally homogeneous torsion-less affine connections defined in open domains of a 2-dimensional manifold depend essentially on at most 4 parameters (see Theorem 2.4).

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## 1. Introduction

The field of affine differential geometry is well-established and still in quick development (see e.g. [6]). Also, many basic facts about affine transformation groups and affine Killing vector fields are known from the literature (see [2], [1]). Yet, it is remarkable that the seemingly easy problem to classify all locally homogeneous torsion-less connections in plane domains was not solved until recently. (For dimension three it seems to be a really hard problem.)

At the present time, two different versions of such a classification in dimension two have been worked out (see [5], [7] for the direct reference and [3], [4] for related topics published earlier).

In [7], Theorem 1.1, the following basic result has been proved:

**Theorem 1.1.** *Let $\nabla$ be a torsion-less locally homogeneous affine connection on a 2-dimensional manifold $M$. Then, either $\nabla$ is a Levi-Civita connection of constant curvature or, in a neighborhood $U$ of each point $p \in M$ there is a system $(u, v)$ of local coordinates and constants $a, b, c, d, e, f$ such that $\nabla$ is expressed in $U$ by one of the following formulas:*
**Type A***:*

$$\nabla_{\partial_u}\partial_u = a\partial_u + b\partial_v,\ \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v,\ \nabla_{\partial_v}\partial_v = e\partial_u + f\partial_v. \tag{1}$$

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**Type B***:*

$$\nabla_{\partial_u}\partial_u = \frac{a\partial_u + b\partial_v}{u},\ \nabla_{\partial_u}\partial_v = \frac{c\partial_u + d\partial_v}{u},\ \nabla_{\partial_v}\partial_v = \frac{e\partial_u + f\partial_v}{u}\,. \tag{2}$$

Here a "Levi-Civita connection" involves also the Lorentzian case. Further, let us remark that all connections of type A are of special type (Ricci symmetric and projectively flat). Hence the connections of type B cannot be reduced to those of type A. Yet, it is still not known if both classes have a non-empty intersection.

The aim of this short note is to draw some consequences from the basic theorem above and to show "how large" the family of all homogeneous torsion-less connections in the plain domains is.

## 2. The main result

We shall start with following

**Lemma 2.1.** *Let a torsion-less affine connection on* $\mathbb{R}^2$ *be of type* A, *i.e., with respect to a local coordinate system* $(u, v)$ *it is expressed by formula* (1). *Then there is a* (*nonsingular*) *linear transformation* $x = pu + qv$, $y = ru + sv$ *such that, with respect to the new system of local coordinates* $(x, y)$, *at least two of the Christoffel symbols are zero.*

Proof: Consider any linear transformation as above and the inverse transformation $u = Px + Qy$, $v = Rx + Sy$. Then $D = PS - RQ \neq 0$ and

$$p = S/D,\ q = -Q/D,\ r = -R/D,\ s = P/D. \tag{3}$$

Using the expressions (3), we get for the new coordinate vector fields $\partial_x, \partial_y$

$$\nabla_{\partial_x}\partial_x = \overline{a}\partial_x + \overline{b}\partial_y, \quad \nabla_{\partial_x}\partial_y = \overline{c}\partial_x + \overline{d}\partial_y, \quad \nabla_{\partial_y}\partial_y = \overline{e}\partial_x + \overline{f}\partial_y \tag{4}$$

where

$$\begin{aligned}
D\bar{a} &= aP^2S - bP^2Q + 2cPRS - 2dPQR + eR^2S - fQR^2\,,\\
D\bar{b} &= -aP^2R + bP^3 - 2cPR^2 + 2dP^2R - eR^3 + fPR^2\,,\\
D\bar{c} &= aPQS - bPQ^2 + cS(PS + QR) - dQ(PS + QR) + eRS^2 - fQRS\,,\\
D\bar{d} &= -aPQR + bP^2Q - cR(PS + QR) + dP(PS + QR) - eR^2S + fPRS\,,\\
D\bar{e} &= aQ^2S - bQ^3 + 2cQS^2 - 2dQ^2S + eS^3 - fQS^2\,,\\
D\bar{f} &= -aQ^2R + bPQ^2 - 2cQRS + 2dPQS - eRS^2 + fPS^2\,.
\end{aligned} \tag{5}$$

Our goal is to choose constants $P, Q, R, S$ such that $D = PS - RQ \neq 0$ and at least two quantities of $\overline{a}, \overline{b}, \dots, \overline{f}$ are zero. We shall start with the general case in which we can satisfy $\overline{a} = \overline{d} = 0$. Here we have first from (5)

$$\overline{a} + \overline{d} = (a + d)P + (c + f)R \tag{6}$$

and thus $\overline{a} + \overline{d} = 0$ holds if we put

$$P = c + f,\ R = -(a + d). \tag{7}$$

Substituting this for $P$ and $R$ in the first equation of (5) we see that $\overline{a} = 0$ reduces to

$$\begin{aligned}\big((c+f)^2 b - 2(a+d)(c+f)d + (a+d)^2 f\big)\ Q -\\ - \big((c+f)^2 a - 2(a+d)(c+f)c + (a+d)^2 e\big)\ S = 0\,.\end{aligned} \tag{8}$$

This will be satisfied if we put

$$\begin{aligned} Q &= (c+f)^2 a - 2(a+d)(c+f)c + (a+d)^2 e\,,\\ S &= (c+f)^2 b - 2(a+d)(c+f)d + (a+d)^2 f\,.\end{aligned} \tag{9}$$

Hence the equations $\overline{a} = \overline{d} = 0$ will be solved if the values $P, Q, R, S$ defined by (7) and (9) satisfy $D = PS - RQ \neq 0$, which is equivalent to

$$(c+f)^3 b + (a+d)^3 e - (a+d)(c+f)(ac - 2af + 4cd + df) \neq 0. \tag{10}$$

Thus, it remains the case

$$(c+f)^3 b + (a+d)^3 e - (a+d)(c+f)(ac - 2af + 4cd + df) = 0. \tag{11}$$

We shall distinguish the following subcases which imply (11) and exclude each other:

I) $a + d = 0,\ b = 0,$
II) $a + d = 0,\ b \neq 0,\ c + f = 0,$
III) $a + d \neq 0,\ e = e_{\mathrm{spec}},$

where

$$e_{\mathrm{spec}} = \frac{(c+f)\big((a+d)(ac - 2af + 4cd + df) - (c+f)^2 b\big)}{(a+d)^3}\ . \tag{12}$$

**Case I.** Here we can assume $a \neq 0, f \neq 0$, because otherwise a pair of Christoffel symbols, $(a, b)$ or $(b, f)$, would be already zero. Substituting

$$d = -a,\ b = 0 \tag{13}$$

and

$$S = 0,\ P = 1,\ Q = 1,\ R = 2a/f \tag{14}$$

into (5), we see that $D = PS - RQ = -2a/f \neq 0$ and $\overline{a} = \overline{e} = 0$.

**Case II.** Here we substitute

$$d = -a,\ f = -c \tag{15}$$

and

$$P = 1,\ Q = a,\ R = 0,\ S = b \tag{16}$$

into (5) and we see that $\overline{a} = \overline{d} = 0$, $D = PS - RQ = b \neq 0$.

**Case III.** First we substitute $P$ and $R$ from (7) into the second formula of (5) and we find that the equality $\overline{b} = 0$ is equivalent to the equality (11). Hence if we put here $e = e_{\rm spec}$ from (12), the component $\overline{b}$ will vanish identically for arbitrary $Q$ and $S$. If we make the same substitutions in the third formula of (5), we see that $\overline{c} = 0$ is equivalent to

$$\begin{aligned}&(a+d)\Big((a+d)Q + (c+f)S\Big)\Big\{\Big((a+d)\big((a+d)d - (c+f)b\big)\Big)Q + \\ &+ \Big((a+d)(2af - 3cd - df) + (c+f)^2 b\Big)S\Big\} = 0.\end{aligned} \tag{17}$$

Now, $a + d \neq 0$ and we must also have $(a+d)Q + (c+f)S \neq 0$ because otherwise $D = PS - RQ = 0$, a contradiction. Therefore, we shall try to satisfy

$$\Big((a+d)\big((a+d)d - (c+f)b\big)\Big)Q + \Big((a+d)(2af - 3cd - df) + (c+f)^2 b\Big)S = 0 \tag{18}$$

and we put to this purpose

$$Q = -(a+d)(2af - 3cd - df) - (c+f)^2 b,\ S = (a+d)\big((a+d)d - (c+f)b\big). \tag{19}$$

If we substitute for $P, R, Q, S$ their expressions from (7) and (19) into $D = PS - RQ$, we see that $D \neq 0$ is equivalent to the inequality

$$(c+f)^2 b + (a+d)(af - 2cd - df) \neq 0. \tag{20}$$

Thus, this inequality guarantees that $\overline{b}$ and $\overline{c}$ vanish for some regular transformation of coordinates.

It remains the new singular case

$$(c+f)^2 b + (a+d)(af - 2cd - df) = 0. \tag{21}$$

Suppose now that $c + f = 0$ in (21). Then we obtain $(a+d)^2 c = 0$ and because $a + d$ is still nonzero, we get $c = f = 0$, which is a trivial case. Hence the only nontrivial case to satisfy (21) is

$$a + d \neq 0,\ c + f \neq 0,\ e = e_{\rm spec},\ b = b_{\rm spec}\,, \tag{22}$$

where

$$b_{\mathrm{spec}} = (a+d)(-af + 2cd + df)/(c+f)^2. \tag{23}$$

(Here the new form of $e_{\mathrm{spec}}$ will be not relevant.) Using a linear transformation with

$$P = c + f,\ R = -(a+d),\ Q = 1,\ S = 0, \tag{24}$$

we obtain $PS - QR = a + d \neq 0$. If we substitute for $b, P, Q, R, S$ from (23), (24) into the first formula of (5), we see at once that $\overline{a} = 0$. Hence we get $\overline{a} = \overline{b} = 0$ in this case and the proof of lemma is finished. □

We can now see that the same procedure as in Lemma 2.1 can be applied, with a slight modification but with the same equations (5), also to the connections of type B. We obtain

**Lemma 2.2.** *Let $\nabla$ be a connection of type* B *in a neighborhood $U \subset M$. Then, using a non-singular linear transformation of local coordinates, $\nabla$ can be expressed in the form*

$$\nabla_{\partial_u}\partial_u = \frac{a\partial_u + b\partial_v}{\alpha u + \beta v}\,,\ \nabla_{\partial_u}\partial_v = \frac{c\partial_u + d\partial_v}{\alpha u + \beta v}\,,\ \nabla_{\partial_v}\partial_v = \frac{e\partial_u + f\partial_v}{\alpha u + \beta v}\,, \tag{25}$$

*where at least two of the constants $a, b, \dots, f$ are zero and $\alpha, \beta$ are again constants, not both equal to zero.*

**Theorem 2.3.** *Let $\nabla$ be a torsion-less locally homogeneous affine connection on a 2-dimensional manifold $M$. If $\nabla$ is of type* A*, then, with respect to a convenient system of local coordinates, at least four of the constants $a, b, \dots, f$ are equal to $0, 1$, or $-1$. If $\nabla$ is of type* B *and expressed in the form* (25)*, then the same result holds.*

Proof: Let first $\nabla$ be of type A. Assume that at least two of the Christoffel symbols $a, b, \dots, f$ are already equal to zero. Let us now consider a new linear transformation $u = Px,\ v = Sy,\ PS \neq 0,\ R = Q = 0$. Then the system (5) reduces to

$$\overline{a} = aP,\ \overline{b} = bP^2/S,\ \overline{c} = cS,\ \overline{d} = dP,\ \overline{e} = eS^2/P,\ \overline{f} = fS. \tag{26}$$

If at least three of the coefficients $a, b, \dots, f$ are nonzero, then we can choose $P$ and $S$ in such a way that at least two new coefficients are equal to 1 or $-1$, respectively. The zero coefficients remain zero.

Let now $\nabla$ be of type B and written in the form (25). Then after putting $\overline{\alpha} = P\alpha$, $\overline{\beta} = S\beta$ in the transformed formulas we can still use (26) for the transformation of $a, b, \dots, f$ into $\overline{a}, \overline{b}, \dots, \overline{f}$. Hence the result follows. □

(One can check easily that our procedure does not work if the connection $\nabla$ is expressed in the form (2).)

We conclude our Note with the main theorem, which follows from the previous results:

**Theorem 2.4.** *Let $M$ be a 2-dimensional manifold. The set $\mathcal{A}$ of all equivalence classes of torsion-less locally homogeneous affine connections on open domains of $M$ with respect to local affine diffeomorphisms is a union of finite number of (possibly intersecting) subsets $\mathcal{A}_i$ $(i = 1, \dots, r)$ such that*
(a) *if $\mathcal{A}_i$ corresponds to a Levi-Civita connection of constant curvature then it consist of one element;*
(b) *if $\mathcal{A}_i$ is of type* A, *then it depends on at most* 2 *arbitrary parameters;*
(c) *if $\mathcal{A}_i$ is of type* B, *then it depends on at most* 4 *arbitrary parameters.*

Instead of a full proof we shall only give a short explanation. In case (a), we obtain 5 classes $\mathcal{A}_i$ consisting of one element. One element (representing any flat connection) corresponds to both Euclidean and flat Lorentzian 2-space. The other four elements correspond to the different signs of nonzero constant curvatures combined with the different signatures of the corresponding metrics.

In case (b), inspecting carefully the proof of Lemma 2.1, we find just four non-equivalent cases for vanishing of two Christoffel symbols: $a = b = 0$, $a = c = 0$, $a = d = 0$ and $a = e = 0$. (We see that all these cases still have a non-empty intersection.) Now, using the proof of Theorem 2.3, we obtain a new decomposition into more subcases. These subcases correspond to the subsets $\mathcal{A}_i$ from Theorem 2.4.

In case (c) we proceed in an analogous way.

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